Transactions of NAS of Azerbaijan, Issue Mathematics
Series of Physical-Technical and Mathematical Sciences

Parametric Marcinkiewicz integral operator and higher order commutators on generalized weighted Morrey spaces

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Received: 10.05.2017 / Revised: 12.09.2017 / Accepted: 10.10.2017

Abstract. *In this paper, we study the boundedness of parametric Marcinkiewicz integral operator and its higher order commutator with rough kernels on generalized weighted Morrey spaces.*

Keywords. Parametric Marcinkiewicz integrals, generalized weighted Morrey spaces, higher order commutator.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35.

1 Introduction

Suppose that S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

(iii) $\Omega \in L^1(S^{n-1})$.

The parametric Marcinkiewicz integral is defined by

$$\mu^\rho(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2},$$

where $0 < \rho < n$. When $\rho = 1$, we simply denote it by $\mu(f)$. It is well-known that the operator $\mu(f)$ is defined by Stein in [13].

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For $m \in \mathbb{N}$, $b \in BMO(\mathbb{R}^n)$, the higher-order commutator of parametric Marcinkiewicz integral is defined as follows

$$\mu^{\rho}_{b^m}(f)(x) = \left(\int_0^{\infty} \left| \frac{1}{t^{\rho}} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} (b(x)-b(y))^m f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}.$$

The classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ were introduced by Morrey [10] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Mizuhara [9] introduced generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ (see, also [11,4]); Komori and Shirai [8] defined the weighted Morrey spaces $L^{p,\kappa}(\omega)$; Guliyev [3] gave a concept of generalized weighted Morrey space $\mathcal{M}^p_{\varphi}(w,\mathbb{R}^n)$ which could be viewed as extension of both $L^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,\kappa}(\omega)$.

Let $1 \leq p < \infty$ and let φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and let w be a non-negative measurable function on $\mathbb{R}^n$. Following [3], we denote the generalized weighted Morrey space $\mathcal{M}^p_{\varphi}(w,\mathbb{R}^n)$, the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\mathcal{M}^p_{\varphi}(w,\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L^p(w,B(x,r))},$$

where

$$\|f\|_{L^p(w,B(x,r))} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy \right)^{\frac{1}{p}}.$$

Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x.

In this paper, we consider the boundedness of parametric Marcinkiewicz integral operator and its higher order commutator with rough kernels on generalized weighted Morrey spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Background materials

Even though the A_p class is well-known, for completeness, we offer the definition of A_p weight functions.

Definition 2.1 *For, $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_p weight if*

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{p'}{p}} dx \right)^{\frac{p'}{p}} < \infty, \tag{2.1}$$

where the supremum is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. A locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|} \int_B w(y)dy \leq Cw(x), \qquad a.e\ \ x \in B$$

for some constant $C > 0$. We define $A_\infty = \bigcup_{p\geq 1} A_p$.

For any $w \in A_\infty$ and any Lebesgue measurable set E, we write $w(E) = \int_E w(x)dx$.
For any $w \in A_p$, by (2.1) we have

$$\left(w^{-\frac{p'}{p}}(B)\right)^{1/p'} = \|w^{-\frac{1}{p}}\|_{L^{p'}(B)} \le C|B|\,(w(B))^{-1/p} \tag{2.2}$$

and

$$w^{1-p'} \in A_{p'}. \tag{2.3}$$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 2.2 *Suppose that* $b \in L^1_{\rm loc}(\mathbb{R}^n)$, *let*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_{B(x,r)}|dy,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L^1_{\rm loc}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}.$$

The following results concerning the boundedness of Marcinkiewicz integral and its higher-order commutator on weighted L^p space are known.

Theorem 2.1 *[12] Suppose that* $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying* (1.1)-(1.2) *and* $0 < \rho < n$. *Then, for every* $q' < p < \infty$ *and* $w \in A_{p/q'}$, *there is a constant* C *independent of* f *such that*

$$\|\mu^\rho(f)\|_{L^p(w,\mathbb{R}^n)} \le C\|f\|_{L^p(w,\mathbb{R}^n)}.$$

Theorem 2.2 *[12] Suppose that* $b \in BMO(\mathbb{R}^n)$, $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying* (1.1)-(1.2) *and* $0 < \rho < n$. *Then, for every* $q' < p < \infty$ *and* $w \in A_{p/q'}$, *there is a constant* C *independent of* f *such that*

$$\|\mu^\rho_{b^m}(f)\|_{L^p(w,\mathbb{R}^n)} \le C\|f\|_{L^p(w,\mathbb{R}^n)}.$$

In the next sections where we prove our main estimates, we use the following lemma.

