

On boundedness of the operator Riesz function and potential generated by Gegenbauer differential operator on Morrey space

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Abstract. We consider the generalized shift operator, associated with the Gegenbauer differential operator $G_\lambda = (x^2 - 1)^{\frac{1}{2}-\lambda} \frac{d}{dx} (x^2 - 1)^{\lambda+\frac{1}{2}} \frac{d}{dx}$. The maximal operator M_G (G-maximal operator) and the Riesz potential I_G^α (G-Riesz potential), associated with the generalized shift operator are investigated. At first, we prove that the G-maximal operator M_G is bounded from the G-Morrey space $L_{p,\lambda,\gamma}$ to $L_{p,\lambda,\gamma}$ for all $1 < p < \infty$, and is bounded from the G-Morrey space $L_{1,\lambda,\gamma}$ to the weak G-Morrey space $WL_{q,\lambda,\gamma}$ for $0 \leq \alpha < 2\lambda + 1$. Moreover we prove that the G-Riesz potential I_G^α $0 < \alpha < 2\gamma + 1$ is bounded from the G-Morrey space $L_{p,\lambda,\gamma}$ to $L_{q,\lambda,\gamma}$ if and only if $\frac{\alpha}{2\lambda+1-\gamma} = \frac{1}{p} - \frac{1}{q}$, $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$.

Also we prove that the G-Riesz potential I_G^α is bounded from the G-Morrey space $L_{1,\lambda,\gamma}$ to the weak G-Morrey space $WL_{q,\lambda,\gamma}$ if and only if $\frac{\alpha}{2\lambda+1-\gamma} = 1 - \frac{1}{q}$.

Keywords. G-Riesz potential, G-maximal function, G-Morrey space.

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1 Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball centered at x of radius r . Let $f \in L_1^{loc}(\mathbb{R}^n)$. The maximal operator M and the Riesz potential I^α are defined by

$$Mf(x) = \sup_{t>0} \frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y)| dy,$$

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y) dy}{|x - y|^{n-\alpha}}, \quad 0 < \alpha < n,$$

where $|B(x, t)|$ is the Lebesgue measure of the ball $B(x, t)$.

The operators M and I^α play important role in real and harmonic analysis.

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In the theory of partial differential equations Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play important role. They were introduced by C. Morrey in 1938 [9] and defined as follows: For $0 \leq \lambda \leq n$, $1 \leq p < \infty$, $f \in L_{p,\lambda}(\mathbb{R}^n)$ if $f \in L_p^{loc}(\mathbb{R}^n)$ and

$$\|f\|_{L_{p,\lambda}} \equiv \|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in (\mathbb{R}^n), r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

If $\lambda = 0$, then $L_{p,\lambda}(\mathbb{R}^n)$, $L_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ if $\lambda = n$, then $L_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \{0\}$ is the set of all functions equivalent to 0 on \mathbb{R}^n .

Also by $WL_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in (\mathbb{R}^n), r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

F. Chiarenza and M. Frasca [2] studied the boundedness of the maximal operator M in Morrey spaces $L_{p,\lambda}$. Their results can be summarized as follows:

Theorem A. Let $0 < \alpha < n$ and $0 \leq \lambda < n$, $1 \leq p < \infty$.

1) If $1 < p < \infty$ then H is bounded from $L_{p,\lambda}$ to $L_{p,\lambda}$,

2) If $p = 1$, then M is bounded $L_{1,\lambda}$ to $WL_{1,\lambda}$.

The classical result by Hardy-Littlewood-Sobolev states that $1 < p < q < \infty$, then I^α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = \frac{n}{p} - \frac{n}{q}$ and for $p = 1 < q < \infty$ is bounded from $L_{1,\lambda}$ to $WL_{1,\lambda}$ if and only if $\alpha = n - \frac{n}{q}$. D.R. Adams [1] studied the boundedness of the Riesz potential in Morrey spaces and proved the follows statement.

Theorem B. Let $0 < \alpha < n$ and $0 \leq \lambda < n$, $1 \leq p < \frac{n-\lambda}{\alpha}$.

1) If $1 < p < \frac{n-\lambda}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of I^α from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

2) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of I^α from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{q,\lambda}(\mathbb{R}^n)$.

If $\alpha = \frac{n}{p} - \frac{n}{q}$, then $\lambda = 0$ and the statement of theorem B reduced to the abovementioned result by Hardy-Littlewood-Sobolev.

In the paper [5] authors consider the generalized shift operator, generated by the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^n \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \dots, \gamma_k > 0$$

in terms of which the B -maximal operator and the B Riesz potential are investigated in the B -Morrey space.

They obtain necessary and sufficient conditions for the operator I_γ^α to be bounded from B -Morrey space $L_{p,\lambda,\gamma}$ to $L_{q,\lambda,\gamma}$ to weak B -Morrey space $WL_{q,\lambda,\gamma}$.

The results obtained in the paper are analogue of [5].

2 Definitions and auxiliary results

Analogy by [5] we introduce the following notation.

Definition 2.1 Let $1 \leq p < \infty$, $0 \leq \gamma \leq 1 + 2\lambda$. We denote by $L_{p,\lambda,\gamma}(R_+, G)$ Gegenbauer-Morrey (G -Morrey space) associated with the Gegenbauer differential operator G_λ as the

set of locally integrable functions $f(x)$, $x \in R_+ = [0, \infty)$, with the finite norm

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{x \in R_+, r > 0} \left(r^{-\gamma} \int_{H(x,r)} |f(cht)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}},$$

where $H(x, r) = (x - r, x + r) \cap R_+$, i.e.

$$H(x, r) = \begin{cases} (0, x + r), & 0 \leq x < r, \\ (x - r, x + r), & r \leq x < \infty. \end{cases}$$

Definition 2.2 Let $1 \leq p < \infty$, $0 \leq \gamma \leq 1 + 2\lambda$. We denote by $WL_{p,\lambda,\gamma}(R_+, G)$ the weak $L_{p,\lambda,\gamma}$ space defined as the set of locally integrable functions $f(x)$, $x \in R_+$ with the finite norm

$$\|f\|_{WL_{p,\lambda,\gamma}(R_+, G)} = \sup_{r > 0} r \sup_{t > 0, x \in R_+} (t^{-\gamma} |\{y \in H(x, t) : |f(chy)| > r\}|_{\gamma})^{\frac{1}{p}}.$$

Notice that

$$WL_{p,\lambda}(R_+, G) = WL_{p,\lambda,0}(R_+, G),$$

$$L_{p,\lambda,\gamma}(R_+, G) \subset WL_{p,\lambda,\gamma}(R_+, G)$$

and $\|f\|_{WL_{p,\lambda,\gamma}} \leq \|f\|_{L_{p,\lambda,\gamma}}$.

In [7] the Gegenbauer maximal function is defined as follows:

$$M_G^* f(chx) = \sup_{r > 0} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r A_{cht} |f(chx)| sh^{2\lambda} t dt,$$

$$A_{cht} f(chx) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda) \Gamma(\frac{1}{2})} \int_0^{\pi} f(chxcht - shxsht \cos \varphi) (\sin \varphi)^{2\lambda-1} d\varphi$$

is of the Gegenbauer generalized shift function,

$$M_G f(chx) = \sup_{r > 0} \frac{1}{|H(x, r)|_{\lambda}} \int_{H(x,r)} |f(cht)| sh^{2\lambda} t dt,$$

where

$$|H(0, r)|_{\lambda} = \int_0^r sh^{2\lambda} t dt, \quad |H(x, r)|_{\lambda} = \int_{H(x,r)} sh^{2\lambda} t dt,$$

moreover

$$M_G^* f(chx) \lesssim M_G f(chx). \quad (2.1)$$

Here $A \lesssim B$ denotes that exists the constant $c > 0$ such that $0 < A \leq cB$, moreover C can depend on some parameters.

