

Inverse Spectral And Inverse Nodal Problems For Sturm-Liouville Equations With Point δ' -Interaction

Manaf Dzh. Manafov

Received: 07.10.2016 / Revised: 01.05.2017/ Accepted: 08.08.2017

Abstract. Inverse spectral and inverse nodal problems are studied for Sturm-Liouville equations with δ' -interaction. We obtain uniqueness and reconstruction using the nodal set of eigenfunctions for the given problem.

Keywords. Sturm-Liouville equations · Inverse spectral and inverse nodal problems · Point δ' -interaction

1 Introduction

We consider the Sturm-Liouville boundary value problem (BVP) L :

$$ly := -y'' + q(x)y = \lambda y, \quad x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \quad (1.1)$$

$$U(y) := y'(0) = 0, \quad V(y) := y(\pi) = 0, \quad (1.2)$$

$$I(y) := \begin{cases} y\left(\frac{\pi}{2} + 0\right) - y\left(\frac{\pi}{2} - 0\right) = \alpha y'\left(\frac{\pi}{2}\right), \\ y'\left(\frac{\pi}{2} + 0\right) = y'\left(\frac{\pi}{2} - 0\right) \equiv y'\left(\frac{\pi}{2}\right), \end{cases} \quad (1.3)$$

where $q(x)$, $\alpha \neq 0$ are real, and $q(x) \in W_2^1(0, \pi)$, λ is a spectral parameter.

Without loss of generality we assume that

$$\int_0^\pi q(x)dx = 0. \quad (1.4)$$

Notice that, we can understand problem (1.1) and (1.3) as studying the equation

$$-y'' + \left(\alpha \delta' \left(x - \frac{\pi}{2}\right) + q(x)\right)y = \lambda y, \quad x \in (0, \pi), \quad (1.5)$$

where $\delta'(x)$ is the Dirac derivative function (see [1]).

On the separable Hilbert space $L^2([0, \pi])$ consider the linear differential operator

$$L : y(x) \rightarrow -y''(x) + q(x)y(x)$$

M. Dzh. Manafov

Adiyaman University, Faculty of Science and Arts, Department of Mathematics, Adiyaman, 02040, Turkey
Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, AZ 1141, Azerbaijan
E-mail: mmanafov@adiyaman.edu.tr

with a dense domain

$$D(L) = \left\{ y(x) \in W_2^2 \left[(0, \pi) \setminus \left\{ \frac{\pi}{2} \right\} \right] \cap W_2^0[(0, \pi)] : y\left(\frac{\pi}{2} + 0\right) - y\left(\frac{\pi}{2} - 0\right) = \alpha y'\left(\frac{\pi}{2}\right), \right. \\ \left. y'\left(\frac{\pi}{2} + 0\right) = y'\left(\frac{\pi}{2} - 0\right), y'(0) = 0, y(\pi) = 0 \right\}.$$

We will consider inverse problems of recovering $q(x)$ the given spectral and nodal characteristics. We denote the BVP (1.1)-(1.3) by $L = L(q)$.

Inverse spectral problems consist in recovering operators from their spectral characteristics. Such problems play an important role in mathematics and have many applications in natural sciences (see, for example, monographs [4, 8, 12–14, 17]). Inverse nodal problems consist in constructing operators from the given nodes (zeros) of eigenfunctions (see [5, 7, 11, 15, 16]).

In this present paper we obtain some results on inverse spectral and inverse nodal problems and establish connections between them.

2 Inverse Spectral Problems

In this section we study so-called incomplete inverse problem of recovering the potential $q(x)$ from a part of the spectrum BVP L . The technique employed is similar to those used in [17].

Let $y(x)$ and $z(x)$ be continuously differentiable functions on the intervals $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi)$. Denote $\langle y, z \rangle := yz' - y'z$. If $y(x)$ and $z(x)$ satisfy the matching conditions (1.3), then

$$\langle y, z \rangle_{x=\frac{\pi}{2}-0} = \langle y, z \rangle_{x=\frac{\pi}{2}+0}, \quad (2.1)$$

i.e. the function $\langle y, z \rangle$ is continuous on $(0, \pi)$.

