Transactions of NAS of Azerbaijan, Issue Mathematics
Series of Physical-Technical and Mathematical Sciences

Necessary conditions of the extremum in nondifferentiable programming problems

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Received: 21.06.2017 / Revised: 02.12.2017 / Accepted: 08.01.2018

Abstract. *In the paper using the classes of $S-(\alpha,\beta,\nu,\delta,\omega)$ and $S-(\beta,\delta)$ locally Lipschitz mappings at the point, necessary and sufficient conditions for the extremum of the second order extremal problems with restrictions are derived.*

Keywords. locally Lipshitz mapping, sublinear function, cone, convex set.

Mathematics Subject Classification (2010): 05C35 · 52A20

1 Introduction

Study of the smooth extremal problems with restrictions (conditional extremum problems) is based on Lagrange's principle offered by J.L. Lagrange at the end of the 18th century. Strict justification of the Lagrange principle for the wide class of problems demanded serious efforts of many mathematicians and was generally finished in the second half of the XX century. The convex extremal problems with restrictions are well studied in book [5], [7]. Nonsmooth extremal problems with restrictions are considered in book [4],[6] and in the classes of locally Lipschitz functions necessary conditions of the first order are obtained. In this work necessary conditions of extremum in the first and second orders for the nonsmooth and in particular for smooth extremal problems with restrictions are proved. Note that when obtaining the necessary conditions for the extremum the essential role plays the classes of $S-(\alpha,\beta,\nu,\delta,\omega)$ and $S-(\beta,\delta)$ locally Lipschitz mappings at the point (see [9]-[12]).

In the work using subdifferential of the marginal function (see [3], p. 212) the second order necessary conditions of the extremum for extremal problems with restrictions are derived. Using the classes of $S-(\alpha,\beta,\nu,\delta,o(\beta))$ and $S-(\beta,\delta)$ locally Lipschitz mappings at the point, in Banach space the second order sufficient conditions of the weak minimum with restrictions are also obtained.

2 Class the locally Lipschitz functions of higher order

Let X and Y be Banach spaces, $C\subset X$, $F:X\to Y$, $S:X\to Y$, $f:X\to R$, $\varphi:X\to R$, $\alpha>0$, $\nu>0$, $\beta\geq\alpha\nu$, $K>0$, $\delta>0$, $o:R_+\to R_+$, $\tilde{o}:R_+\to R_+$

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and $\omega : R_+ \to R_+$, where $o(0) = 0$, $\tilde{o}(0) = 0$, $\omega(0) = 0$, $R_+ = [0, +\infty)$. Let's put $B = \{y \in X : \ \|y\| \le 1\}$, $B(x, \delta) = \{y \in X : \ \|y - x\| \le \delta\}$.

The mapping F is said to be $S - (\alpha, \beta, \nu, \delta, \omega)$ locally Lipschitz with the constant K at the point $\bar{x} \in X$, if F satisfies the condition

$$\|F(\bar{x} + x + y) - F(\bar{x} + x) - S(x + y) + S(x)\| \le K \|y\|^{\nu} \left(\|x\|^{\beta - \alpha\nu} + \|y\|^{\frac{\beta - \alpha\nu}{\alpha}}\right) + \omega(\|x\|)$$

at $x, y \in \delta B$. If $\omega(t) \equiv 0$, then the mapping F is said to be $S - (\alpha, \beta, \nu, \delta)$ locally Lipschitz with the constant K at the point $\bar{x}$ (see [10]). If $\omega(t) \equiv 0$ and $S(x) \equiv 0$, then the mapping F is said to be $(\alpha, \beta, \nu, \delta)$ locally Lipschitz with the constant K at the point $\bar{x}$.

If there is the function $o : R_+ \to R_+$, where $\lim\limits_{t \downarrow 0} \frac{o(t)}{t} = 0$, such that $\omega(\|x\|) = o(\|x\|^{\beta})$, then $S - (\alpha, \beta, \nu, \delta, \omega)$ locally Lipschitz with the constant K at the point $\bar{x}$ the mapping F we call $S - (\alpha, \beta, \nu, \delta, o(\beta))$ locally Lipschitz with the constant K at the point $\bar{x}$.

We call the mapping $F : X \to Y$ satisfying the condition

$$\|F(\bar{x} + y) - F(\bar{x}) - S(y)\| \le K \|y\|^{\beta}$$

at $y \in \delta B$, $S - (\beta, \delta)$ locally Lipschitz with the constant K at the point $\bar{x}$.

If there is the function $\tilde{o} : R_+ \to R_+$, where $\lim\limits_{t \downarrow 0} \frac{\tilde{o}(t)}{t} = 0$, such that

$$\|F(\bar{x} + y) - F(\bar{x}) - S(y)\| \le \tilde{o}(\|y\|^{\beta})$$

at $y \in \delta B$, then we call the mapping $F : X \to Y$ $S - (\tilde{o}(\beta), \delta)$ locally Lipschitz at the point $\bar{x}$.

If $F(x) = f(x)$, we put $S(x) = \varphi(x)$. Further we consider that $S(0) = 0$ and $\varphi(0) = 0$ (if $S(0) \ne 0$, it is necessary to consider the function $\tilde{S}(x) = S(x) - S(0)$).

If there is the function $o : R_+ \to R_+$, where $\lim\limits_{t \downarrow 0} \frac{o(t)}{t} = 0$, such that

$$f(\bar{x} + y) - f(\bar{x}) - \varphi(y) \le o(\|y\|^{\beta})$$

at $y \in \delta B$, then we call the function f $\varphi - (o(\beta), \delta)$ locally semi-Lipschitz at the point $\bar{x}$.

The $\varphi - (\alpha, \beta, \nu, \delta, \omega)$ locally semi-Lipschitz function at the point $\bar{x}$ is defined in the same way.

In [9]-[11] $S - (\alpha, \beta, \nu, \delta, \omega)$ locally Lipschitz mapping with the constant K at the point are defined and a number of their properties are also studied.

Let $M \subset X,\ x_0 \in M$. Let's put $d_M(x) = \inf \{\|y - x\| : y \in M\}$. It is easily checked that

$$|d_M^n(x_0 + x + y) - d_M^n(x_0 + x)| \le 2 \cdot 9^{n-1} \|y\| \left(\|x\|^{n-1} + \|y\|^{n-1}\right)$$

at $x,\ y \in X$.

By $B(X^2, R)$ we designate the set of all continuous bilinear symmetric functions from $X \times X$ in R. The set $M \subset B(X^2, R)$ is called bounded, if there is a number $L > 0$ such that $|x_2^*(x, y)| \le L \|x\| \cdot \|y\|$ at $x_2^* \in M$.

Let the function $f : X \to R$ be representable in the form

$$f(x_0 + x) = f(x_0) + \sup_{x_1^* \in M_1} x_1^*(x) + \sup_{x_2^* \in M_2} x_2^*(x, x) + o(\|x\|^2),$$

where $M_1 \subset X^*$ and $M_2 \subset B(X^2, R)$ are bounded sets, $\frac{o(\lambda)}{\lambda} \to 0$ at $\lambda \downarrow 0$. Let's put $\varphi(x) = \sup\limits_{x_1^* \in M_1} x_1^*(x) + o(\|x\|^2)$. Then it is easily checked that f satisfies $\varphi - (1, 2, 1, \delta)$, $\delta > 0$ locally Lipschitz condition at the point x_0.