Symbol $A \approx B$ denote that $A \lesssim B$ and $B \lesssim A$.

The following result are analogue of the according inequality to [4].

Theorem 2.1 For every nonnegative function $g \in L_{1,\lambda}^{loc}(R_+)$ for every $1 \leq p < \infty$ and $0 < t < \infty$ the inequality

$$\int_{H(x,r)} (M_G f(chy))^p g(chy) sh^{2\lambda} y dy \leq \int_{H(x,r)} |f(chy)|^p M_G g(chy) sh^{2\lambda} y dy$$

is true.

Proof. By Hölder's inequality, we have

$$\begin{aligned} (M_G f(chy))^p &= \left(\sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \int_{H(y,r)} |f(cht)| sh^{2\lambda} t dt \right)^p \\ &\leq \sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \left(\int_{H(y,r)} |f(cht)|^p sh^{2\lambda} t dt \right) \left(\int_{H(y,r)} sh^{2\lambda} t dt \right)^{\frac{p}{q}} \\ &= \sup_{r>0} \frac{|H(y,r)|_\lambda^{\frac{p}{q}}}{|H(y,r)|_\lambda^p} \int_{H(y,r)} |f(cht)|^p sh^{2\lambda} t dt \\ &= \sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \int_{H(y,r)} |f(cht)|^p sh^{2\lambda} t dt. \end{aligned}$$

From this we get

$$\begin{aligned} &\int_{H(x,r)} (M_G f(chy))^p g(chy) sh^{2\lambda} y dy \\ &\leq \int_{H(x,r)} \left(\sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \int_{H(y,r)} |f(cht)|^p sh^{2\lambda} t dt \right) g(chy) sh^{2\lambda} y dy \\ &= \int_{H(x,r)} |f(chy)|^p \left(\sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \int_{H(y,r)} g(cht) sh^{2\lambda} t dt \right) sh^{2\lambda} y dy \\ &= \int_{H(x,r)} |f(chy)|^p M_G g(chy) sh^{2\lambda} y dy. \quad \square \end{aligned}$$

Theorem 2.2 The following Chebyshev type inequality

$$|\{y \in H(x,r) : M_G f(chy) > \alpha\}|_\lambda \leq \frac{1}{\alpha} \int_{H(x,r)} M_G f(chy) sh^{2\lambda} y dy$$

for all $\alpha > 0$ and $t > 0$ is valid.

Proof. Since

$$M_G f(chy) \geq \alpha \mathcal{X}_{\{M_G f(chy) > \alpha\}}(chy),$$

then

$$\begin{aligned} \int_{H(x,r)} M_G f(chy) sh^{2\lambda} y dy &\geq \alpha \int_{H(x,r)} \mathcal{X}_{\{M_G f(chy) > \alpha\}}(chy) sh^{2\lambda} y dy \\ &= \alpha |\{y \in H(x,r) : M_G f(chy) > \alpha\}|_\lambda. \end{aligned}$$

The assertion of the theorem, follows from the last inequality.

Further we will need some auxiliary assertions.

Lemma 2.1 [7] For $0 < \lambda < \frac{1}{2}$ the following relations

$$|H(0,r)|_\lambda \approx \begin{cases} (sh \frac{r}{2})^{2\lambda+1}, & 0 < r < 2, \\ (sh \frac{r}{2})^{4\lambda}, & 2 \leq r < \infty \end{cases}$$

is true.

Lemma 2.2 For $r > 0$, $0 < \lambda < \frac{1}{2}$ and $x \in [0, \infty)$ the following relations

$$M_G \mathcal{X}_{H(x,r)}(chy) \approx \begin{cases} \left(\frac{sh \frac{r}{2}}{sh \frac{|x-y|+r}{2}} \right)^{2\lambda+1}, & 0 < x+r < 2, \\ \left(\frac{sh \frac{r}{2}}{sh \frac{|x-y|+r}{2}} \right)^{4\lambda}, & 2 \leq x+r < \infty \end{cases}$$

is true.

Proof. Let $0 \leq x < r$. Then

$$\begin{aligned} M_G \mathcal{X}_{H(0,r)}(chx) &= \sup_{r>0} \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} \mathcal{X}_{H(0,r)}(cht) sh^{2\lambda} t dt \\ &= \sup_{r>0} \frac{|(0,r) \cap (0,x+r)|_\lambda}{|(0,x+r)|_\lambda} = \sup_{r>0} \frac{\int_0^r sh^{2\lambda} t dt}{\int_0^{x+r} sh^{2\lambda} t dt}. \end{aligned}$$

We estimat $|H(x,r)|_\lambda$. Let $0 \leq x < r$ and $0 < x+r < 2$. Then

$$\begin{aligned} |H(x,r)|_\lambda &= \int_0^{x+r} sh^{2\lambda} t dt \geq \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda} t dt \\ &\geq \frac{x+r}{2} \left(sh \frac{x+r}{2} \right)^{2\lambda} \geq e^{-1} \left(sh \frac{x+r}{2} \right)^{2\lambda+1}, \end{aligned} \quad (2.2)$$

since $sh t \leq et$ for $0 \leq t \leq 1$. On the other hand,

$$\begin{aligned}
|H(x, r)|_\lambda &= \int_0^{x+r} sh^{2\lambda} t dt = 2^{2\lambda} \int_0^{x+r} \left(sh \frac{t}{2} ch \frac{t}{2} \right)^{2\lambda} dt \\
&= 2^{2\lambda+1} \int_0^{x+r} \left(sh \frac{t}{2} \right)^{2\lambda} \left(ch \frac{t}{2} \right)^{2\lambda-1} d\left(sh \frac{t}{2} \right) \\
&\leq 2^{2\lambda+1} \int_0^{x+r} \left(sh \frac{t}{2} \right)^{2\lambda} d\left(sh \frac{t}{2} \right) \leq \frac{2^{2\lambda+1}}{2\lambda+1} \left(sh \frac{x+r}{2} \right)^{2\lambda+1}. \quad (2.3)
\end{aligned}$$

From (2.2) and (2.3) for $0 \leq x < r$ and $0 < x+r < 2$ as follows

$$|H(x, r)|_\lambda \approx \left(sh \frac{x+r}{2} \right)^{2\lambda+1}. \quad (2.4)$$

Now let $0 \leq x < r$ and $2 \leq x+r < \infty$. Then

$$\begin{aligned}
|H(x, r)|_\lambda &= \int_0^{x+r} sh^{2\lambda} t dt \geq \int_{\frac{x+r}{2}}^{x+r} \frac{sh^{2\lambda} t d(sht)}{cht} \\
&\geq \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda-1} t d(sht) = \frac{1}{4\lambda} \left(sh^{2\lambda}(x+r) - sh^{2\lambda} \frac{x+r}{2} \right) \\
&\geq \frac{1}{4\lambda} \left(sh^{2\lambda}(x+r) - 4^{-\lambda} sh^{2\lambda}(x+r) \right) = \frac{1}{4\lambda} (1 - 4^{-\lambda}) sh^{2\lambda}(x+r) \\
&= \frac{4^\lambda - 1}{4\lambda \cdot 4^\lambda} \left(2sh \frac{x+r}{2} ch \frac{x+r}{2} \right)^{2\lambda} \geq \frac{4^\lambda - 1}{4\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda}. \quad (2.5)
\end{aligned}$$

On the other hand

$$\begin{aligned}
|H(x, r)|_\lambda &= \int_0^{x+r} sh^{2\lambda} t dt = 2^{2\lambda+1} \int_0^{x+r} \frac{\left(sh \frac{t}{2} \right)^{2\lambda} d\left(sh \frac{t}{2} \right)}{\left(ch \frac{t}{2} \right)^{1-2\lambda}} \\
&\leq 2^{2\lambda+1} \int_0^{x+r} \left(sh \frac{t}{2} \right)^{4\lambda-1} d\left(sh \frac{t}{2} \right) = \frac{2^{2\lambda+1}}{4\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda} \quad (2.6)
\end{aligned}$$

From (2.5) and (2.6) it follows that

$$|H(x, r)|_\lambda \approx \left(sh \frac{x+r}{2} \right)^{4\lambda}. \quad (2.7)$$

by $0 \leq x < r$ and $2 \leq x+r < \infty$

Combining (2.4) and (2.7) we obtain

$$|H(x, r)|_\lambda \approx \begin{cases} \left(sh \frac{x+r}{2} \right)^{2\lambda+1}, & 0 < x+r < 2, \\ \left(sh \frac{x+r}{2} \right)^{4\lambda}, & 2 \leq x+r < \infty, \end{cases} \quad (2.8)$$

by $0 \leq x < r$.