Lemma 2.1 *[3]*
i) Let $w \in A_\infty$ *and* $b \in BMO(\mathbb{R}^n)$. *Let also* $1 \le p < \infty$, $x \in \mathbb{R}^n$, $m > 0$ *and* $r_1, r_2 > 0$. *Then*

$$\left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{mp} w(y)dy\right)^{1/p} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)^m \|b\|_*^m,$$

where C *is independent of* $f,\ w,\ x,\ r_1,\ r_2$ *and* $b_{B(x,r_2),w} = \frac{1}{w(B(x,r_2))}\int_{B(x,r)} b(y)w(y)dy$.

ii) Let $w \in A_p$ *and* $b \in BMO(\mathbb{R}^n)$. *Let also* $1 < p < \infty$, $x \in \mathbb{R}^n$, $m > 0$ *and* $r_1, r_2 > 0$. *Then*

$$\left(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{mp'} w^{1-p'}(y)dy\right)^{1/p'} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)^m \|b\|_*^m,$$

where C *is independent of* $f,\ w,\ x,\ r_1,\ r_2$.

3 Local Guliyev Estimates

Inspiring by the ideas of [3] (see, also [7]) and [2] we prove the following local estimates for the operators μ^ρ and $\mu^\rho_{b^m}$.

Lemma 3.1 *Suppose that $\Omega \in L^q(S^{n-1})$ $(q > 1)$ satisfying* (1.1)-(1.2) *and* $0 < \rho < n$. *Then, for every $q' < p < \infty$ and $w \in A_{p/q'}$, there is a constant C independent of f such that*

$$\|\mu^\rho(f)\|_{L^p(w,B(x_0,r))} \le C\,(w(B(x_0,r)))^{1/p} \int_{2r}^{\infty} \|f\|_{L^p(w,B(x_0,t))}\,(w(B(x_0,t)))^{-\frac{1}{p}}\,\frac{dt}{t}. \tag{3.1}$$

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ and $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{c_{(2B)}}(y), \quad r > 0,$$

and have

$$\|\mu^\rho(f)\|_{L^p(w,B)} \le \|\mu^\rho(f_1)\|_{L^p(w,B)} + \|\mu^\rho(f_2)\|_{L^p(w,B)}.$$

Since $f_1 \in L^p(w,\mathbb{R}^n)$ and from the boundedness of μ^ρ in $L^p(w,\mathbb{R}^n)$ (Theorem 2.1) it follows that

$$\|\mu^\rho(f_1)\|_{L^p(w,B)} \le \|\mu^\rho(f_1)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|f_1\|_{L^p(w,\mathbb{R}^n)} = \|f\|_{L^p(w,2B)}.$$

By using Hölder's inequality at (2.1), we have

$$|B| \lesssim (w(B))^{1/p}\,\|w^{-\frac{1}{p}}\|_{L^{p'}(B)}.$$

Then, for $q' < p < \infty$,

$$\begin{aligned}
\|\mu^\rho(f_1)\|_{L^p(w,B)} &\lesssim |B|\|f\|_{L^p(\omega,2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\
&\lesssim |B| \int_{2r}^{\infty} \|f\|_{L^p(\omega,B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim (w(B))^{1/p}\,\|w^{-\frac{1}{p}}\|_{L^{p'}(B)} \int_{2r}^{\infty} \|f\|_{L^p(\omega,B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim (w(B))^{1/p} \int_{2r}^{\infty} \|f\|_{L^p(\omega,B(x_0,t))}\,\|w^{-\frac{1}{p}}\|_{L^{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}.
\end{aligned}$$

By (2.2), we get

$$\|\mu^\rho(f_1)\|_{L^p(w,B)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(\omega,B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \tag{3.2}$$

Note that, using spherical coordinates we have

$$\|\Omega(x-\cdot)\|_{L^q(B(x_0,t))} \lesssim \|\Omega\|_{L^q(S^{n-1})}\,|B(0,t+|x-x_0|)|^{\frac{1}{q}}. \tag{3.3}$$