If the functions f_τ, $\tau\in\Omega$, satisfy (α,β,ν,δ,ω) Lipschitz condition with the constant L_τ at the point x_0, and $L = \sup\{L_\tau : \tau \in \Omega\} < +\infty$, then $f(x) = \sup\{f_\tau(x) : \tau \in \Omega\}$ also satisfies (α,β,ν,δ,ω) locally Lipschitz condition with the constant L at the point x_0 (see [11], p. 200).

Let's designate $I = \{0, 1, \ldots, m\}$ and $J = \{1, \ldots, m\}$.

3 On an existence of the solution

Let $C \subset X$, $x_0 \in C$. Let's put

$$T_C(x_0) = \{\upsilon \in X : \forall h_n > 0,\ h_n \to 0, \forall x_n \to x_0, \exists u_n \to \upsilon,\ x_n + h_n u_n \in C\},$$

$$T(x_0; C) = \{\upsilon \in X : \exists h_n > 0,\ h_n \to 0,\ \exists u_n \to \upsilon,\ x_0 + h_n u_n \in C\}.$$

If $C \subset X$ is aconvex set, then $T_C(x_0) = T(x_0; C) = cl \bigcup_{\lambda>0} \frac{C-x_0}{\lambda}$ (see [3]).

Lemma 3.1 *If X and Y are Banach spaces, $\Lambda : X \to Y$ is a linear continuous surjective operator, $\varphi_i : X \to R$ are sublinear continuous functions at $i \in I$, $C \subset X$ is a convex set, $x_0 \in C$, $\max\limits_{0\le i\le m} \varphi_i(x) \ge 0$ at $x \in \{z \in T_C(x_0) : \Lambda z = 0\}$, then*

$$g(y,a) = \begin{cases} \inf\limits_{x\in T_C(x_0),\ \Lambda x+y=0}\ \max\limits_{0\le i\le m} (\varphi_i(x) + a_i), & if\ T_C(x_0) \bigcap \Lambda^{-1}(-y) \ne \emptyset, \\ +\infty, & if\ T_C(x_0) \bigcap \Lambda^{-1}(-y) = \emptyset \end{cases}$$

is a sublinear function in $Y \times R^{m+1}$, and if besides $int\, T_C(x_0) \bigcap Ker\, \Lambda \ne \emptyset$, then $T_C(x_0) \bigcap \Lambda^{-1}(-y) \ne \emptyset$ at $y \in Y$ and $g(y, a)$ is a sublinear continuous function in $Y \times R^{m+1}$.

Proof. Let $e(x, a) = \max\limits_{0\le i\le m} (\varphi_i^1(x) + a_i))$, where $\varphi_i : X \to R$ are sublinear continuous functions at $i \in I$. Let's designate

$$V(x,y,a) = e(x,a) + \delta_{T_C(x_0)}(x) + \delta_{graph(\Lambda^{-1})}(-y,x) = \begin{cases} e(x,a) : x \in T_C(x_0) \bigcap \Lambda^{-1}(-y), \\ +\infty : \ x \notin T_C(x_0) \bigcap \Lambda^{-1}(-y). \end{cases}$$

It is easily checked that $V(x, y, a)$ is a sublinear lower semicontinuous function and $g(y, a) = \inf\limits_{x\in X} V(x, y, a)$ is a sublinear function.

Let $int\, T_C(x_0) \bigcap Ker\, \Lambda \ne \emptyset$. By the condition there is the mapping $M : Y \to X$ (generally speaking, discontinuous nonlinear), satisfying the conditions $\Lambda \circ M = I_Y$ and $\|M(y)\| \le c\|y\|$ for some $c > 0$ (see [1], p. 128).

As $\varphi_i : X \to R$ are sublinear continuous functions at $i \in I$, according to Hermander's theorem (see [8]) $\varphi_i(x) = \max\limits_{p\in\partial\varphi_i(0)} \langle p, x\rangle$ at $x \in X$, where $\partial\varphi_i(0)$ is a weakly compact set in X^*. Then there is a number $L_i > 0$ such that $\|p\|_* \le L_i$ at $p \in \partial\varphi_i(0)$. Having put $L = \max\limits_{0\le i\le m} L_i$ we have that $\|p\|_* \le L$ at $p \in \bigcup_{i=0}^m \partial\varphi_i(0)$. If $d = \max\limits_{0\le i\le m} a_i$, then

$$g(y,a) = \inf_{x\in T_C(x_0),\ \Lambda x+y=0}\ \max_{0\le i\le m} (\varphi_i(x) + a_i) \le \inf_{x\in T_C(x_0),\ \Lambda x+y=0} (L\|x\| + d).$$

As from $\Lambda x + y = 0$ follows that $x = M(-y) + z$, where $z \in Ker\ \Lambda$, then the system $x \in T_C(x_0)$, $\Lambda x + y = 0$ has the solution x if and only if $x = M(-y) + z \in T_C(x_0)$ at some $z \in Ker\Lambda$. Let $\bar{x} \in int\, T_C(x_0) \bigcap Ker\, \Lambda$. As $\|M(y)\| \le c\|y\|$ at $y \in Y$, there exists $\delta > 0$ such that $M(-y) + \bar{x} \in T_C(x_0)$ at $\|y\| \le \delta$, $y \in Y$. Then $\lambda(M(-y) + \bar{x}) \in T_C(x_0)$ at $\|y\| \le \delta$, $y \in Y$ and $\lambda \ge 0$. It is clear that $\Lambda(\lambda(M(-y) + \bar{x})) + \lambda y = 0$ at

$\|y\| \leq \delta$, $y \in Y$ and $\lambda \geq 0$. From here we have that $\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar{x}) \in T_C(x_0)$ and $\Lambda(\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar{x}))+y=0$ at $y \in Y$, $y \neq 0$. Therefore $T_C(x_0)\bigcap \Lambda^{-1}(-y) \neq \emptyset$ at $y \in Y$. Then

$$g(y,a) \leq \inf_{x\in T_C(x_0),\ \Lambda x+y=0} (L\|x\|+d) = \inf_{M(-y)+z\in T_C(x_0),\ z\in Ker\Lambda} (L\|M(-y)+z\|+d)$$

$$\leq (L\left\|\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar{x})\right\|+d) \leq d+Lc\|y\|+L\|\bar{x}\|\frac{\|y\|}{\delta}$$

at $y \in Y$. As $g(0,0)=0$, from here it follows that $g(y,a)$ is a continuous function.

The lemma is proved.

Let the conditions of the first part of Lemma 3.1 be satisfied. If in Lemma 3.1 $T_C(x_0)\bigcap \Lambda^{-1}(-y) \neq \emptyset$ at $y \in Y = R^n$, then $g(y,a)$ is a continuous sublinear function in $R^n \times R^{m+1}$. If $C = X$ and $Im\Lambda = Y$, it follows from Lemma 3.1 that $g(y,a)$ is a continuous sublinear function.