Let $\varphi(x, \lambda)$ be solution of equation (1.1) satisfying the initial conditions $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = 0$ and the matching conditions (1.3). Then $U(\varphi) = 0$. Denote

$$\Delta(\lambda) := -V(\varphi) = -\varphi(\pi, \lambda). \quad (2.2)$$

By virtue of (2.1) and the Liouville's formula (see [2, p.83]), $\Delta(\lambda)$ does not depend on x . The function $\Delta(\lambda)$ is called characteristic function on L .

Lemma 2.1 *The BVP L has a countable set of eigenvalues $\{\lambda_n\}_{n \geq 1}$. All eigenvalues are real, simple and $n \rightarrow \infty$*

$$\sqrt{\lambda_n} = n + \frac{1}{\pi n} \left(w_0 + (-1)^{n-1} w_1 \right) + \frac{1}{\pi n^2} w_2 + o\left(\frac{1}{n^2}\right), \quad (2.3)$$

$$w_0 = \frac{2}{\alpha} + \frac{1}{2} \int_0^\pi q(t) dt, \quad w_1 = \frac{1}{2} \left(\int_{\frac{\pi}{2}}^\pi q(t) dt - \int_0^{\frac{\pi}{2}} q(t) dt \right), \\ w_2 = -\frac{1}{4\alpha} \left\{ 4 \int_0^\pi q(t) dt + [q(0) + (\alpha - 1)q\left(\frac{\pi}{2}\right) - \alpha q(\pi)] \right. \\ \left. + \left[\alpha \left(\int_0^\pi q(t) dt \right)^2 + \left(\frac{1}{2} - \alpha \right) \left(\int_0^{\frac{\pi}{2}} q(t) dt \right)^2 \right] \right\}. \quad (2.4)$$

Proof. Let $\lambda = k^2$, $\tau := Imk$. From (2.2), in order to find eigenvalues, we have to construct a solution $\varphi(x, \lambda)$ for all interval $(0, \pi)$. For this reason we construct the following equation

$$-y'' + \left(\alpha \delta' \left(x - \frac{\pi}{2} \right) + q(x) \right) y = \lambda y, \quad x \in (0, \pi),$$

which has a solution $\varphi(x, \lambda)$. For $|\lambda| \rightarrow \infty$ uniformly in x one has:

$$\begin{aligned} \varphi(x, \lambda) &= \cos kx + \frac{1}{2k} \int_0^x q(t) dt \cdot \sin kx \\ &\quad + \frac{1}{4k^2} \left\{ (q(x) - q(0)) - \frac{1}{2} \left[\int_0^x q(t) dt \right]^2 \right\} \cos kx \\ &\quad + o \left(\frac{1}{k^2} \exp(|\tau|x) \right), \quad x < \frac{\pi}{2}. \end{aligned} \quad (2.5)$$

$$\begin{aligned} \varphi'(x, \lambda) &= -k \sin kx + \frac{1}{2} \int_0^x q(t) dt \cdot \cos kx \\ &\quad + \frac{1}{4k} \left\{ -(q(x) - q(0)) + \frac{1}{2} \left[\int_0^x q(t) dt \right]^2 \right\} \sin kx \\ &\quad + o \left(\frac{1}{k} \exp(|\tau|x) \right), \quad x < \frac{\pi}{2}. \end{aligned} \quad (2.6)$$

Using the matching conditions (1.3) we write $\varphi(x, \lambda)$ in $(\frac{\pi}{2}, \pi)$:

$$\begin{aligned} \varphi(x, \lambda) &= -k \frac{\alpha}{2} [\sin kx + \sin k(\pi - x)] + \left[\left[1 + \frac{\alpha}{4} \int_0^x q(t) dt \right] \cos kx \right. \\ &\quad \left. + \frac{\alpha}{4} \left[\int_0^{\frac{\pi}{2}} q(t) dt - \int_{\frac{\pi}{2}}^x q(t) dt \right] \cos k(\pi - x) \right. \\ &\quad \left. + \frac{1}{4k} \left\{ 2 \int_0^x q(t) dt + \frac{1}{2} \left[q(0) + (\alpha - 1) q \left(\frac{\pi}{2} \right) - \alpha q(x) \right] \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \left[\alpha \left(\int_0^x q(t) dt \right)^2 - \left(\alpha - \frac{1}{2} \right) \left(\int_0^{\frac{\pi}{2}} q(t) dt \right)^2 \right] \right\} \sin kx \right. \\ &\quad \left. + \frac{1}{4k} \left\{ \frac{1}{2} \left[q(0) + (\alpha - 1) q \left(\frac{\pi}{2} \right) - \alpha q(x) \right] + \frac{1}{2} \left[\alpha \left(\int_{\frac{\pi}{2}}^x q(t) dt \right)^2 \right. \right. \right. \\ &\quad \left. \left. \left. - \alpha \int_0^{\frac{\pi}{2}} q(t) dt \int_{\frac{\pi}{2}}^x q(t) dt + \frac{1}{2} \left(\int_0^{\frac{\pi}{2}} q(t) dt \right)^2 \right] \right\} \sin k(\pi - x) \right. \\ &\quad \left. + o \left(\frac{1}{k} \exp(|\tau|x) \right) \right). \end{aligned} \quad (2.7)$$