It's clear that $x \in B$, $y \in^{c} (2B)$ implies $\frac{1}{2}|x_0 - y| \leq |x-y| \leq \frac{3}{2}|x_0 - y|$. Then by the Minkowski inequality, we get

$$
\begin{aligned}
|\mu^{\rho}(f_2)(x)| &\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |f_2(y)| \left(\int_{|x-y|}^{\infty} \frac{dt}{t^{1+2\rho}} \right)^{1/2} dy \\
&\lesssim \int_{^c(2B)} \frac{|f(y)||\Omega(x-y)|}{|x-y|^n} dy \\
&\lesssim \int_{^c(2B)} \frac{|f(y)||\Omega(x-y)|}{|x_0-y|^n} dy.
\end{aligned} \tag{3.4}
$$

By Fubini's theorem we have

$$
\begin{aligned}
\int_{^c(2B)} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} dy &\approx \int_{^c(2B)} |\Omega(x-y)| \, |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx \int_{2r}^{\infty} \int_{2r \leq |x_0-y| < t} |\Omega(x-y)| \, |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)| \, |f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned} \tag{3.5}
$$

Applying Hölder's inequality and (3.3), we get

$$
\int_{^c(2B)} \frac{|\Omega(x-y)| \, |f(y)|}{|x_0-y|^n} dy \lesssim \|\Omega\|_{L^q(S^{n-1})} \int_{2r}^{\infty} \|f\|_{L^{q'}(B(x_0,t))} |B(0, t+|x-x_0|)|^{\frac{1}{q}} \frac{dt}{t^{n+1}}. \tag{3.6}
$$

Note that for $t > 2r$ and $|x - x_0| < r$ we have $t + |x - x_0| < t + r < \frac{3}{2}t$. Since $q' < p < \infty$, $v = \frac{p}{q'} > 1$ and $w \in A_v$, from the Hölder's inequality and (2.2) we get that

$$
\begin{aligned}
\int_{^c(2B)} \frac{|\Omega(x-y)| \, |f(y)|}{|x_0-y|^n} dy &\lesssim \int_{2r}^{\infty} \|f\|_{L^p(\omega, B(x_0,t))} \, \|w^{-\frac{1}{v}}\|_{L^{v'}(B(x_0,t))}^{\frac{1}{q'}} \, t^{\frac{n}{q}} \, \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L^p(\omega, B(x_0,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \, |B(x_0,t)|^{\frac{1}{q'}} \, t^{\frac{n}{q}} \, \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L^p(\omega, B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}.
\end{aligned}
$$

Therefore

$$
|\mu^{\rho}(f_2)(x)| \lesssim \int_{2r}^{\infty} \|f\|_{L^p(\omega, B(x_0,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}. \tag{3.7}
$$

Moreover, for all $p \in (1, \infty)$, the inequality

$$
\|\mu^{\rho}(f_2)\|_{L^p(\omega,B)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(\omega, B(x_0,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t} \tag{3.8}
$$

holds. Combining (3.2) and (3.8), the proof of Lemma 3.1 is completed.

Lemma 3.2 *Suppose that $b \in BMO(\mathbb{R}^n)$, $m \in \mathbb{N}$, $\Omega \in L^q(S^{n-1})$ $(q > 1)$ satisfying satisfying* (1.1)-(1.2) *and $0 < \rho < n$. Then, for every $q' < p < \infty$ and $w \in A_{p/q'}$, there is a constant C independent of f such that*

$$
\|\mu^{\rho}_{b^m}(f)\|_{L^p(w,B(x_0,r))} \leq C \|b\|_*^m \, (w(B(x_0,r)))^{1/p} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\|f\|_{L^p(w,B(x_0,t))}}{(w(B(x_0,t)))^{\frac{1}{p}}} \frac{dt}{t}. \tag{3.9}
$$

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{c_{(2B)}}(y), \quad r > 0,$$

and have

$$\|\mu^{\rho}_{b^m}(f)\|_{L^p(w,B)} \le \|\mu^{\rho}_{b^m}(f_1)\|_{L^p(w,B)} + \|\mu^{\rho}_{b^m}(f_2)\|_{L^p(w,B)}.$$