Lemma 3.2 *If X is a reflexive Banach space, $\Lambda : X \to Y$ is a linear continuous surjective operator, $\varphi_i : X \to R$ are sublinear continuous functions at $i \in I$, $C \subset X$ is a convex set, $x_0 \in C$, $int\,T_C(x_0)\bigcap Ker\,\Lambda \neq \emptyset$ and there exists $d > 0$ such that $\max\limits_{0\leq i\leq m} \varphi_i(x) \geq d$ at $x \in T_C(x_0)$ and $\|x\| = 1$, then exists the solution of the problem $\inf\limits_{x\in T_C(x_0),\ \Lambda x+y=0} \max\limits_{0\leq i\leq m}(\varphi_i(x)+a_i)$, i.e. there is the element $\bar{x} \in T_C(x_0)\bigcap \Lambda^{-1}(-y)$ such that $\max\limits_{0\leq i\leq m}(\varphi_i(\bar{x})+a_i) = \inf\limits_{x\in T_C(x_0),\ \Lambda x+y=0} \max\limits_{0\leq i\leq m}(\varphi_i(x)+a_i)$.*

Proof. It is easily checked that $T_C(x_0)\bigcap \Lambda^{-1}(-y)$ is a closed convex set in X. Let's designate $\alpha = \min\limits_{0\leq i\leq m} a_i$. As

$$\max_{0\leq i\leq m}(\varphi_i(x)+a_i) \geq \max_{0\leq i\leq m}(\varphi_i(x)+\alpha) = \max_{0\leq i\leq m}\|x\|\varphi_i(\frac{x}{\|x\|})+\alpha \geq d\|x\|+\alpha$$

at $x \in T_C(x_0)$. Therefore $\max\limits_{0\leq i\leq m}(\varphi_i(x)+a_i) \to +\infty$ at $x \in T_C(x_0)\bigcap \Lambda^{-1}(-y)$, $\|x\| \to +\infty$. Then validity of Lemma 3.2 follows from Proposition 2.1.2[13]. The lemma is proved.

Let's put $g_{(a,y)}(x) = \max\limits_{0\leq i\leq m}(\varphi_i(x)+a_i)+\delta_{T_C(x_0)\bigcap \Lambda^{-1}(-y)}(x)$, where $(a,y) \in R^{m+1}\times Y$, $N_C(x_0) = T_C(x_0)^-$ (see [3]) and $p \in X^*$.

Lemma 3.3 *If $\Lambda : X \to Y$ is a linear continuous surjective operator, $\varphi_i : X \to R$ are sublinear continuous functions at $i \in I$, $C \subset X$ is a set, $x_0 \in C$, $T_C(x_0)$ is Clarks tangent cone of to the set C at the point x_0, $int\,T_C(x_0)\bigcap Ker\,\Lambda \neq \emptyset$, $\max\limits_{0\leq i\leq m}\varphi_i(x) \geq 0$ at $x \in T_C(x_0)\bigcap Ker\,\Lambda$, then*

$$g^*_{(a,y)}(p) = \begin{cases} \inf\{\langle p-x^*,\bar{x}\rangle - \sum\limits_{i=0}^{m}\lambda_i(\langle q_i,\bar{x}\rangle+a_i) : \lambda_i \geq 0,\ \sum\limits_{i=0}^{m}\lambda_i = 1, \\ q_i \in \partial\varphi_i(0),\ x^* \in N_C(x_0),\ p-\sum\limits_{i=0}^{m}\lambda_i q_i - x^* \in Im\Lambda^*\} : if\ p \in K, \\ \quad +\infty : \qquad\qquad if\ p \notin K, \end{cases}$$

where $K = co\bigcup_{i=0}^{m}\partial\varphi_i(0)+N_C(x_0)+Im\Lambda^*$ and $\bar{x} \in T_C(x_0)\bigcap Ker\,\Lambda$ any point.

Proof. Note that $T_C\,(x_0)\bigcap\varLambda^{-1}(-y)$ is the closed convex set in X.

As $g_{(a,y)}(x)=\max\limits_{0\le i\le m}(\varphi_i(x)+a_i)+\delta_{T_C(x_0)\bigcap\varLambda^{-1}(-y)}(x)$, then

$$g^*_{(a,y)}(p)=\sup_{x\in T_C\,(x_0),\ \varLambda x+y=0}\{\langle p,x\rangle\ -\max_{0\le i\le m}(\varphi_i(x)+a_i)\}$$

$$=\sup_{x\in T_C\,(x_0),\ \varLambda x+y=0}\ \min_{0\le i\le m}(\langle p,x\rangle\ -\varphi_i(x)-a_i)$$

$$=-\inf_{x\in T_C\,(x_0),\ \varLambda x+y=0}\ \max_{0\le i\le m}(\varphi_i(x)-\langle p,x\rangle\ +a_i).$$

Denote $K=\{p\in X^*:\ \max\limits_{0\le i\le m}(\varphi_i(x)-\langle p,x\rangle\ )\ge 0\ \ for\ x\in T_C(x_0)\bigcap Ker\,\varLambda\}$. Note that $K=co\bigcup_{i=0}^m\partial\varphi_i(0)+(T_C(x_0)\bigcap Ker\,\varLambda)^-$. Therefore $K=co\bigcup_{i=0}^m\partial\varphi_i(0)+T_C(x_0)^-+(Ker\,\varLambda)^\perp$ (see [3], Theorem 4.1.16). As $(Ker\,\varLambda)^\perp=Im\varLambda^*$ (see [1], Lemma 2.1.7), then $K=co\bigcup_{i=0}^m\partial\varphi_i(0)++N_C(x_0)+Im\varLambda^*$.

Suppose $\bar p\notin K$. Then exist $z\in T_C(x_0)\bigcap Ker\,\varLambda$ such that $\max\limits_{0\le i\le m}(\varphi_i(z)-\langle p,z\rangle\ )<0$.

Let $\bar x\in int\,T_C(x_0)\bigcap Ker\,\varLambda\ne\emptyset$. From the proof Lemma 3.1 we get that $\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x)\in T_C(x_0)$ and $\varLambda(\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x))+y=0$ at $y\in Y$. Note that $\lambda z\in T_C(x_0)\bigcap Ker\,\varLambda$, $\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x)+\lambda z\in\in T_C(x_0)$ and $\varLambda((\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x)+\lambda z)+y=0$ for $\lambda\ge 0$. Therefore

$$\inf_{x\in T_C\,(x_0),\ \varLambda x+y=0}\ \max_{0\le i\le m}(\varphi_i(x)-\langle p,x\rangle\ +a_i))\le\inf_{\lambda\ge 0}\ \max_{0\le i\le m}(\varphi_i(\lambda z+\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x)$$

$$-\left\langle p,\lambda z+\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x)\right\rangle+a_i)\le\inf_{\lambda\ge 0}\ \max_{0\le i\le m}(\varphi_i(\lambda z)+\varphi_i(\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x))-\langle p,\lambda z\rangle$$

$$-\left\langle p,\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x)\right\rangle+a_i)\le\inf_{\lambda\ge 0}\ \max_{0\le i\le m}(\varphi_i(\lambda z)-\langle p,\lambda z\rangle)$$

$$+\max_{0\le i\le m}(\varphi_i(\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x)-\left\langle p,\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x)\right\rangle+a_i)\le\ \max_{0\le i\le m}(\varphi_i(z)$$

$$-\langle p,z\rangle\ )\sup_{\lambda\ge 0}\lambda+\max_{0\le i\le m}(\varphi_i(\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x))-\left\langle p,\frac{\|y\|}{\delta}(M(-\frac{\delta y}{\|y\|})+\bar x)\right\rangle+a_i)=-\infty.$$

Suppose $p\in K$. Let's put $\varphi_i^1(x)=\varphi_i(x)-\langle p,x\rangle$, $e(x,z)=\max\limits_{0\le i\le m}(\varphi_i^1(x)+z_i))$, $I:R^{m+1}\to R^{m+1}$, $Iz=z$, $a=(a_0,a_1,\dots,a_m)$.