It follows from (2.7) that for $|\lambda| \rightarrow \infty$

$$\Delta(\lambda) = \frac{\alpha}{2} \left(k \sin k\pi - w_0 \cos k\pi + w_1 - \frac{w_2}{k} \sin k\pi \right) + o \left(\frac{1}{k} \exp(|\tau|\pi) \right). \quad (2.8)$$

Denote $G_\delta = \{k : |k - n| \geq \delta, n = 0, \pm 1, \pm 2, \dots\}$, $\delta > 0$. Let us show that

$$|\sin k\pi| \geq C_\delta \exp(|\tau|\pi), \quad k \in G_\delta, \quad (2.9)$$

$$|\Delta(\lambda)| \geq C_\delta |k| \exp(|\tau|\pi), \quad k \in G_\delta, |k| \geq k^*, \quad (2.10)$$

for sufficiently large $k^* = k^*(\delta)$.

Let $k = \sigma + i\tau$. It is sufficient to prove (2.9) for the domain

$$D_\delta = \left\{ k : \sigma \in \left[-\frac{1}{2}, \frac{1}{2} \right], \tau \geq 0, |k| \geq \delta \right\}.$$

Denote $\theta(k) = |\sin k\pi| \exp(-|\tau|\pi)$. Let $k \in D_\delta$. For $\tau \geq 1$, $\theta(k) \geq C_\delta$. Since $\sin k\pi = (\exp(ik\pi) - \exp(-ik\pi))/2i$, we have for $\tau \geq 1$, $\theta(k) \geq \frac{1}{4}$. Thus, (2.9) is proved. Further, using (2.8) we get for $k \in G_\delta$,

$$\Delta(\lambda) = \frac{\alpha}{2} k \sin k\pi \left(1 + O\left(\frac{1}{k}\right) \right),$$

and consequently (2.10) is valid.

Denote

$$\Gamma_n = \left\{ \lambda : |\lambda| = \left(n + \frac{1}{2} \right)^2 \right\}.$$

By virtue of (2.8)

$$\Delta(\lambda) = f(\lambda) + g(\lambda), \quad f(\lambda) = k \sin k\pi, \quad |g(\lambda)| \leq C \exp(|\tau|\pi).$$

According to (2.9), $|f(\lambda)| > |g(\lambda)|$, $\lambda \in \Gamma_n$, for sufficiently large n ($n \geq n^*$). Applying Rouché's theorem (see [3]) to the circle $\sigma_n(\delta) = \{k : |k - n| \leq \delta\}$, we conclude that for sufficiently large n , in $\sigma_n(\delta)$ there is exactly one zero of $\Delta(k^2)$. Since $\delta > 0$ is arbitrary, we must have

$$k_n = n + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \rightarrow \infty. \quad (2.11)$$

Substituting (2.11) into (2.8) we get

$$n \sin \varepsilon_n \pi - \left(w_0 \cos \varepsilon_n \pi + (-1)^{n-1} w_1 \right) + \frac{1}{n} w_2 = 0. \quad (2.12)$$

Then $\sin \varepsilon_n \pi = O\left(\frac{1}{n}\right)$, i.e. $\varepsilon_n = O\left(\frac{1}{n}\right)$. Using (2.12) once more we obtain more precisely

$$\varepsilon_n = \frac{1}{\pi n} \left(w_0 + (-1)^{n-1} w_1 \right) + \frac{1}{\pi n^2} w_2. \quad (2.13)$$

Substituting (2.13) into (2.11), we get (2.3). Since the BVP L is selfadjoint (see [10]), all eigenvalue $\{\lambda_n\}_{n \geq 1}$ are real and simple.