Since $f_1 \in L^p(w, \mathbb{R}^n)$ and from the boundedness of $\mu^{\rho}_{b^m}$ in $L^p(w, \mathbb{R}^n)$ (Theorem 2.2) it follows that

$$\|\mu^{\rho}_{b^m}(f_1)\|_{L^p(w,B)} \le \|\mu^{\rho}_{b^m}(f_1)\|_{L^p(w,\mathbb{R}^n)} \le C\|b\|^m_* \|f_1\|_{L^p(w,\mathbb{R}^n)} = C\|b\|^m_* \|f\|_{L^p(w,2B)}.$$

As the proof of (3.2), we get

$$\|\mu^{\rho}_{b^m}(f_1)\|_{L^p(w,B)} \le C\|b\|^m_* w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(\omega,B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}. \tag{3.10}$$

We now turn to deal with the term $\|\mu^{\rho}_{b^m}(f_2)\|_{L^p(\omega,B)}$. For any given $x \in B$, we have

$$\begin{aligned} |\mu^{\rho}_{b^m}(f_2)| &\le C|(b(x) - b_{B,w})^m||\mu^{\rho}(f_2)(x)| + C|\mu^{\rho}((b - b_{B,w})^m f_2)(x)| \\ &= I_1 + I_2. \end{aligned}$$

By (3.7), we have

$$I_1 \lesssim |(b(x) - b_{B,w})^m| \int_{2r}^{\infty} \|f\|_{L^p(\omega,B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.$$

Then from Lemma 2.1 we get

$$\|I_1\|_{L^p(w,B)} \lesssim \|b\|^m_* w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(\omega,B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.$$

When $\Omega \in L^q(S^{n-1})$, it follows from (3.4), (3.5) and (3.6) that

$$I_2 \lesssim \int_{2r}^{\infty} \|(b - b_{B,w})^m f\|_{L^{q'}(B(x_0,t))} t^{\frac{n}{q}} \frac{dt}{t^{n+1}}.$$

Set $v = \frac{p}{q'} > 1$. Since $w \in A_v$, from (2.3), we know $w^{1-v'} \in A_{v'}$. By Hölder's inequality

$$I_2 \lesssim \int_{2r}^{\infty} \|f\|_{L^p(\omega,B(x_0,t))} \|(b - b_{B,w})^m\|_{L^{v'q'}(w^{1-v'},B(x_0,t))}\, t^{\frac{n}{q}} \frac{dt}{t^{n+1}}.$$

Since $w^{1-v'} \in A_{v'}$, from (2.2), we know

$$\left(w^{1-v'}(B(x_0,t))\right)^{\frac{1}{v'q'}} \le Ct^{\frac{n}{q'}} \left(w(B(x_0,t))\right)^{-\frac{1}{p}}. \tag{3.11}$$

Using (3.11) and Lemma 2.1, we obtain

$$\begin{aligned} \|(b - b_{B,w})^m\|_{L^{v'q'}(w^{1-v'},B(x_0,t))} &= \left(\int_{B(x_0,t)} |b(y) - b_{B,w}|^{mv'q'} w^{1-v'}(y) dy\right)^{\frac{1}{v'q'}} \\ &\lesssim \|b\|^m_* \left(1 + \ln\frac{t}{r}\right)^m \left(w^{1-v'}(B(x_0,t))\right)^{\frac{1}{v'q'}} \\ &\lesssim \|b\|^m_* \left(1 + \ln\frac{t}{r}\right)^m t^{\frac{n}{q'}} \left(w(B(x_0,t))\right)^{-\frac{1}{p}}. \end{aligned}$$

Hence

$$I_2 \lesssim \|b\|_*^m \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^m \|f\|_{L^p(\omega,B(x_0,t))} \, (w(B(x_0,t)))^{-\frac{1}{p}} \, \frac{dt}{t}.$$

Therefore,

$$\|I_2\|_{L^p(w,B)} \lesssim \|b\|_*^m w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(\omega,B(x_0,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}. \tag{3.12}$$

Combining (3.10) and (3.12), the proof of Lemma 3.2 is completed.

Remark 3.1 For the case $\rho = 1$ and $m = 1$ the local estimate (3.9) was proved in [6, Lemma 5.2]. But there are some gaps in that proof. We also fill the gaps of proof of [6, Lemma 5.2] in the proof of Lemma 3.2.