Then

$$q(y,a)=-g^*_{(a,y)}(p)=\inf_{x\in T_C\,(x_0),\ \varLambda x+y=0}\ \max_{0\le i\le m}(\varphi_i^1(x)+a_i)$$

$$=\inf_{x\in T_C\,(x_0),\,z\in R^{m+1},\ x\in\varLambda^{-1}(-y),\ z\in I^{-1}(a)}\ \max_{0\le i\le m}(\varphi_i^1(x)+z_i).$$

As $0\in T_C(x_0)$ and $dom\ e=X\times R^{m+1}$, it is easily checked that

$$((0,0),(0,0))\in int(\{((x,z),(-\varLambda x,Iz)):(x,z)\in(T_C(x_0)\times R^{m+1})\}$$

$$-dom\,e\times(Y\times R^{m+1}))=int(\{((x,z),(-\varLambda x,Iz)):(x,z)\in$$

$$\in(T_C(x_0)\times R^{m+1})\}+(X\times R^{m+1})\times(Y\times R^{m+1}))=(X\times R^{m+1})\times(Y\times R^{m+1}).$$

Let's denote by $L=(-\varLambda,I)$. Then from Corollary 4.5.4 [3] it follows that the following statements are equivalent: $(y^*,b)\in\partial q(0,0)$ and $\exists\,((p,\alpha),(0,0))\in\partial e(x,z)_{((x,z),(y,a))=((0,0),(0,0))}$; $L^*(y^*,b)\in(p,\alpha)+(N_C(x_0),0)$, i.e. $\exists\,(p,\alpha)\in\partial e(0,0)\ ;\ \ -\varLambda^*y^*\in p+N_C(x_0),\ \ b=\alpha$.

Let's denote

$e(x,a)=\max\limits_{0\le i\le m}(\varphi_i^1(x)+a_i)=\max\limits_{0\le i\le m}(\varphi_i^1(x)+\langle(0,\dots 0,1,0\dots,0),(0,\dots 0,a_i,0\dots,0)\rangle)$.

From theorem Dubovitzkii- Milyutin (see [1], p.234) follows that

$$\partial e(0,0)=co\{(p_0,1,0,\dots 0),(p_1,0,1,0,\dots 0),\dots,(p_m,0,\dots 0,1):p_i\in\partial\varphi_i^1(0)\}$$
$$=\left\{\left(\sum_{i=0}^m\lambda_ip_i,\lambda_0,\lambda_1,\dots,\lambda_m\right):\ \lambda_i\ge 0,\ \sum_{i=0}^m\lambda_i=1,\ p_i\in\partial\varphi_i^1(0)\right\}.$$

Therefore

$$\partial q(0,0)=\left\{(y^*,\lambda_0,\lambda_1,\dots,\lambda_m):\ \lambda_i\ge 0,\ \sum_{i=0}^m\lambda_i=1,\ p_i\in\partial\,\varphi_i^1(0),\ x^*\in N_C(x_0)\ ,\right.$$
$$\left. y^*\in Y^*,\ \sum_{i=0}^m\lambda_ip_i+x^*+\Lambda^*y^*=0\right\}.$$

According to Hermander's theorem (see [8]) we have that $q(y,a)=\sup\limits_{(y^*,\lambda)\in\partial q(0,0)}(\langle y^*,y\rangle+\sum\limits_{i=0}^m\lambda_ia_i)$. Therefore

$$q(y,a)=\inf_{x\in T_C\,(x_0),\ \Lambda x+y=0}\ \max_{0\le i\le m}(\varphi_i^1(x)+a_i)=\sup_{(y^*,\lambda)\in\partial q(0,0)}(\langle y^*,y\rangle+\sum_{i=0}^m\lambda_ia_i)$$
$$=\sup\left\{\langle y^*,y\rangle+\sum_{i=0}^m\lambda_ia_i:\ \lambda_i\ge 0,\ \sum_{i=0}^m\lambda_i=1,\ p_i\in\partial\varphi_i^1(0),\ x^*\in N_C(x_0)\ ,\right.$$
$$\left. y^*\in Y^*,\ \sum_{i=0}^m\lambda_ip_i+x^*+\Lambda^*y^*=0\right\}.$$

As $\sum\limits_{i=0}^m\lambda_ip_i+x^*+\Lambda^*y^*=0$, then $\langle\Lambda^*y^*,x\rangle=-\left\langle\sum\limits_{i=0}^m\lambda_ip_i+x^*,x\right\rangle$ for $x\in X$. Therefore $\langle y^*,\Lambda x\rangle=-\left\langle\sum\limits_{i=0}^m\lambda_ip_i+x^*,x\right\rangle$ for $x\in X$. If $x\in T_C\,(x_0)\bigcap\Lambda^{-1}(-y)$, then $y=-\Lambda x$. Then we have that $\langle y^*,y\rangle=\left\langle\sum\limits_{i=0}^m\lambda_ip_i+x^*,x\right\rangle$ for $x\in T_C\,(x_0)\bigcap\Lambda^{-1}(-y)$. Then

$$g^*_{(a,y)}(p)=-q(y,a)=-\sup\left\{\left\langle\sum_{i=0}^m\lambda_ip_i+x^*,x\right\rangle+\sum_{i=0}^m\lambda_ia_i:\ \lambda_i\ge 0,\ \sum_{i=0}^m\lambda_i=1\ ,\right.$$
$$\left. p_i\in\partial\varphi_i^1(0),\ x^*\in N_C(x_0),-\sum_{i=0}^m\lambda_ip_i-x^*\in Im\Lambda^*\right\}$$

for $x\in T_C\,(x_0)\bigcap\Lambda^{-1}(-y)$. As $p_i\in\partial\varphi_i^1(0)=\partial\varphi_i(0)-p$, then $p_i=q_i-p$, where $q_i\in\partial\varphi_i(0)$. Therefore

$$g^*_{(a,y)}(p)=-\sup\left\{\left\langle\sum_{i=0}^m\lambda_i(q_i-p)+x^*,x\right\rangle+\sum_{i=0}^m\lambda_ia_i:\ \lambda_i\ge 0,\ \sum_{i=0}^m\lambda_i=1\ ,\right.$$