Now we consider the case then $r \leq x < \infty$. Then $H(x, r) = (x - r, x + r)$. Let $0 < x + r < 2$. From (2.2) we have.

$$|H(x, r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda} t dt \geq \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda} t dt \geq e^{-1} \left(sh \frac{x+r}{2} \right)^{2\lambda+1}. \quad (2.9)$$

On the other hand from (2.3) we obtain

$$|H(x, r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda} t dt \leq \int_0^{x+r} sh^{2\lambda} t dt \leq \frac{2^{\lambda+1}}{2\lambda+1} \left(sh \frac{x+r}{2} \right)^{2\lambda+1}. \quad (2.10)$$

Remain consider the case, then $2 \leq x + r < \infty$. By (2.5) we have

$$|H(x, r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda} t dt \geq \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda} t dt \geq \frac{4^\lambda - 1}{4\lambda} \left(sh^{2\lambda} \frac{x+r}{2} \right)^{4\lambda}. \quad (2.11)$$

On the other hand by (2.6) we get

$$|H(x, r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda} t dt \leq \int_0^{x+r} sh^{2\lambda} t dt \leq \frac{2^{\lambda+1}}{4\lambda} \left(sh \frac{x+r}{2} \right)^{4\lambda}. \quad (2.12)$$

Combining (2.9)-(2.12) by $r \leq x < \infty$, we obtain

$$|H(x, r)|_\lambda \approx \begin{cases} \left(sh \frac{x+r}{2} \right)^{2\lambda+1}, & 0 < x + r < 2, \\ \left(sh \frac{x+r}{2} \right)^{4\lambda}, & 2 \leq x + r < \infty, \end{cases} \quad (2.13)$$

Now from (2.8) and (2.13) we have

$$|H(x, r)|_\lambda \approx \begin{cases} \left(sh \frac{x+r}{2} \right)^{2\lambda+1}, & 0 < x + r < 2, \\ \left(sh \frac{x+r}{2} \right)^{4\lambda}, & 2 \leq x + r < \infty, \end{cases} \quad (2.14)$$

by $0 \leq x < \infty$ and $0 < r < \infty$. Now let $r \leq x < \infty$. Then

$$\begin{aligned} M_G \mathcal{X}_{H(0,r)}(chx) &= \sup_{r>0} \frac{1}{|H(x, r)|_\lambda} \int_{H(x,r)} \mathcal{X}_{H(0,r)}(cht) sh^{2\lambda} t dt \\ &= \sup_{r>0} \frac{|(0, r) \cap (x - r, x + r)|_\lambda}{|(x - r, x + r)|_\lambda} = \sup_{r>0} \frac{\int_{x-r}^r sh^{2\lambda} t dt}{\int_{x-r}^{x+r} sh^{2\lambda} t dt}. \end{aligned} \quad (2.15)$$

Let $0 < r < x \leq 2$, then

$$\int_{x-r}^r sh^{2\lambda} t dt \geq \int_{\frac{r}{2}}^r sh^{2\lambda} t dt \geq \frac{r}{2} sh^{2\lambda} \frac{r}{2} \geq \frac{1}{e} \left(sh \frac{r}{2} \right)^{2\lambda+1}. \quad (2.16)$$

On the other hand

$$\begin{aligned} \int_{x-r}^r sh^{2\lambda} t dt &\leq \int_0^r sh^{2\lambda} t dt = 2^{2\lambda+1} \int_0^r \frac{(sh\frac{t}{2})^{2\lambda} d(sh\frac{t}{2})}{(ch\frac{t}{2})^{1-2\lambda}} \\ &\leq 2^{2\lambda+1} \int_0^r \left(sh\frac{t}{2} \right)^{2\lambda} d\left(sh\frac{t}{2} \right) = \frac{2^{\lambda+1}}{2\lambda+1} \left(sh\frac{r}{2} \right)^{2\lambda+1}. \end{aligned} \quad (2.17)$$

Thus from (2.16) and (2.17) we obtain that

$$\int_{x-r}^r sh^{2\lambda} t dt \approx \left(sh\frac{r}{2} \right)^{2\lambda+1}, \quad (2.18)$$

where $0 < r < x \leq 2$.

Now let $2 \leq r < x < \infty$, then

$$\begin{aligned} \int_{x-r}^r sh^{2\lambda} t dt &\geq \int_{\frac{r}{2}}^r sh^{2\lambda} t dt = 2^{2\lambda+1} \int_{\frac{r}{2}}^r \frac{(sh\frac{t}{2})^{2\lambda} d(sh\frac{t}{2})}{(ch\frac{t}{2})^{1-2\lambda}} \\ &\geq \frac{2^{2\lambda+1}}{2\lambda+1} \left(ch\frac{r}{2} \right)^{2\lambda-1} \left[\left(sh\frac{r}{2} \right)^{2\lambda+1} - \left(sh\frac{r}{4} \right)^{2\lambda+1} \right] \\ &\geq \frac{2^{\lambda+1}}{2\lambda+1} \left(2sh\frac{r}{2} \right)^{2\lambda-1} \left[\left(sh\frac{r}{2} \right)^{2\lambda+1} - \left(2^{-(2\lambda+1)} sh\frac{r}{2} \right)^{2\lambda+1} \right] \\ &= \frac{2^{4\lambda}(2^{2\lambda+1}-1)}{(2\lambda+1)2^{2\lambda+1}} \left(sh\frac{r}{2} \right)^{2\lambda+1} \left(sh\frac{r}{2} \right)^{2\lambda-1} = \frac{2^{2\lambda-1}(2^{2\lambda+1}-1)}{2\lambda+1} \left(sh\frac{r}{2} \right)^{4\lambda} \\ &\geq \frac{2^{2\lambda-1}}{2\lambda+1} \left(sh\frac{r}{2} \right)^{4\lambda}. \end{aligned} \quad (2.19)$$

On the other hand

$$\begin{aligned} \int_{x-r}^r sh^{2\lambda} t dt &\leq 2^{2\lambda+1} \int_0^r \frac{(sh\frac{t}{2})^{2\lambda} d(sh\frac{t}{2})}{(ch\frac{t}{2})^{1-2\lambda}} \\ &\leq 2^{\lambda+1} \int_0^r \left(sh\frac{t}{2} \right)^{4\lambda-1} d\left(sh\frac{t}{2} \right) = \frac{2^{2\lambda-1}}{\lambda} \left(sh\frac{r}{2} \right)^{4\lambda}. \end{aligned} \quad (2.20)$$

Combining (2.19) and (2.20) we have

$$\int_{x-r}^r sh^{2\lambda} t dt \approx \left(sh\frac{r}{2} \right)^{4\lambda}, \quad (2.21)$$

where $2 \leq r < x < \infty$.