Together with L we consider a BVP $\tilde{L} = L(\tilde{q})$ of the same form, but with different coefficient \tilde{q} . The following theorem has been proved in [6] for the classical Sturm-Liouville equation. We show it also holds for (1.1)-(1.3).

Theorem 2.1 *If for any $n \in \mathbb{N}$*

$$\lambda_n = \tilde{\lambda}_n, \quad \langle y_n, \tilde{y}_n \rangle_{x=\frac{\pi}{2}-0} = 0,$$

then $q(x) = \tilde{q}(x)$ almost everywhere (a.e.) on $(0, \pi)$.

Proof. Since

$$-y''(x, \lambda) + q(x)y(x, \lambda) = \lambda y(x, \lambda), \quad -\tilde{y}''(x, \lambda) + \tilde{q}(x)\tilde{y}(x, \lambda) = \lambda \tilde{y}(x, \lambda),$$

$$y(0, \lambda) = 1, \quad y'(0, \lambda) = 0 \quad \tilde{y}(0, \lambda) = 1, \quad \tilde{y}'(0, \lambda) = 0$$

it follows from (2.1) that

$$\int_0^{\frac{\pi}{2}} r(x)y(x, \lambda)\tilde{y}(x, \lambda) dx = \langle y, \tilde{y} \rangle_{x=\frac{\pi}{2}-0}, \quad (2.14)$$

where $r(x) = q(x) - \tilde{q}(x)$. Since $\langle y_n, \tilde{y}_n \rangle_{x=\frac{\pi}{2}-0} = 0$ for $n \in \mathbb{N} \cup \{0\}$, it follows from (2.14) that

$$\int_0^{\frac{\pi}{2}} r(x)y(x, \lambda_n)\tilde{y}(x, \lambda_n) dx = 0, \quad n \in \mathbb{N} \cup \{0\}. \quad (2.15)$$

For $x \leq \frac{\pi}{2}$ the following representation holds (see [8, 12]):

$$y(x, \lambda) = \cos kx + \int_0^x K(x, t) \cos kt dt,$$

where $K(x, t)$ is a continuous function which does not depend on λ . Hence

$$2y(x, \lambda)\tilde{y}(x, \lambda) = 1 + \cos 2k\pi x + \int_0^x V(x, t) \cos 2kt dt, \quad (2.16)$$

where $V(x, t)$ is a continuous function which does not depend on λ . Substituting (2.16) into (2.15) and taking the relation $\int_0^\pi r(x) dx = 0$ into account, we calculate

$$\int_0^{\frac{\pi}{2}} \left(r(x) + \int_x^{\frac{\pi}{2}} V(t, x)r(t) dt \right) \cos 2k_n x dx = 0, \quad n \in \mathbb{N} \cup \{0\},$$

which implies from the completeness of the function cosine, that

$$r(x) + \int_x^{\frac{\pi}{2}} V(t, x)r(t) dt = 0 \text{ a.e. on } \left(0, \frac{\pi}{2}\right).$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution, it follows that $r(x) = 0$ a.e. on $(0, \frac{\pi}{2})$. To prove that $q(x) = \tilde{q}(x)$ a.e. on $(\frac{\pi}{2}, \pi)$ we will consider the supplementary problem \widehat{L} :

$$-y''(x, \lambda) + q_1(x)y(x, \lambda) = \lambda y(x, \lambda), \quad q_1(x) = q(\pi - x), \quad 0 < x < \frac{\pi}{2}$$

$$U(y) := y'(0, \lambda) = 0$$

$$y\left(\frac{\pi}{2} - 0, \lambda\right) + \alpha y'\left(\frac{\pi}{2} - 0, \lambda\right) = y\left(\frac{\pi}{2} + 0, \lambda\right), \quad y'\left(\frac{\pi}{2} - 0, \lambda\right) = y'\left(\frac{\pi}{2} + 0, \lambda\right).$$

It follows from (2.1) that

$$\langle y_n, \tilde{y}_n \rangle_{x=\frac{\pi}{2}+0} = 0.$$

A direct calculation implies that $\widehat{y}_n(x) := y_n(\pi - x)$ is the solution to the supplementary problem \widehat{L} and $\widehat{y}_n(\frac{\pi}{2} - 0) = y_n(\frac{\pi}{2} + 0)$. Thus for the supplementary problem \widehat{L} the assumption conditions in Theorem 2.1 are still satisfied. If we repeat the above arguments then yields $r(\pi - x) = 0$ on $0 < x < \frac{\pi}{2}$, that is $q(x) = \tilde{q}(x)$ a.e. on $(\frac{\pi}{2}, \pi)$.