$$q_i \in \partial\varphi_i(0),\ x^* \in N_C(x_0),\ -\sum_{i=0}^{m}\lambda_i p_i - x^* \in Im\Lambda^* \Big\}$$

$$= -\sup\Big\{ -\langle p, x\rangle + \langle x^*, x\rangle + \sum_{i=0}^{m}\lambda_i(\langle q_i, x\rangle + a_i) : \lambda_i \geq 0\ ,$$

$$\sum_{i=0}^{m}\lambda_i = 1,\ q_i \in \partial\varphi_i(0),\ x^* \in N_C(x_0),\ p - \sum_{i=0}^{m}\lambda_i q_i - x^* \in Im\Lambda^* \Big\}$$

for $x \in T_C\,(x_0) \bigcap \Lambda^{-1}(-y)$. Then we have that

$$g^*_{(a,y)}(p) = \begin{cases} \inf\{\langle p - x^*, x\rangle - \sum\limits_{i=0}^{m}\lambda_i(\langle q_i, x\rangle + a_i) : \lambda_i \geq 0,\ \sum\limits_{i=0}^{m}\lambda_i = 1,\ q_i \in \\ \in \partial\varphi_i(0), x^* \in N_C(x_0),\ p - \sum\limits_{i=0}^{m}\lambda_i q_i - x^* \in Im\Lambda^*\} : \ if\ p \in K, \\ +\infty : \qquad\qquad if\ p \notin K. \end{cases}$$

The lemma is proved.

Let's put $g_{(a,y)}(x) = \max\limits_{0\leq i\leq m}(\varphi_i(x) + a_i) + \delta_{T_C(x_0)\bigcap \Lambda^{-1}(-y)}(x)$. Let the condition of Lemma 3.3 be satisfied. Then we have that $\{x \in X : g_{(a,y)}(x) = \inf\{g_{(a,y)}(z) : z \in X\}\} = \partial g^*_{(a,y)}(0)$ (see [5], Theorem 6.4.5). Therefore if $\partial g^*_{(a,y)}(0) \neq \emptyset$ then, exists there is the element $\bar{x} \in T_C\,(x_0) \bigcap \Lambda^{-1}(-y)$ such that

$$\max_{0\leq i\leq m}(\varphi_i(\bar{x}) + a_i) = \inf_{x\in T_C\,(x_0),\ \Lambda x + y = 0}\ \max_{0\leq i\leq m}(\varphi_i(x) + a_i).$$

Let's note that, if $0 \in int\, dom g^*_{(a,y)}$, then $\partial g^*_{(a,y)}(0) \neq \emptyset$ (see[3]). If $X = R^n$ and $0 \in ri\, dom g^*_{(a,y)}$, then $\partial g^*_{(a,y)}(0) \neq \emptyset$ (see[7]).

Corollary 3.1 *If the condition of Lemma 3.3 be satisfied and* $0 \in int(co\bigcup_{i=0}^m \partial\varphi_i(0) + N_C(x_0) + Im\Lambda^*)$*, then* $\partial g^*_{(a,y)}(0) \neq \emptyset$.

Lemma 3.4 *Let X and Y be Hilbert spaces, $\Lambda : X \to Y$ be a linear continuous surjective operator, $\varphi_i : X \to R$ be sublinear continuous functions at $i \in I$, $C \subset X$ be a set, $x_0 \in C$, $T_C(x_0)$ be Clarks tangent cone of to the set C at the point x_0, $\bar{r}_1(x) = \max\limits_{0\leq i\leq m}\varphi_i(x) + \delta_{T_C(x_0)}(x)$, $\bar{r}_2(x) = \delta_{Ker\ \Lambda}(x)$, $int\, T_C\,(x_0) \bigcap Ker\,\Lambda \neq \emptyset$ and $0 \in int(dom\,\bar{r}^*_1 + dom\,\bar{r}^*_2)$. Then there exists the solution of the problem* $\inf\limits_{x\in T_C\,(x_0),\ \Lambda x+y=0}\ \max\limits_{0\leq i\leq m}(\varphi_i(x) + a_i)$.

Proof. Let's consider the problem

$$g_{(a,y)}(x) = \max_{0\leq i\leq m}(\varphi_i(x) + a_i) + \delta_{T_C(x_0)\bigcap \Lambda^{-1}(-y)}(x) \to \min,\ x \in X.$$

Let's denote $r_1(x) = \max\limits_{0\leq i\leq m}(\varphi_i(x) + a_i) + \delta_{T_C(x_0)}(x)$, $r_2(x) = \delta_{\Lambda^{-1}(-y)}(x)$, then the condition of Lemma 3.4 we have that $0 \in dom\, r_1 - dom\, r_2$. If $0 \in int(dom\, r^*_1 + dom\, r^*_2)$, $\upsilon = \inf\{r_1(x) + r_2(x) :\ x \in X\}$ and $\upsilon_* = \inf\{r^*_1(-p) + r^*_2(p) : p \in X^*\}$, then from Fenxel theorem (see[2]) there is the element $\bar{x} \in T_C\,(x_0) \bigcap \Lambda^{-1}(-y)$ such that

$$\max_{0\leq i\leq m}(\varphi_i(\bar{x}) + a_i) = \inf_{x\in T_C\,(x_0),\ \Lambda x + y = 0}\ \max_{0\leq i\leq m}(\varphi_i(x) + a_i).$$

As $\max\limits_{0\leq i\leq m}\varphi_i(x)+\min\limits_{0\leq i\leq m}a_i+\delta_{T_C(x_0)}(x)\leq r_1(x)\leq\max\limits_{0\leq i\leq m}\varphi_i(x)+\max\limits_{0\leq i\leq m}a_i+\delta_{T_C(x_0)}(x)$, then

$$\sup_{x\in X}\{\langle p,x\rangle-\max_{0\leq i\leq m}\varphi_i(x)-\max_{0\leq i\leq m}a_i-\delta_{T_C(x_0)}(x)\}\leq r_1^*(p)$$

$$\leq\sup_{x\in X}\{\langle p,x\rangle-\max_{0\leq i\leq m}\varphi_i(x)-\min_{0\leq i\leq m}a_i-\delta_{T_C(x_0)}(x)\},$$

where $p\in X^*$.Therefore

$$\sup_{x\in X}\{\langle p,x\rangle-\max_{0\leq i\leq m}\varphi_i(x)-\delta_{T_C(x_0)}(x)\}-\max_{0\leq i\leq m}a_i\leq r_1^*(p)$$

$$\leq\sup_{x\in X}\{\langle p,x\rangle-\max_{0\leq i\leq m}\varphi_i(x)-\delta_{T_C(x_0)}(x)\}-\min_{0\leq i\leq m}a_i.$$

Then we have that $domr_1^*=dom\bar{r}_1^*$, where $\bar{r}_1(x)=\max\limits_{0\leq i\leq m}\varphi_i(x)+\delta_{T_C(x_0)}(x)$.