Now from (2.18) and (2.21) we obtain

$$\int_{x-r}^r sh^{2\lambda} t dt \approx \begin{cases} \left(sh\frac{r}{2} \right)^{2\lambda+1}, & 0 < r < x \leq 2, \\ \left(sh\frac{r}{2} \right)^{4\lambda}, & 2 \leq r < x < \infty. \end{cases} \quad (2.22)$$

Further from (2.14) and (2.22) it follows that

$$M_G \mathcal{X}_{H(0,r)}(chx) \approx \begin{cases} \left(\frac{sh \frac{r}{2}}{sh \frac{x+r}{2}} \right)^{2\lambda+1}, & 0 < x+r < 2, \\ \left(\frac{sh \frac{r}{2}}{sh \frac{x+r}{2}} \right)^{4\lambda}, & 2 \leq x+r < \infty. \end{cases} \quad (2.23)$$

Further

$$M_G \mathcal{X}_{H(0,t)}(chx) = \sup_{r>0} \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} \mathcal{X}_{H(0,t)}(chu) sh^{2\lambda} u du.$$

From this we have

$$\begin{aligned} M_G \mathcal{X}_{H(0,t)}(ch(y-z)) &= \sup_{r>0} \frac{1}{|H(y-z,r)|_\lambda} \int_{H(y-z,r)} \mathcal{X}_{H(0,t)}(chu) sh^{2\lambda} u du \\ &= \sup_{r>0} \frac{1}{|H(y-z,r)|_\lambda} \int_{H(y,r)} \mathcal{X}_{H(0,t)}(ch(v-z)) sh^{2\lambda}(v-z) dv \\ &= \sup_{r>0} \frac{1}{|H(y-z,r)|_\lambda} \int_{H(y,r)} \mathcal{X}_{H(z,t)}(chv) sh^{2\lambda} v dv \\ &= M_G \mathcal{X}_{H(z,t)}(chy). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{X}_{H(0,t)}(ch(y-z)) &= \begin{cases} 1, & y-z \in H(0,t), \\ 0, & y-z \notin H(0,t), \end{cases} \\ \Leftrightarrow \mathcal{X}_{H(z,t)}(chy) &= \begin{cases} 1, & y \in H(z,t), \\ 0, & y \notin H(z,t). \end{cases} \end{aligned} \quad (2.24)$$

The assertion of Lemma 2.2 it follows from (2.23) and (2.24)

Theorem 2.3 1) If $f \in L_{1,\lambda,\gamma}(R_+, G)$, $0 \leq \gamma < 1 + 2\lambda$, then $M_G f \in WL_{1,\lambda,\gamma}(R_+, G)$ and

$$\|M_G f\|_{WL_{1,\lambda,\gamma}} \lesssim \|f\|_{L_{1,\lambda,\gamma}}.$$

2) If $f \in L_{p,\lambda,\gamma}(R_+, G)$, $1 < p < \infty$, $0 \leq \gamma < 1 + 2\lambda$, then $M_G f \in WL_{p,\lambda,\gamma}(R_+, G)$ and

$$\|M_G f\|_{L_{p,\lambda,\gamma}} \lesssim \|f\|_{L_{p,\lambda,\gamma}}.$$

Proof. 1) By defenition

$$\|M_G f\|_{WL_{1,\lambda,\gamma}(R_+, G)} = \sup_{r>0} r \sup_{t>0, x \in R_+} (t^{-\gamma} |\{y \in H(x,t) : M_G f(chy) > r\}|_\lambda).$$

Now using the Theorem 2.2 and also the Theorem 2.1 by $p = 1$ and $g(chy) \equiv 1$ we get

$$\|M_G f\|_{WL_{1,\lambda,\gamma}(R_+, G)} \leq \sup_{t>0, x \in R_+} \left(t^{-\gamma} \int_{H(x,r)} |f(chy)| sh^{2\lambda} y dy \right) = \|f\|_{L_{1,\lambda,\gamma}}.$$

2) Using the theorem by $p > 1$ and $g(chy) \equiv 1$, second asseption of the theorem we obtain.

3 The Xardy - Littlewood - Sobolev inequality for the G -Riesr potential in Morry space

We consider the G -Riesr potential) see [7])

$$I_G^\alpha f(chx) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(cht) dr \right) A_{cht} f(chx) sh^{2\lambda} t dt,$$

where $h_r(cht) = \int_1^\infty e^{-\gamma(\gamma+2\lambda)r} P_\gamma^\lambda(cht) (\gamma^2 - 1)^{\lambda-\frac{1}{2}} d\gamma$, and $A_{cht} f(chx) \equiv A_{cht}^\lambda f(chx)$.

Theorem 3.1 Let $0 < \alpha < 2\lambda + 1$, $0 < \gamma < 2\lambda + 1 - \alpha$ and $1 \leq p < \frac{2\lambda+1-\gamma}{\alpha}$.

1) If $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1-\gamma}$ is necessary and sufficient for the boundedness of I_G^α from $L_{p,\lambda,\gamma}(R_+, G)$ to $L_{q,\lambda,\gamma}(R_+, G)$.

2) If $p = 1 < \frac{2\lambda+1-\gamma}{\alpha}$, then condition $1 - \frac{1}{p} = \frac{\alpha}{2\lambda+1-\gamma}$ is necessary and sufficient for the boundedness of I_G^α from $L_{1,\lambda,\gamma}(R_+, G)$ and $WL_{q,\lambda,\gamma}(R_+, G)$.

Proof. 1). Sufficiency. Let $0 < \alpha < 1 + 2\lambda$, $0 < \gamma < 2\lambda + 1 - \alpha$, $f \in L_{p,\lambda,\gamma}(R_+, G)$ and $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$. For I_G^α the following estimate holds (see.[7] , Corollary 3.1)

$$\begin{aligned} |I_G^\alpha f(chx)| &\lesssim \int_0^\infty A_{cht} (shx)^{\alpha-2\lambda-1} |f(cht)| sh^{2\lambda} t dt \\ &= \left(\int_0^r + \int_r^\infty \right) A_{cht} |f(chx)| (shx)^{\alpha-2\lambda-1} sh^{2\lambda} t dt \\ &= A_1(x, r) + A_2(x, r). \end{aligned} \quad (3.1)$$

We estimate $A_1(x, r)$. Let $0 < r < 2$. Then taking into account (2.1) we get

$$\begin{aligned} A_1(x, r) &\lesssim \int_0^r \frac{A_{cht} |f(chx)| sh^{2\lambda} t dt}{(sh t)^{2\lambda+1-\alpha}} \\ &\lesssim \sum_{k=0}^\infty \int_{2^{-(k+1)r}}^{2^{-k}r} \frac{A_{cht} |f(chx)| sh^{2\lambda} t dt}{(sh t)^{2\lambda+1-\alpha}} \\ &\lesssim \sum_{k=0}^\infty \left(sh \frac{r}{2^{k+1}} \right)^\alpha \left(sh \frac{r}{2^{k+1}} \right)^{-2\lambda-1} \int_0^{2^{-k}r} A_{cht} |f(chx)| sh^{2\lambda} t dt \\ &\lesssim (shr)^\alpha M_G^* f(chx) \left(\sum_{k=0}^\infty 2^{-(k+1)\alpha} \right) \lesssim (shr)^\alpha M_G f(chx). \end{aligned} \quad (3.2)$$