3 Inverse Nodal Problems

In this section, we consider the inverse nodal problems with δ' - interaction. We obtain uniqueness theorem and a procedure of recovering the potential $q(x)$ on the whole interval $(0, \pi)$ from a dense subset of nodal points. We recall that these results were given for regular Sturm-Liouville problems defined on the interval $(0, 1)$ in [7].

The eigenfunctions of the *BVP L* have the form $y_n(x) = \varphi(x, \lambda_n)$. We note that $y_n(x)$ are real-valued functions. Substituting (2.2) into (2.4) and (2.6) we obtain the following asymptotic formulae for $n \rightarrow \infty$ uniformly in x :

$$y_n(x) = \cos nx + \frac{1}{2n} \left(\int_0^x q(t) dt - \frac{1}{\pi} (w_0 + (-1)^{n-1} w_1) x \right) \cos nx + o\left(\frac{1}{n}\right), \quad x < \frac{\pi}{2}, \quad (3.1)$$

$$\begin{aligned} y_n(x) = & \frac{\alpha}{2} n [(-1)^n - 1] \sin nx + \left\{ 1 - \frac{\alpha}{4} ((-1)^n - 1) \int_0^x q(t) dt \right. \\ & + (-1)^n \frac{\alpha}{2} \int_0^{\frac{\pi}{2}} q(t) dt - \frac{\alpha}{2} \left[(-1)^n \left(1 - \frac{x}{\pi} \right) + \frac{x}{\pi} \right] (w_0 + (-1)^{n-1} w_1) \left. \right\} \cos nx \\ & + \frac{1}{4n} \left\{ \frac{2}{\pi} (w_0 + (-1)^{n-1} w_1) [\alpha ((-1)^n - 1) \right. \\ & + \alpha (w_0 + (-1)^{n-1}) \left[\left(\frac{x}{\pi} \right)^2 - (-1)^n \left(1 - \frac{x}{\pi} \right)^2 \right] - \left(4 + \alpha \int_0^x q(t) dt \right) \left. \right] \\ & + 2 \int_0^x q(t) dt + \frac{1}{2} [1 - (-1)^n] \left[q(0) + (\alpha - 1) q\left(\frac{\pi}{2}\right) - \alpha q(x) \right] \\ & + \frac{1}{2} \left[\alpha \left(\int_0^x q(t) dt \right)^2 - \left(\alpha - \frac{1}{2} \right) \left(\int_0^{\frac{\pi}{2}} q(t) dt \right)^2 \right] - (-1)^n \frac{1}{2} \left[\alpha \left(\int_{\frac{\pi}{2}}^x q(t) dt \right)^2 \right. \\ & \left. - \alpha \int_0^{\frac{\pi}{2}} q(t) dt \int_{\frac{\pi}{2}}^x q(t) dt + \frac{1}{2} \left(\int_0^{\frac{\pi}{2}} q(t) dt \right)^2 \right] \left. \right\} \sin nx \\ & + \frac{\alpha}{2n} \left[(-1)^n \left(\frac{x}{\pi} - 1 \right) - \frac{x}{\pi} \right] w_2 \cos nx + o\left(\frac{1}{n}\right), \quad x > \frac{\pi}{2}. \quad (3.2) \end{aligned}$$

We used the same method employed in [[9], ch.1, pp. 14-18] to prove the eigenfunction $y_n(x)$ corresponding eigenvalue λ_n exactly having $\left[\frac{n}{2}\right] + n - 1$ simple zeros in $(0, \pi)$. More precisely, $\left[\frac{n}{2}\right]$ of $\left[\frac{n}{2}\right] + n - 1$ simple zeros lies in $(0, \frac{\pi}{2})$ and the rest is in $(\frac{\pi}{2}, \pi)$:

$$0 < x_n^1 < x_n^2 < \dots < x_n^{\left[\frac{n}{2}\right]} < \frac{\pi}{2} < x_n^{\left[\frac{n}{2}\right]+1} < \dots < x_n^{\left[\frac{n}{2}\right]+n-1} < \pi.$$