Let $p\in X^*$. From proof of Lemma 3.1 we have that

$$r_2^*(p)=\sup_{x\in X}\{\langle p,x\rangle-\delta_{\Lambda^{-1}(-y)}(x)\}=\sup_{x\in\Lambda^{-1}(-y)}\langle p,x\rangle=\sup_{x\in Ker\Lambda+M(-y)}\langle p,x\rangle$$
$$=\sup_{z\in Ker\Lambda}\langle p,z+M(-y)\rangle=\langle p,M(-y)\rangle+\delta_{(Ker\,\Lambda)^{\perp}}(p).$$

Therefore $dom\,r_2^*=dom\,\bar{r}_2^*=(Ker\,\Lambda)^{\perp}$, where $\bar{r}_2(x)=\delta_{Ker\,\Lambda}(x)$. If $0\in int(dom\,\bar{r}_1^*+dom\,\bar{r}_2^*)$, then from Fenxel theorem (see[2]) there is the element $\bar{x}\in T_C\,(x_0)\bigcap\Lambda^{-1}(-y)$ such that

$$\max_{0\leq i\leq m}(\varphi_i(\bar{x})+a_i)=\inf_{x\in T_C\,(x_0),\;\Lambda x+y=0}\;\max_{0\leq i\leq m}(\varphi_i(x)+a_i).$$

The lemma is proved.

Lemma 3.5 *If $\Lambda:X\to Y$ is a linear continuous surjective operator, $\varphi_i:X\to R$ are sublinear continuous functions at $i\in I$, $C\subset X$ is a set, $x_0\in C$, $T_C(x_0)$ is Clarks tangent cone of to the set C at the point x_0, $int\,T_C(x_0)\bigcap Ker\,\Lambda\neq\emptyset$, then there exists the solution of the problem $\inf\limits_{x\in T_C\,(x_0),\;\Lambda x+y=0}\;\max\limits_{0\leq i\leq m}(\varphi_i(x)+a_i)$ if and only if, there exist the elements $z_1^*\in dom\,\varphi^*$, $z_2^*\in(Ker\,\Lambda)^{\perp}$ and $z_3^*\in T_C\,(x_0)^-$ such that $z_1^*+z_2^*+z_3^*=0$ and $\partial h_1^*(z_1^*)\bigcap\Lambda^{-1}(-y)\bigcap\partial\delta_{T_C(x_0)^-}(z_3^*)\neq\emptyset$, where $\varphi(x)=\max\limits_{0\leq i\leq m}\varphi_i(x)$, $h_1(x)=\max\limits_{0\leq i\leq m}(\varphi_i(x)+a_i)$.*

Proof. Let $g_{(a,y)}(x)=\max\limits_{0\leq i\leq m}(\varphi_i(x)+a_i)+\delta_{T_C(x_0)\bigcap\Lambda^{-1}(-y)}(x)$, $h_1(x)=\max\limits_{0\leq i\leq m}(\varphi_i(x)+a_i)$, $h_2(x)=\delta_{\Lambda^{-1}(-y)}(x)$ and $h_3(x)=\delta_{T_C(x_0)}(x)$. From proof of Lemma 3.1 we have that $int\,T_C(x_0)\bigcap\Lambda^{-1}(-y)\neq\emptyset$ at $y\in Y$. If $p\in dom\,g_{(a,y)}^*$, then by Theorem 6.5.8 [5] exists there is the elements $p_1\in dom\,h_1^*$, $p_2\in dom\,h_2^*$ and $p_3\in dom\,h_3^*$ such that $p=p_1+p_2+p_3$ and
$g_{(a,y)}^*(p)=h_1^*(p_1)+h_2^*(p_2)+h_3^*(p_3)$.

Let $p\in X^*$. As $\max\limits_{0\leq i\leq m}\varphi_i(x)+\min\limits_{0\leq i\leq m}a_i\leq h_1(x)\leq\max\limits_{0\leq i\leq m}\varphi_i(x)+\max\limits_{0\leq i\leq m}a_i$, then

$$\sup_{x\in X}\left\{\langle p,x\rangle-\max_{0\leq i\leq m}\varphi_i(x)-\max_{0\leq i\leq m}a_i\right\}\leq h_1^*(p)\leq\sup_{x\in X}\left\{\langle p,x\rangle-\max_{0\leq i\leq m}\varphi_i(x)-\min_{0\leq i\leq m}a_i\right\}.$$

Therefore

$$\sup_{x\in X}\left\{\langle p,x\rangle-\max_{0\le i\le m}\varphi_i(x)\right\}-\max_{0\le i\le m}a_i\le h_1^*(p)\le\sup_{x\in X}\left\{\langle p,x\rangle-\max_{0\le i\le m}\varphi_i(x)\right\}-\min_{0\le i\le m}a_i.$$

Then we have that $dom\, h_1^*=dom\,\varphi^*$, where $\varphi(x)=\max\limits_{0\le i\le m}\varphi_i(x)$.

If $p\in X^*$, then from proof of Lemma 3.4 we have

$$h_2^*(p)=\sup_{x\in X}\left\{\langle p,x\rangle-\delta_{\Lambda^{-1}(-y)}(x)\right\}=\langle p,M(-y)\rangle+\delta_{(Ker\,\Lambda)^\perp}(p).$$

If $p\in X^*$, then $h_3^*(p)=\sup\limits_{x\in X}\left\{\langle p,x\rangle-\delta_{T_C(x_0)}(x)\right\}=\sup\limits_{x\in T_C(x_0)}\langle p,x\rangle=\delta_{T_C(x_0)^-}(p)$.

If $p\in dom\, g^*_{(a,y)}$, then by Corollary 6.5.9 [5] there exist the elements $p_1\in dom\,\varphi^*$, $p_2\in(Ker\,\Lambda)^\perp$ and $p_3\in T_C\,(x_0)^-$ such that $p_1+p_2+p_3=p$ and

$$g^*_{(a,y)}(p)=\inf\{h_1^*(x_1^*)+h_2^*(x_2^*)+h_3^*(x_3^*):p=x_1^*+x_2^*+x_3^*\}$$

$$=h_1^*(p_1)+\langle p_2,M(-y)\rangle+\delta_{(Ker\,\Lambda)^\perp}(p_2)+\delta_{T_C(x_0)^-}(p_3).$$

Then by Proposition 6.6.4[5] we have that $\partial\, g^*_{(a,y)}(0)\ne\emptyset$ if and only if, there exist the elements $z_1^*\in dom\,\varphi^*$, $z_2^*\in(Ker\,\Lambda)^\perp$ and $z_3^*\in T_C\,(x_0)^-$ such that $z_1^*+z_2^*+z_3^*=0$ and

$$\partial h_1^*(z_1^*)\bigcap(Ker\Lambda+M(-y))\bigcap\partial\delta_{T_C(x_0)^-}(z_3^*)\ne\emptyset.$$

The lemma is proved.

Note that if $z_3^*\in T_C\,(x_0)^-$, then $\partial\delta_{T_C(x_0)^-}(z_3^*)=N_{T_C(x_0)^-}(z_3^*)$ (see [3]).