Let $2 \leq r < \infty$ and $0 < \alpha < 4\lambda$. From the proof of Corollary 3.1 on [7] it follows that

$$\begin{aligned}
A_1(x, r) &= \int_0^r \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt}{(cht)^{2\lambda+1-\alpha}} \\
&\leq \int_0^r \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt}{(cht)^{4\lambda-\alpha}} \leq \int_0^r \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt}{(sht)^{4\lambda-\alpha}} \\
&\leq \sum_{k=0}^{\infty} \int_{2^{-(k+1)r}}^{2^{-kr}} \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt}{(sht)^{4\lambda-\alpha}} \\
&\lesssim \sum_{k=0}^{\infty} \left(sh \frac{r}{2^{k+1}} \right)^\alpha \left(sh \frac{r}{2^{k+1}} \right)^{-4\lambda} \int_0^{2^{-kr}} A_{cht} |f(chx)| sh^{2\lambda} t dt \\
&\lesssim M_G f(chx) \sum_{k=0}^{\infty} \left(sh \frac{r}{2^{k+1}} \right) \lesssim (shr)^\alpha M_G f(chx), 0 < \alpha < 4\lambda. \quad (3.3)
\end{aligned}$$

Now let $4\lambda \leq \alpha$. From the proof of Corollary 3.1 on [7] it follows that $|I_G^\alpha f(chx)| \lesssim 1$, so we have

$$\begin{aligned}
|A_1(x, r)| &\lesssim \int_0^r A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt \\
&= \frac{(sh \frac{r}{2})^{4\lambda}}{(sh \frac{r}{2})^{4\lambda}} \int_0^r A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt \\
&\leq (sh \frac{r}{2})^{4\lambda} M_G^* f(chx) \leq (shr)^\alpha M_G f(chx), 4\lambda \leq \alpha < 2\lambda + 1. \quad (3.4)
\end{aligned}$$

From (3.2), (3.3) and (3.4) we have

$$A_1(x, r) \lesssim (shr)^\alpha M_G f(chx), 0 < \alpha < 2\lambda + 1, 0 < r < \infty. \quad (3.5)$$

Further by Hölder's inequality

$$\begin{aligned}
A_{cht} |f(chx)| &= \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda) \Gamma(\frac{1}{2})} \int_0^\pi |f(chxcht - shxsht \cos \varphi)| (\sin \varphi)^{2\lambda-1} d\varphi \\
&\leq \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda) \Gamma(\frac{1}{2})} \left(\int_0^\pi |f(chxcht - shxsht \cos \varphi)|^p (\sin \varphi)^{2\lambda-1} d\varphi \right)^{\frac{1}{p}} \left(\int_0^\pi (\sin \varphi)^{2\lambda-1} d\varphi \right)^{\frac{1}{q}} \\
&\leq \left(\frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda) \Gamma(\frac{1}{2})} \right)^{\frac{1}{p}} \left(\int_0^\pi |f(chxcht - shxsht \cos \varphi)|^p (\sin \varphi)^{2\lambda-1} d\varphi \right)^{\frac{1}{p}}.
\end{aligned}$$

From this it follows that

$$(A_{cht} |f(chx)|)^p \leq A_{cht} |f(chx)|^p.$$

But then (2.1) it follows that

$$\begin{aligned} \frac{1}{|H(0,r)|_\lambda} \int_0^r (A_{cht}|f(chx)|)^p sh^{2\lambda} t dt &\lesssim \frac{1}{|H(0,r)|_\lambda} \int_0^r A_{cht}|f(chx)|^p sh^{2\lambda} t dt \\ &\lesssim \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} |f(chx)|^p sh^{2\lambda} t dt. \end{aligned}$$

Now taking into account of Lemma 2.1 and relation (2.14), we obtain

$$\begin{aligned} \int_0^r (A_{cht}|f(chx)|)^p sh^{2\lambda} t dt &\lesssim \frac{|H(0,r)|_\lambda}{|H(x,r)|_\lambda} \int_{H(x,r)} |f(chx)|^p sh^{2\lambda} t dt \\ &\lesssim \int_{H(x,r)} |f(chx)|^p sh^{2\lambda} t dt, \end{aligned}$$

from this it follows that

$$\|A_{cht}f\|_{L_{p,\lambda,\gamma}} \lesssim \|f\|_{L_{p,\lambda,\gamma}}.$$

We estimate $A_2(x,r)$. By Hölder's inequality

$$\begin{aligned} A_2(x,r) &\leq \left(\int_r^\infty (A_{cht}|f(chx)|)^p (sht)^{-\beta} sh^{2\lambda} t dt \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_r^\infty (sht)^{(\frac{\beta}{p} + \alpha - 2\lambda - 1)p'} sh^{2\lambda} t dt \right)^{\frac{1}{p'}} = A_{2.1} \cdot A_{2.2}. \end{aligned} \quad (3.6)$$

Let $\gamma < \beta < 2\lambda + 1 - p\alpha$. For $A_{2.1}$ we have

$$\begin{aligned} A_{2.1} &\leq \left(\sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1} r} (A_{cht}|f(chx)|)^p (sht)^{-\beta} sh^{2\lambda} t dt \right)^{\frac{1}{p}} \\ &\lesssim \left(\sum_{j=0}^{\infty} (sh2^j r)^{\gamma-\beta} (sh2^j r)^{-\gamma} \int_{2^j r}^{2^{j+1} r} (A_{cht}|f(chx)|)^p sh^{2\lambda} t dt \right)^{\frac{1}{p}} \\ &\lesssim \left(\sum_{j=0}^{\infty} (sh2^j r)^{\gamma-\beta} (2^j r)^{-\gamma} \int_{2^j r}^{2^{j+1} r} (A_{cht}|f(chx)|)^p sh^{2\lambda} t dt \right)^{\frac{1}{p}} \\ &\lesssim (shr)^{\frac{\gamma-\beta}{p}} \|A_{cht}f\|_{L_{p,\lambda,\gamma}} \left(\sum_{j=0}^{\infty} 2^{(\gamma-\beta)j} \right)^{\frac{1}{p}} \lesssim (shr)^{\frac{\gamma-\beta}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \end{aligned} \quad (3.7)$$

And for $A_{2,2}$ we have

$$\begin{aligned} A_{2,2} &\leq \left(\int_r^\infty (sht)^{\left(\frac{\beta}{p} + \alpha - 2\lambda - 1\right)p'} sh^{2\lambda} t dt \right)^{\frac{1}{p'}} \lesssim \left(\int_r^\infty (sht)^{\left(\frac{\beta}{p} + \alpha - 2\lambda - 1\right)p' + 2\lambda} d(sht) \right)^{\frac{1}{p'}} \\ &\lesssim (shr)^{\frac{\beta}{p} + \alpha - 2\lambda - 1 + \frac{2\lambda + 1}{p'}} = (shr)^{\frac{\beta}{p} + \alpha - 2\lambda - 1 + (2\lambda + 1)\left(1 - \frac{1}{p}\right)} \lesssim (shr)^{\frac{\beta}{p} + \alpha - \frac{2\lambda + 1}{p}}. \end{aligned} \quad (3.8)$$

Taking into account (3.7) and (3.8) in (3.6) we obtain

$$A_2(x, r) \lesssim (shr)^{\alpha + \frac{\gamma - 2\lambda - 1}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \quad (3.9)$$

Now, from (3.5), (3.8) and (3.1) we get

$$|I_G^\alpha f(x)| \lesssim \left((shr)^{\alpha + \frac{\gamma - 2\lambda - 1}{p}} \|f\|_{L_{p,\lambda,\gamma}} + (shr)^\alpha M_G f(chx) \right). \quad (3.10)$$

The right-hand attains its minimum at

$$shr = \left(\frac{2\lambda + 1 - \gamma}{p\alpha} - 1 \right)^{\frac{p}{2\lambda + 1 - \gamma}} \left(\frac{\|f\|_{L_{p,\lambda,\gamma}}}{M_G f(chx)} \right)^{\frac{p}{2\lambda + 1 - \gamma}}. \quad (3.11)$$