A set $X_L := \left\{ x_n^j \right\}_{n \geq 2, j=1, \left[\frac{n}{2}\right]+n-1}$ is called the set of all nodal points of the *BVP L*. Let

$$X_L^k := \left\{ x_{2m+k}^j \right\}_{m \geq 1, j=3m-k-1}, \quad k = 0, 1. \quad \text{Clearly, } X_L^0 \cup X_L^1 = X_L.$$

Set

$$\mu_n^0 := 0, \quad \mu_n^{\left[\frac{n}{2}\right]+n} := \pi, \quad \mu_n^j := \frac{j}{n} \pi \quad \text{for } n = 2m + 1$$

and

$$\gamma_n^j := \mu_n^j - \frac{\pi}{2n} \quad \text{for } n = 2m.$$

Inverse nodal problems consist in recovering the problem $q(x)$ from the given set X_L of nodal points or from a certain its part.

Taking (3.1)-(3.2) into account, we obtain the following asymptotic formulae for nodal points as $n \rightarrow \infty$ uniformly in j :

for $x_n^j \in (0, \frac{\pi}{2})$:

$$x_n^j = \gamma_n^j + \frac{1}{2n^2} \left(\int_0^{\gamma_n^j} q(t) dt - \frac{1}{\pi} (w_0 - w_1) \gamma_n^j \right) + o\left(\frac{1}{n^2}\right), \quad n = 2m, \quad (3.3)$$

$$x_n^j = \gamma_n^j + \frac{1}{2n^2} \left(\int_0^{\gamma_n^j} q(t) dt - \frac{1}{\pi} (w_0 + w_1) \gamma_n^j \right) + o\left(\frac{1}{n^2}\right), \quad n = 2m + 1, \quad (3.4)$$

for $x_n^j \in (\frac{\pi}{2}, \pi)$:

$$x_n^j = \gamma_n^j + \frac{\beta}{2n^2} \left(\int_0^{\gamma_n^j} q(t) dt - \frac{1}{2\pi} (w_0 - w_1) (2\gamma_n^j - \pi) + d_1 \right) + o\left(\frac{1}{n^2}\right), \quad n = 2m, \quad (3.5)$$

$$\frac{x_n^j}{n} = \mu_n^j + \frac{1}{2n^2} \left(\int_0^{\mu_n^j} q(t) dt - \frac{1}{2\pi} (w_0 + w_1) (2\mu_n^j - \pi) + d_2 \right) + o\left(\frac{1}{n^2}\right), \quad n = 2m + 1, \quad (3.6)$$

where

$$\beta = \frac{8}{\alpha\pi} - \frac{4}{\pi} \int_0^{\frac{\pi}{2}} q(t) dt - 1, \quad (3.7)$$

$$d_1 = \frac{4}{\pi} \left(\int_0^{\frac{\pi}{2}} q(t) dt \right)^2 - \frac{4}{\alpha}, \quad d_2 = \frac{1}{2} \int_0^{\frac{\pi}{2}} q(t) dt - \frac{1}{\alpha}.$$

Using these formula we arrive at the following assertion.

Theorem 3.1 Fix $k \in \{0, 1\}$ and $x \in [0, \pi]$. Let $\{x_n^j\} \subset X_L^k$ be chosen such that $\lim_{n \rightarrow \infty} x_n^j = x$. Then there exists a finite limit

$$g_k(x) := \lim_{n \rightarrow \infty} 2n \begin{bmatrix} nx_n^j - j\pi + \frac{\pi}{2}, & \text{if } x_n^j \in (0, \frac{\pi}{2}), \\ \frac{1}{\beta} \left(nx_n^j - j\pi + \frac{\pi}{2} \right), & \text{if } x_n^j \in (\frac{\pi}{2}, \pi), n = 2m, \\ x_n^j - j\pi, & \text{if } x_n^j \in (\frac{\pi}{2}, \pi), n = 2m + 1 \end{bmatrix}, \quad (3.8)$$

and

$$g_k(x) = \int_0^x q(t) dt - \frac{1}{\pi} (w_0 + (-1)^{k-1} w_1) x, \quad x \leq \frac{\pi}{2},$$

$$g_k(x) = \int_0^x q(t) dt - \frac{1}{2\pi} (w_0 + (-1)^{k-1} w_1) (2x - \pi) + d_k, \quad x \geq \frac{\pi}{2} \quad (3.9)$$

where d_1 and d_2 are defined by (3.7).

Let us now formulate a uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem.