Let the condition of Lemma 3.5 be satisfied. Then by Proposition 6.6.3 [5] we have that $\partial\, g^*_{(a,y)}(0)\supset\partial h_1^*(z_1^*)\bigcap(Ker\Lambda+M(-y))\bigcap T_C(x_0)$ for $z_1^*\in dom\,\varphi^*\bigcap(Ker\,\Lambda)^\perp$. Therefore if $\bigcup_{z_1^*\in dom\,\varphi^*\bigcap(Ker\,\Lambda)^\perp}\partial h_1^*(z_1^*)\bigcap(Ker\Lambda+M(-y))\bigcap T_C(x_0)\ne\emptyset$, then $\partial\, g^*_{(a,y)}(0)\ne\emptyset$.

As $\varphi_i:X\to R$ are sublinear continuous functions at $i\in I$, then $z\in\partial h_1^*(z_1^*)$ if and only if $z_1^*\in\partial h_1(z)$. Therefore if $dom\,\varphi^*\subset(Ker\,\Lambda)^\perp$, then $\partial\, g^*_{(a,y)}(0)\ne\emptyset$.

Let's note that if $g_{(a,y)}(x)=\langle p_0,x\rangle+a_0+\delta_{T_C(x_0)\bigcap\Lambda^{-1}(-y)}(x)$ and $h(x)=\langle p_0,x\rangle+a_0$, where $p_0\in X^*$, then $\partial h^*(p_0)=X$ and $\Lambda^{-1}(-y)\bigcap T_C(x_0)\subset\partial g^*_{(a,y)}(0)$. Therefore if $\Lambda^{-1}(-y)\bigcap T_C(x_0)\ne\emptyset$, then $\partial\, g^*_{(a,y)}(0)\ne\emptyset$.

Let $\varphi_i:X\to R$ are sublinear continuous functions and $a_i\in R$ at $i\in I$, $h_1(x)=\max\limits_{0\le i\le m}(\varphi_i(x)+a_i)$. Then by analogy Lemma 3.3 we have that

$$h_1^*(p)=\begin{cases}\inf\{-\sum\limits_{i=0}^m\lambda_ia_i:\lambda_i\ge0,\ \sum\limits_{i=0}^m\lambda_i=1,\ q_i\in\partial\varphi_i(0),\ p=\sum_{i=0}^m\lambda_iq_i\}: & if\ p\in P,\\ +\infty: & if\ p\notin P,\end{cases}$$

where $P=co\bigcup_{i=0}^m\partial\varphi_i(0)$, $dom\,h_1^*=P$ and $p\to h_1^*(p)$ is proper convex function.

4 Necessary condition of the second order

Let X and Y be Banach spaces, $f_i : X \to R$, $i \in I$, $F : X \to Y$, $C \subset X$.

Let's consider the problem

$$f_0(x) \to \min, f_i(x) \leq 0, i = 1, \dots, m, F(x) = 0,\ x \in C. \tag{4.1}$$

If $C \subset X$ is a convex set and $x_0 \in C$, we will designate that $\tilde{C} = intC \bigcup \{x_0\}$, $S_{\tilde{C}}(x_0) = \bigcup_{\lambda>0} \frac{\tilde{C}-x_0}{\lambda}$. Let's note that if $intC \neq \emptyset$ and C is a convex set, then $T_C(x_0) = clS_{\tilde{C}}(x_0)$.

Theorem 4.1 *If X and Y are Banach spaces, x_0 is the local minimum point in problem (4.1), $\beta > 1$, the function f_i satisfies $\varphi_i - (\beta, \delta)$ locally semi-Lipschitz condition with the constant K at the point x_0, where $i \in I$, $\varphi_i : X \to R$ are sublinear continuous functions at $i \in I$, $f_i(x_0) = 0$ at $i \in J$, the mapping $F : X \to Y$ is strictly differentiable at the point x_0 and $F'(x_0)X = Y$, C is a convex set, $intC \neq \emptyset$, then there exist simultaneously non-zero $\alpha_0 \geq 0,\ \alpha_1 \geq 0, \dots, \alpha_m \geq 0$ and $y^* \in Y^*$ such that $\sum_{i=0}^m \alpha_i\varphi_i(x) + \langle y^*, F'(x_0)x\rangle \geq 0$ at $x \in T_C(x_0)$.*

Proof. Let's designate $\Lambda = F'(x_0)$. Let's show that the system

$$\varphi_0(x) < 0, \varphi_1(x) < 0, \dots, \varphi_m(x) < 0, \Lambda x = 0$$

has no solution on $S_{\tilde{C}}(x_0)$. Let's assume opposite. Let there exist $\bar{x} \in S_{\tilde{C}}(x_0)$ such that $\varphi_i(\bar{x}) < 0$ at $i \in I$ and $\Lambda\bar{x} = 0$. As $\Lambda\bar{x} = F'(x_0)\bar{x} = 0$ and $F'(x_0)X = Y$, according to Lyusternik's theorem there exists $\varepsilon > 0$ and the mapping $r : [0, \varepsilon] \to X$ such that $\frac{r(t)}{t} \to 0$ at $t \downarrow 0$ and $F(x_0 + t\bar{x} + r(t)) = 0$ at $t \in [0, \varepsilon]$. By the condition the function f_i satisfies $\varphi_i - (\beta, \delta)$ locally semi-Lipschitz condition with the constant K at the point x_0, where $i \in I$, $\beta > 1$. Then by the condition we have that

$$f_i(x_0 + t\bar{x} + r(t)) - f_i(x_0) - \varphi_i(t\bar{x} + r(t)) \leq K \left\|t\bar{x} + r(t)\right\|^\beta$$

at $t \in [0, \varepsilon]$, $\|t\bar{x} + r(t)\| \leq \delta$, $i \in I$. As φ_i is a continuous function, there exists $0 < \delta_0 \leq \frac{1}{2}\delta$ such that $\left|\varphi_i\left(\bar{x} + \frac{r(t)}{t}\right) - \varphi_i(\bar{x})\right| \leq \frac{1}{2}\left|\varphi_i(\bar{x})\right|$ at $\left\|\frac{r(t)}{t}\right\| \leq \delta_0$ and $i \in I$. Then we receive that $\varphi_i(\bar{x} + \frac{r(t)}{t}) \leq \frac{1}{2}\varphi_i(\bar{x})$ at $\left\|\frac{r(t)}{t}\right\| \leq \delta_0$ and $i \in \{0, 1, \dots, m\}$. As $\frac{r(t)}{t} \to 0$ at $t \downarrow 0$, then there exists λ, where $0 < \lambda < 1$, such that $\left\|\frac{r(t)}{t}\right\| \leq \delta_0$ at $t \in (0, \lambda]$. Then $\|t\bar{x} + r(t)\| \leq \delta$ at $t \in [0, \lambda_1]$, where $\lambda_1 = \min\{\lambda, \frac{1}{2\|\bar{x}\|}\delta, \varepsilon\}$. Therefore we have that

$$f_i(x_0+t\bar{x}+r(t))-f_i(x_0) \leq 0,5\ t\varphi_i(\bar{x})+Kt^\beta \left\|\bar{x} + \frac{r(t)}{t}\right\|^\beta \leq 0,5\ t\varphi_i(\bar{x})+Kt^\beta(\|\bar{x}\|+\delta_0)^\beta$$