Now, taking into account (3.11) and (3.10), we have

$$\begin{aligned} |I_G^\alpha f(chx)| &\lesssim \left(\frac{M_G f(chx)}{\|f\|_{L_{p,\lambda,\gamma}}} \right)^{1 - \frac{p\alpha}{2\lambda + 1 - \gamma}} \|f\|_{L_{p,\lambda,\gamma}} \\ &= (M_G f(chx))^{1 - \frac{p\alpha}{2\lambda + 1 - \gamma}} \|f\|_{L_{p,\lambda,\gamma}}^{\frac{p\alpha}{2\lambda + 1 - \gamma}}. \end{aligned} \quad (3.12)$$

Now from (3.12) by Theorem 2.3 we have

$$\begin{aligned} \int_{H(x,r)} |I_G^\alpha f(chx)|^q sh^{2\lambda} t dt &\lesssim \\ &\lesssim \|f\|_{L_{p,\lambda,\gamma}}^{\frac{\alpha pq}{2\lambda + 1 - \gamma}} \left(\int_{H(x,r)} (M_G f(chx))^{q - \frac{\alpha pq}{2\lambda + 1 - \gamma}} sh^{2\lambda} t dt \right) \\ &\lesssim r^\gamma \|f\|_{L_{p,\lambda,\gamma}}^{\frac{\alpha pq}{2\lambda + 1 - \gamma}} \|f\|_{L_{p,\lambda,\gamma}}^{q - \frac{\alpha pq}{2\lambda + 1 - \gamma}} = r^\gamma \|f\|_{L_{p,\lambda,\gamma}}^q, \end{aligned} \quad (3.13)$$

where $0 < r < 2$.

From (3.13), we obtain

$$\|I_G^\alpha f\|_{q,\lambda,\gamma} \lesssim \|f\|_{L_{p,\lambda,\gamma}},$$

i.e. I_G^α is bounded from $L_{p,\lambda,\gamma}(R_+, G)$ to $L_{q,\lambda,\gamma}(R_+, G)$.

Necessity. Let $1 < p < \frac{2\lambda + 1 - \gamma}{\alpha}$, $f \in L_{p,\lambda,\gamma}(R_+, G)$ and I_G^α be bounded from $L_{p,\lambda,\gamma}(R_+, G)$ to $L_{q,\lambda,\gamma}(R_+, G)$.

Moreover assume that $f(x) > 0$ is increasing. The dilation function $f_t(chx)$ we define as follows:

$$\begin{cases} f(ch(tht)x) \leq f_t(chx) \leq f(ch(ctht)x), 0 < t < 1, \\ f(ch(tht)x) \leq f_t(chx) \leq f(ch(sht)x), 1 \leq t < \infty. \end{cases} \quad (3.14)$$

Then for $1 \leq t < \infty$ we have

$$\begin{aligned}
\|f_t\|_{L_{p,\lambda,\gamma}} &= \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f_t(chy)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\geq \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f(ch(tht)y)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\quad [(tht)y = u, y = (ctht)u, dy = (ctht)du] \\
&= (ctht)^{\frac{1}{p}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xtht, rtht)} |f(chu)|^p sh^{2\lambda} (ctht)u du \right)^{\frac{1}{p}} \\
&\geq (ctht)^{\frac{2\lambda+1}{p}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xtht, rtht)} |f(chu)|^p sh^{2\lambda} u du \right)^{\frac{1}{p}} \\
&= (ctht)^{\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{rtht}{r} \right)^{\frac{2\lambda-1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} = (ctht)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \\
&= \left(\frac{cht}{sht} \right)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \geq \frac{(cht)^\alpha}{(sht)^{\frac{2\lambda+1-\gamma}{p}}} \|f\|_{L_{p,\lambda,\gamma}} \\
&\geq (sht)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \tag{3.15}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|f_t\|_{L_{p,\lambda,\gamma}} &= \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f_t(chy)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\leq \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f(ch(sht)y)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \quad [(sht)y = u, dy = \frac{du}{sht}] \\
&= (sht)^{-\frac{1}{p}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xsht, rsht)} |f(chu)|^p sh^{2\lambda} \frac{u}{sht} du \right)^{\frac{1}{p}} \\
&\leq (sht)^{-\frac{2\lambda+1}{p}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xsht, rsht)} |f(chu)|^p sh^{2\lambda} u du \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&= (sht)^{-\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{r sht}{r} \right)^{\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \\
&= (sht)^{-\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}} \leq (sht)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \tag{3.16}
\end{aligned}$$

Consider the case when $0 < t < 1$. From (3.14) we have

$$\begin{aligned}
\|f_t\|_{L_{p,\lambda,\gamma}} &= \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f_t(chy)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\leq \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f(ch(ctht)y)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\quad [(ctht)y = u, dy = (tht)du] \\
&= (tht)^{\frac{1}{p}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xctht, rctht)} |f(chu)|^p sh^{2\lambda}(tht)udu \right)^{\frac{1}{p}} \\
&\leq (tht)^{\frac{2\lambda+1}{p}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xctht, rctht)} |f(chu)|^p sh^{2\lambda}udu \right)^{\frac{1}{p}} \\
&= (tht)^{\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{rctht}{r} \right)^{\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} = (tht)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \\
&= \left(\frac{sht}{cht} \right)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \lesssim \frac{1}{(cht)^{\frac{2\lambda+1-\gamma}{p}-\alpha}} \|f\|_{L_{p,\lambda,\gamma}} \\
&\leq (sht)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \tag{3.17}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|f_t\|_{L_{p,\lambda,\gamma}} &= \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f_t(chy)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\geq \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f(ch(tht)y)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\quad [(tht)y = u, dy = (ctht)du] \\
&= (ctht)^{\frac{1}{p}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xtht, rtht)} |f(chu)|^p sh^{2\lambda}(ctht)udu \right)^{\frac{1}{p}} \\
&\geq (ctht)^{\frac{2\lambda+1}{p}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xtht, rtht)} |f(chu)|^p sh^{2\lambda}udu \right)^{\frac{1}{p}} \\
&= (ctht)^{\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{rtht}{r} \right)^{\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} = (ctht)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \\
&= \left(\frac{cht}{sht} \right)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \geq \frac{(cht)^\alpha}{(sht)^{\frac{2\lambda+1-\gamma}{p}}} \|f\|_{L_{p,\lambda,\gamma}} \\
&\geq (sht)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \tag{3.18}
\end{aligned}$$

From (3.15)-(3.18) it follows

$$\|f_t\|_{L_{p,\lambda,\gamma}} \approx (t)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}, \tag{3.19}$$

where $0 < t < \infty$.

By the definition of (G -Riesz) potential we can write

$$I_G^\alpha f_t(chx) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty u^{\frac{\alpha}{2}-1} h_u(chv) du \right) A_{chv} f_t(chx) sh^{2\lambda} v dv.$$

Using (3.14) we have

$$\begin{aligned}
\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} &\leq \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |I_G^\alpha f_t(chy)|^q sh^{2\lambda} y dy \right)^{\frac{1}{q}} \\
&\leq \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |I_G^\alpha f(ch(ctht)y)|^q sh^{2\lambda} y dy \right)^{\frac{1}{q}} \\
&\quad [(ctht)y = z, dy = (tht)dz] \\
&= (tht)^{\frac{1}{q}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xctht, rctht)} |I_G^\alpha f(chz)|^q sh^{2\lambda}(tht)z dz \right)^{\frac{1}{q}} \\
&\leq (tht)^{\frac{2\lambda+1}{q}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xctht, rctht)} |I_G^\alpha f(chz)|^q sh^{2\lambda}(tht)z dz \right)^{\frac{1}{q}} \\
&= (tht)^{\frac{2\lambda+1}{q}} \left(\sup_{r>0} \frac{rctht}{r} \right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} = (tht)^{\frac{2\lambda+1-\gamma}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} \\
&= \frac{1}{(ctht)^{\frac{2\lambda+1-\gamma}{q}}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} \lesssim (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}, \quad (3.20)
\end{aligned}$$

where $0 < t < 1$.