4 Main Results

Theorem 4.1 *Let $1 < p < \infty$, $\Omega \in L^q(S^{n-1})$ $(q > 1)$ satisfying* (1.1)-(1.2) *and* $0 < \rho < n$. *Let also $w \in A_{p/q'}$ with $q' < p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}}{w\big(B(x,t)\big)^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{4.1}$$

where C does not depend on x and r. Then the operator μ^ρ is bounded from $\mathcal{M}^p_{\varphi_1}(w,\mathbb{R}^n)$ to $\mathcal{M}^p_{\varphi_2}(w,\mathbb{R}^n)$.

Proof. The proof follows from [5, Theorem 3.1] and Lemma 3.1. We can also give the following alternative proof for Theorem 4.1.

Since $f \in \mathcal{M}^p_{\varphi_1}(w,\mathbb{R}^n)$ and the fact $\|f\|_{L^p(w,B(x,t))}$ is a non-decreasing function of t, we get

$$\begin{aligned}
\frac{\|f\|_{L^p(w,B(x,t))}}{\operatorname*{ess\,sup}_{0<t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}} &\le \operatorname*{ess\,sup}_{0<t<s<\infty} \frac{\|f\|_{L^p(w,B(x,t))}}{\varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}} \\
&\le \sup_{s>0,\, x\in\mathbb{R}^n} \frac{\|f\|_{L^p(w,B(x,s))}}{\varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}} \\
&\le \|f\|_{\mathcal{M}^p_{\varphi_1}(w,\mathbb{R}^n)}.
\end{aligned}$$

Since (φ_1, φ_2) satisfies (4.1), we have

$$\begin{aligned}
&\int_r^{\infty} \|f\|_{L^p(w,B(x,t))} \, (w(B(x,t)))^{-\frac{1}{p}} \, \frac{dt}{t} \\
&\le \int_r^{\infty} \frac{\|f\|_{L^p(w,B(x,t))}}{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}} \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}}{(w(B(x,t)))^{\frac{1}{p}}} \frac{dt}{t} \\
&\le \|f\|_{\mathcal{M}^p_{\varphi_1}(w,\mathbb{R}^n)} \int_r^{\infty} \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}}{(w(B(x,t)))^{\frac{1}{p}}} \frac{dt}{t} \\
&\lesssim \|f\|_{\mathcal{M}^p_{\varphi_1}(w,\mathbb{R}^n)} \varphi_2(x,r).
\end{aligned}$$

Then by (3.1) we get

$$\begin{aligned}\|\mu^\rho\|_{\mathcal{M}^p_{\varphi_2}(w,\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r)^{-1/p}) \|\mu^\rho\|_{L^p(w,B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L^p(w,B(x,t))} \left(w(B(x,t))\right)^{-\frac{1}{p}} \frac{dt}{t} \\ &\lesssim \|f\|_{\mathcal{M}^p_{\varphi_1}(w,\mathbb{R}^n)}.\end{aligned}$$

Theorem 4.2 *Let* $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$, $m \in \mathbb{N}$, $\Omega \in L^q(S^{n-1})$ $(q > 1)$ *satisfying* (1.1)-(1.2) *and* $0 < \rho < n$. *Let also* $w \in A_{p/q'}$ *with* $q' < p < \infty$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)^m \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s) w\big(B(x,s)\big)^{\frac{1}{p}}}{w\big(B(x,t)\big)^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r),$$

where C *does not depend on* x *and* r. *Then the operator* $\mu^\rho_{b^m}$ *is bounded from* $\mathcal{M}^p_{\varphi_1}(w,\mathbb{R}^n)$ *to* $\mathcal{M}^p_{\varphi_2}(w,\mathbb{R}^n)$.

Proof. The proof of Theorem 4.2 is similar to the proof of Theorem 4.1.

Remark 4.1 In the case $w = 1$ and $m = 1$, Theorems 4.1 and 4.2 are proved in [1] for $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$, respectively. Since $\mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1) \subsetneq L^q(S^{n-1})$ $(q > 1)$, our results are better than the results of [1].

Acknowledgements. The research of F. Deringoz was partially supported by the grant of Ahi Evran University Scientific Research Project (FEF.A3.16.011). The author is grateful to Professor Yinsheng Jiang for his generosity of sending us the work [12].

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