Theorem 3.2 Fix $k \in \{0, 1\}$. Let $X \subset X_L^k$ be a subset of nodal points which is dense on $(0, \pi)$. Let $X = \tilde{X}$. Then $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$. Thus the specification of X uniquely determines the potential $q(x)$ on $(0, \pi)$. The function $q(x)$ can be constructed via the formula

$$q(x) = g'_k(x) + \frac{1}{\pi} (g_k(\pi) - g_k(0)), \quad (3.10)$$

where $g_k(x)$ is calculated by (3.9).

Proof. Formula (3.10) follows from (3.9), (1.4) and (2.4). Note that by (3.9), we have

$$g'_k(x) = q(x) - \frac{w_0 + (-1)^{k-1} w_1}{\pi}, \quad x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \quad (3.11)$$

hence

$$g_k(\pi) - g_k(0) = \int_0^\pi q(x) dx - (w_0 + (-1)^{n-1} w_1). \quad (3.12)$$

Then (3.10) can be derived directly from (3.11) and (3.12). Note that if $X = \tilde{X}$, then (3.8) yields $g_k(x) \equiv \tilde{g}_k(x)$, $x \in [0, \pi]$. By (3.10), we obtain $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$.

By a similar way, in general the following theorem can be proved.

Theorem 3.3 Let $X \subset X_L$ be a dense on $(0, \pi)$ subset of nodal points. Let $X = \tilde{X}$. Then $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$.

Acknowledgements The author wants to thank the anonymous referees for their valuable suggestions that improving the quality of this article.

References

1. Albeverio, S., Gesztesy, F., Hoegh-Krohn, R., Holden, H. with an appendix by P. Exner, Solvable Models in Quantum Mechanics (second edition), AMS Chelsea Publ. (2005)
2. Coddington, E. A., Levinson, N., Theory of Ordinary Differential Equations, McGraw-Hill, New York (1955)
3. Conway, J. B., Functions of One Complex Variable, 2nd ed., vol. I, Springer-Verlag, New York (1995)
4. Freiling, G., Yurko, V. A., Inverse Sturm-Liouville Problems and Their Applications, NOVA Science Publ., New York (2001)
5. Guo, Y., Wei, G., Inverse Problems: Dense nodal subset on an interior subinterval, J. Diff. Equations, 255, 2002-2017 (2013)
6. Horvath, M., Inverse spectral problems and closed exponential systems, Ann. of Math., 162, 885-918 (2005)
7. Law, C. K., Yang, C. F., Reconstructing the potential function and its derivatives using nodal data, Inverse Problems, 14, 299-312 (1998)
8. Levitan, B. M., Inverse Sturm-Liouville Problems, Nauka, Moscow (1984), Eng. transl.: VNU Science Press, Utrecht (1987)
9. Levitan, B. M., Sargsjan, I. S., Introduction to spectral theory: Self-adjoint ordinary differential operators, American Mathematical Soc., Eng. transl., (1975)
10. Manafov, M. Dzh., Description of the domain of an ordinary differential operator with generalized potentials, Diff. Equations, 32, 716-718 (1996)
11. Manafov, M. Dzh., Kablan, A., Inverse spectral and inverse nodal problems for energy-dependent Sturm-Liouville equations with δ - interaction, Elect. J. of Diff. Equations, 2015, 1-10 (2015)

-
12. Marchenko, V. A., *Sturm-Liouville Operators and Their Applications*, Naukova Dumka, Kiev (1977); Eng. transl.: Birkhäuser (1986)
 13. Pöschel, J., Trubowitz, E., *Inverse Spectral Theory*, Academic Press, New York (1987)
 14. Sadovnichy, V. A., Sultanaev, Y. T., Akhtyamov, A. M., *Inverse Sturm-Liouville Problems with Nonseparated Boundary Conditions*, MSU, Moscow (2009)
 15. Shieh, C.-T., Yurko, V. A., Inverse nodal and inverse spectral problems for discontinuous boundary value problems, *J. Math. Anal. Appl.*, 347, 266-272 (2008)
 16. Yang, C.-F., Inverse nodal problems of discontinuous Sturm-Liouville operator, *J. Diff. Equations*, 254, 1992-2014 (2013)
 17. Yurko, V. A., *Method of Spectral Mappings in the Inverse Problem Theory*, Inverse III-posed Probl. Ser., VSP, Utrecht (2002)