On the other hand,

$$\begin{aligned}
\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} &= \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |I_G^\alpha f_t(chy)|^q sh^{2\lambda} y dy \right)^{\frac{1}{q}} \\
&\geq \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |I_G^\alpha f(ch(tht)y)|^q sh^{2\lambda} y dy \right)^{\frac{1}{q}} [(tht)y = z, dy = (ctht)dz] \\
&\geq (ctht)^{\frac{1}{q}} \sup_{r>0, x \in R_+} \left(r^{-\gamma} \int_{H(xtht, rtht)} |I_G^\alpha f(chz)|^q sh^{2\lambda}(ctht)z dz \right)^{\frac{1}{q}} \\
&\geq (ctht)^{\frac{2\lambda+1}{q}} \left(\sup_{r>0} \frac{rtht}{r} \right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} = \frac{(cht)^{\frac{2\lambda+1-\gamma}{q}}}{(sht)^{\frac{2\lambda+1-\gamma}{q}}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} \\
&\geq (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}. \quad (3.21)
\end{aligned}$$

Combining (3.20) and (3.21) we obtain that

$$\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}, \quad 0 < t < 1. \quad (3.22)$$

Consider the case, when $1 \leq t < \infty$. By (3.14), we have

$$\begin{aligned}
\|f_t\|_{L_{p,\lambda,\gamma}} &= \sup_{r>0,x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f_t(chy)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\leq \sup_{r>0,x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |f(ch(ctht)y)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
&\quad [(ctht)y = u, dy = (tht)du] \\
&= (tht)^{\frac{1}{p}} \sup_{r>0,x \in R_+} \left(r^{-\gamma} \int_{H(xctht,rctht)} |f(chu)|^p sh^{2\lambda}(tht)udu \right)^{\frac{1}{p}} \\
&\leq (tht)^{\frac{2\lambda+1}{p}} \sup_{r>0,x \in R_+} \left(r^{-\gamma} \int_{H(xctht,rctht)} |f(chu)|^p sh^{2\lambda}udu \right)^{\frac{1}{p}} \\
&= (tht)^{\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{rctht}{r} \right)^{\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} = (tht)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \\
&= \left(\frac{sht}{cht} \right)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \lesssim \frac{1}{(cht)^{\frac{2\lambda+1-\gamma}{p}-\alpha}} \|f\|_{L_{p,\lambda,\gamma}} \\
&\leq (sht)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \tag{3.23}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} &\leq \sup_{r>0,x \in R_+} \left(r^{-\gamma} \int_{H(x,r)} |I_G^\alpha f(ch(sht)y)|^q sh^{2\lambda} y dy \right)^{\frac{1}{q}} \\
&\quad [(sht)y = z, dy = \frac{dz}{sht}] \\
&= (sht)^{-\frac{1}{q}} \sup_{r>0,x \in R_+} \left(r^{-\gamma} \int_{H(xsht,rsht)} |I_G^\alpha f(chz)|^q sh^{2\lambda} \left(\frac{z}{sht} \right) dz \right)^{\frac{1}{q}} \\
&= (sht)^{-\frac{2\lambda+1}{q}} \sup_{r>0,x \in R_+} \left(r^\gamma \int_{H(xsht,rsht)} |I_G^\alpha f(chz)|^q sh^{2\lambda} z dz \right)^{\frac{1}{q}} \\
&= (sht)^{-\frac{2\lambda+1}{q}} \left(\sup_{r>0} \frac{rsht}{r} \right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} = (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}. \tag{3.24}
\end{aligned}$$

From (3.23) and (3.24), we have

$$\|I_G^\alpha f t\|_{L_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}, 1 \leq t < \infty. \quad (3.25)$$

Finally, by (3.22), (3.23) and (3.25) we arrive at

$$\|I_G^\alpha f t\|_{L_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}, 0 < t < \infty, \quad (3.26)$$

for all $0 < t < \infty$.

Since I_G^α is bounded from $L_{p,\lambda,\gamma}(R_+, G)$ to $L_{q,\lambda,\gamma}(R_+, G)$, i.e.

$$\|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} \lesssim \|f\|_{L_{p,\lambda,\gamma}},$$

then taking into account (3.26) and (3.19), we obtain

$$\begin{aligned} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} &\approx (sht)^{\frac{2\lambda+1-\gamma}{q}} \|I_G^\alpha f t\|_{L_{q,\lambda,\gamma}} \\ &\leq (sht)^{\frac{2\lambda+1-\gamma}{q}} \|f t\|_{L_{q,\lambda,\gamma}} \leq (sht)^{\alpha+\frac{\gamma-2\lambda-1}{p}} (sht)^{\frac{2\lambda+1-\gamma}{q}} \|f t\|_{L_{q,\lambda,\gamma}} \\ &\leq (sht)^{\alpha+(\gamma-2\lambda-1)(\frac{1}{q}-\frac{1}{p})} \|f\|_{L_{p,\lambda,\gamma}} \\ &= (sht)^{\alpha+(2\lambda+1)(\frac{1}{q}-\frac{1}{p})} (sht)^{\gamma(\frac{1}{p}-\frac{1}{q})} \|f\|_{L_{p,\lambda,\gamma}} \\ &\lesssim \|f\|_{L_{p,\lambda,\gamma}} \begin{cases} (sht)^{\alpha+(2\lambda+1)(\frac{1}{p}-\frac{1}{q})}, & 0 < t < 1, \\ (sht)^{\alpha+(2\lambda+1-\gamma)(\frac{1}{q}-\frac{1}{p})}, & 1 \leq t < \infty. \end{cases} \end{aligned}$$

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{2\lambda+1}$, then at $t \rightarrow 0$ we have $\|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} = 0$ for all $f \in L_{p,\lambda,\gamma}(R_+)$. If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{2\lambda+1-\gamma}$, then at $t \rightarrow \infty$ $\|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} = 0$ for all $f \in L_{p,\lambda,\gamma}$.

Therefore $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1-\gamma}$.

Sufficiently. Let $f \in L_{1,\lambda,\gamma}(G)$. Then

$$\begin{aligned} &|y \in H(0, r) : |I_G^\alpha f(chy)| > 2\beta|_\lambda \\ &\leq |y \in H(0, r) : A_1(y, r) > \beta|_\lambda + |y \in H(x, r) : A_2(y, r) > \beta|_\lambda. \end{aligned}$$

From (3.1) we have

$$\begin{aligned} A_2(y, r) &\leq \int_r^\infty \frac{A_{cht}|f(cht)|sh^{2\lambda}tdt}{(sht)^{2\lambda+1-\alpha}} \leq \sum_{j=0}^\infty \int_{2^j r}^{2^{j+1}r} \frac{A_{cht}|f(cht)|sh^{2\lambda}tdt}{(sht)^{2\lambda+1-\alpha}} \\ &\leq \|A_{cht}f\|_{L_{1,\lambda,\gamma}} \sum_{j=0}^\infty \frac{(2^j r)^\gamma}{(sh2^j r)^{2\lambda+1-\alpha}} \\ &\leq r^\gamma (shr)^{\alpha-2\lambda-1} \|f\|_{L_{1,\lambda,\gamma}} \sum_{j=0}^\infty 2^{(\gamma+\alpha-2\lambda-1)j} \\ &\lesssim (shr)^{\gamma+\alpha-2\lambda-1} \|f\|_{L_{1,\lambda,\gamma}}. \end{aligned} \quad (3.27)$$

Denoting

$$\beta = (shr)^{\gamma+\alpha-2\lambda-1} \|f\|_{L_{1,\lambda,\gamma}}$$

from (3.27) we obtain that $|A_2(y, r)| \leq \beta$ and, consequently, $|y \in H(0, r) : A_2(y, r) > \beta|_\gamma = 0$.

Taking into account inequality (3.5), we obtain

$$\begin{aligned} |\{y \in H(x, r) : A_1(y, r) > \beta\}|_\lambda &\leq \left| \left\{ y \in H(x, r) : M_G f(chy) > \frac{\beta}{c \cdot sh^{\alpha r}} \right\} \right|_\lambda \\ &\leq \frac{sh^{\alpha r}}{\beta} \|f\|_{L_{1,\lambda,\gamma}}. \end{aligned}$$

Thus we have

$$\begin{aligned} |\{y \in H(x, r) : |I_G^\alpha f(chy)| > 2\beta\}|_\lambda &\leq \frac{sh^{\alpha r}}{\beta} \|f\|_{L_{1,\lambda,\gamma}} = (shr)^{\gamma-2\lambda-1} \\ &= \left(\frac{1}{\beta} \|f\|_{L_{1,\lambda,\gamma}} \right)^{\frac{2\lambda+1-\gamma}{2\lambda+1-\alpha-\gamma}} = \left(\frac{1}{\beta} \|f\|_{L_{1,\lambda,\gamma}} \right)^q. \end{aligned}$$

Necessity. We preliminarily establish estimates for $\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}$. From (3.24) at $0 < t < 1$ we have

$$\begin{aligned} \|I_G^\alpha f t\|_{WL_{q,\lambda,\gamma}} &\geq \\ &\geq \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(x,u) : |I_G^\alpha f(ch(tht)y)| > r\}} sh^{2\lambda} y dy \right)^{\frac{1}{q}} \\ &\quad [(tht)y = z, dy = (ctht)dz] \\ &= (ctht)^{\frac{1}{q}} \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(xtht, uht) : |I_G^\alpha f(chz)| > r\}} sh^{2\lambda}(ctht)z dz \right)^{\frac{1}{q}} \\ &= (ctht)^{\frac{1}{q}} \sup_{r>0} r \sup_{u>0} \left(\frac{utht}{u} \right)^{\frac{\gamma}{q}} \\ &\quad \times \sup_{u>0, x \in R_+} \left((utht)^{-\gamma} \int_{\{y \in H(xtht, uht) : |I_G^\alpha f(chz)| > r\}} sh^{2\lambda}(ctht)z dz \right)^{\frac{1}{q}} \\ &\geq (ctht)^{\frac{2\lambda+1}{q}} (tht)^{\frac{\gamma}{q}} \sup_{r>0} r \sup_{u>0, x \in R_+} \left((utht)^{1-\gamma-2\lambda} \int_{\{y \in H(xtht, uht) : |I_G^\alpha f(chz)| > r\}} sh^{2\lambda}z dz \right)^{\frac{1}{q}} \\ &= (tht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} \geq (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \end{aligned} \tag{3.28}$$

On the other hand,

$$\begin{aligned}
\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} &\leq \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(x,u): |I_G^\alpha f(ch(ctht)y)| > r\}} sh^{2\lambda} y dy \right)^{\frac{1}{q}} \\
&\quad [(ctht)y = z, dy = (tht)dz] \\
&= (tht)^{\frac{1}{q}} \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(xctht, uctht): |I_G^\alpha f(chz)| > r\}} sh^{2\lambda}(tht)z dz \right)^{\frac{1}{q}} \\
&\leq (tht)^{\frac{2\lambda+1}{q}} \sup_{u>0} \left(\frac{uctht}{u} \right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} \\
&= (tht)^{\frac{2\lambda+1-\gamma}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} \leq (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \tag{3.29}
\end{aligned}$$

From (3.28) and (3.29) we obtain that for $0 < t < 1$

$$\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \tag{3.30}$$

Consider the case when $1 \leq t < \infty$. From (3.14) we have

$$\begin{aligned}
\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} &\geq \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(x,u): |I_G^\alpha f(ch(tht)y)| > r\}} sh^{2\lambda} y dy \right)^{\frac{1}{q}} \\
&\quad [(tht)y = z, dy = (ctht)dz] \\
&= (ctht)^{\frac{1}{q}} \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(xtht, utht): |I_G^\alpha f(chz)| > r\}} sh^{2\lambda}(ctht)z dz \right)^{\frac{1}{q}} \\
&= (ctht)^{\frac{2\lambda+1}{q}} \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(xtht, utht): |I_G^\alpha f(chz)| > r\}} sh^{2\lambda} z dz \right)^{\frac{1}{q}} \\
&= (ctht)^{\frac{2\lambda+1}{q}} \sup_{u>0} \left(\frac{utht}{u} \right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} \\
&= (tht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} \geq (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \tag{3.31}
\end{aligned}$$

On the other hand

$$\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} \leq \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(x,u): |I_G^\alpha f(ch(sht)y)| > r\}} sh^{2\lambda} y dy \right)^{\frac{1}{q}}$$

$$\begin{aligned}
& [(sht)y = z, dy = \left(\frac{1}{sht}\right)dz] \\
& = (sht)^{\frac{1}{q}} \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(xsht, usht): |I_G^\alpha f(chz)| > r\}} sh^{2\lambda} \left(\frac{1}{sht}\right) zdz \right)^{\frac{1}{q}} \\
& \leq (sht)^{-\frac{2\lambda+1}{q}} \sup_{r>0} r \sup_{u>0, x \in R_+} \left(u^{-\gamma} \int_{\{y \in H(xsht, usht): |I_G^\alpha f(chz)| > r\}} sh^{2\lambda} zdz \right)^{\frac{1}{q}} \\
& = (sht)^{-\frac{2\lambda+1}{q}} \sup_{u>0} \left(\frac{usht}{u}\right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} \\
& \leq (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \tag{3.32}
\end{aligned}$$

From (3.31) and (3.32) we obtain that for $1 \leq t < \infty$

$$\|I_G^\alpha f t\|_{WL_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \tag{3.33}$$

Finally from (3.30) and (3.33) we obtain that for every $0 < t < \infty$

$$\|I_G^\alpha f t\|_{WL_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \tag{3.34}$$

From the boundedness of I_G^α from $L_{1,\lambda,\gamma}(R_+)$ to $WL_{q,\lambda,\gamma}(R_+)$ on using (3.19) and (3.34), we have

$$\begin{aligned}
\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} & \lesssim (sht)^{\frac{2\lambda+1-\gamma}{q}} (sht)^{\alpha+\gamma-2\lambda-1} \|f\|_{L_{1,\lambda,\gamma}} = (sht)^{\alpha-(2\lambda+1-\gamma)(1-\frac{1}{q})} \|f\|_{L_{1,\lambda,\gamma}} \\
& = (sht)^{\alpha-(2\lambda+1)(1-\frac{1}{q})} (sht)^{\gamma(1-\frac{1}{q})} \|f\|_{L_{1,\lambda,\gamma}} \\
& \lesssim \|f\|_{L_{1,\lambda,\gamma}} \begin{cases} (sht)^{\alpha-(2\lambda+1)(1-\frac{1}{q})}, & 0 < t < 1, \\ (sht)^{\alpha+(\gamma-2\lambda-1)(1-\frac{1}{q})}, & 1 \leq t < \infty. \end{cases}
\end{aligned}$$

If $1 < \frac{1}{q} + \frac{\alpha}{2\lambda+1-\gamma}$, then fending $t \rightarrow 0$ $\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} = 0$ for all $f \in L_{1,\lambda,\gamma}(R_+, G)$.

If $1 > \frac{1}{q} + \frac{\alpha}{2\lambda+1-\gamma}$, then fending $t \rightarrow \infty$ we obtain that $\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} = 0$ for all $f \in L_{1,\lambda,\gamma}(R_+, G)$.

Thus $1 - \frac{1}{q} = \frac{\alpha}{2\lambda+1-\gamma}$.

Theorem is proved.

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