at $t \in [0, \lambda_1]$ and $i \in I$. From here we have that $f_0(x_0 + t\bar{x} + r(t)) - f_0(x_0) < 0$, $f_i(x_0 + t\bar{x} + r(t)) < 0$, $i \in J$, $F(x_0 + t\bar{x} + r(t)) = 0$ at rather small $t > 0$. If $\bar{x} \in S_{\tilde{C}}(x_0) = \bigcup_{\lambda>0} \frac{\tilde{C}-x_0}{\lambda}$, then there exists $\lambda_0 > 0$ such that $\bar{x} \in \frac{\tilde{C}-x_0}{\lambda_0}$, i.e. $x_0 + \lambda_0\bar{x} \in \tilde{C}$. Therefore $x_0 + \lambda_0\bar{x} \in int\ C$. Then there exists $\nu_0 > 0$, where $\lambda_0 > \nu_0$, such that $x_0 + \lambda_0\bar{x} + (\lambda_0\frac{r(t)}{t}) \in int\ C$ at $t \in [0, \nu_0]$. Therefore

$$x_0 + t(\bar{x} + \frac{r(t)}{t}) = (1 - \frac{t}{\lambda_0})x_0 + \frac{t}{\lambda_0}(x_0 + \lambda_0\bar{x} + \lambda_0(\frac{r(t)}{t})) \in C,$$

i.e. we have that $x_0 + t\bar{x} + r(t) \in C$ at $t \in [0, \nu_0]$.

As x_0 is a local minimum point in problem (4.1), we receive the contradiction. Therefore the system $\varphi_0(x) < 0$, $\varphi_1(x) < 0$, ..., $\varphi_m(x) < 0$, $\Lambda x = 0$ has no solution on $S_{\tilde{C}}(x_0)$. According to Theorem 5.5.3[12] there exist simultaneously non-zero $\alpha_0 \geq 0$, $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ and $y^* \in Y^*$ such that $\sum_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*, F'(x_0)x\rangle \geq 0$ at $x \in S_{\tilde{C}}(x_0)$.

As $\varphi_i : X \to R$ are continuous functions at $i \in I$ and $\Lambda = F'(x_0)$ is a linear continuous operator, $\sum_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*, F'(x_0)x\rangle \geq 0$ at $x \in clS_{\tilde{C}}(x_0) = T_C(x_0)$. The theorem is proved.

From Theorem 4.1 we have that the zero point minimizes the convex function $\sum_{i=0}^{m} \alpha_i \varphi_i(x) +$ $+ \langle y^*, F'(x_0)x\rangle + \delta_{T_C(x_0)}(x)$ in X. As $\partial \delta_{T_C(x_0)}(0) = N_C(x_0)$, we receive that

$$0 \in \partial \left(\sum_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*, F'(x_0)x\rangle + \delta_{T_C(x_0)}(x) \right)_{x=0} = \sum_{i=0}^{m} \alpha_i \partial \varphi_i(0) + F'(x_0)^* y^* + N_C(x_0).$$

If $intT_C(x_0) \bigcap Ker\, F'(x_0) \neq \emptyset$, then from the proof of Theorem 4.1 we have that $\max_{0 \leq i \leq m} \varphi_i(x) \geq 0$ at $x \in T_C(x_0) \bigcap Ker\, F'(x_0)$.

Let's note that in Theorem 4.1 it is possible to replace the condition: function f_i satisfies $\varphi_i - (\beta, \delta)$ locally semi-Lipschitz condition with the constant K at the point x_0, by the condition: the function f_i satisfies $\varphi_i - (o(1), \delta)$ locally semi-Lipschitz condition at the point x_0.

If $\bar{x} \in I_C(x_0)$ and $r : R_+ \to X$, where $\frac{r(t)}{t} \to 0$ at $t \downarrow 0$, then by definition $I_C(x_0)$ there exists $\alpha_0 > 0$ such that $x_0 + t\bar{x} + r(t) \in C$ at $t \in [0, \alpha_0]$. Therefore from the proof Theorem 4.1 we have that if $C \subset X$ is any set and there exists a hypertangent vector to the set C at the point $x_0 \in C$, Theorem 4.1 remains true if in Theorem 4.1 we replace the cone $S_{\tilde{C}}(x_0)$ by $intT_C(x_0)$, where $T_C(x_0)$ is Clark's tangent cone to the set C at the point x_0(see [4]), i.e. there exist simultaneously non-zero $\alpha_0 \geq 0$, $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ and $y^* \in Y^*$ such that $\sum_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*, F'(x_0)x\rangle \geq 0$ at $x \in intT_C(x_0)$. As $\varphi_i : X \to R$ are sublinear continuous functions at $i \in I$, we will receive from here that $\sum_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*, F'(x_0)x\rangle \geq$ 0 at $x \in T_C(x_0)$.

Theorem 4.1 remains true if $F : X \to Y$ is an affine continuous operator, and $C \subset X$ is a convex set.

Let's consider the problem

$$f_0(x) \to \min, \ f_j(x) \leq 0, \ j = 1, 2, \ldots, m, \quad F(x) = 0, \ x \in C, \tag{4.2}$$

where $f_j : B(x_0, 2\delta) \to R$, $j \in I = \{0, 1, \ldots, m\}$, $F : B(x_0, 2\delta) \to Y$ operator, $C \subset B(x_0, 2\delta)$.

Let's put $\tilde{C} = intC \bigcup \{x_0\}$, $S_{\tilde{C}}(x_0) = \bigcup_{\lambda > 0} \frac{\tilde{C} - x_0}{\lambda}$, $\Lambda x = F'(x_0)x$, $\varphi_i^1 : X \to R$, $g_{(a,y)}(x) = \max_{0 \leq i \leq m} (\varphi_i^1(x)^1 + a_i) + \delta_{T_C(x_0) \bigcap \Lambda^{-1}(-y)}(x)$, where $(a, y) \in R^{m+1} \times Y$ and $H = \{x \in X : \varphi_i^1(x) \leq 0, \quad i \in I, \ \Lambda x = 0, \ x \in S_{\tilde{C}}(x_0)\}$.

Theorem 4.2 *If X and Y are Banach spaces, x_0 is the local minimum point in problem (4.2), $f_i(x_0) = 0$ at $i \in I$, $\beta > 2$, the functions f_i, $i \in I$, satisfy $\varphi_i^1 + \varphi_i^2 - (\beta, \delta)$ locally semi-Lipschitz condition with the constant K at the point x_0, $\varphi_i^1 : X \to R$ are sublinear* *continuous functions at $i \in I$, $\varphi_i^2 : X \to R$ are positive homogeneous functions of degree 2 and satisfy $(1,2,1,\delta,o(2))$ locally Lipschitz condition with the constant K at the zero point at $i \in I$, the mapping $F : X \to Y$ is strictly differentiable at the point x_0, $F'(x_0)X = Y$ and F satisfies $F'(x_0)x + S(x) - (\beta,\delta)$ locally Lipschitz condition with the constant K at the point x_0, $S : X \to Y$ is a positive homogeneous operator of degree 2 and satisfies $(1,2,1,\delta,\tilde{o}(2))$ locally Lipschitz condition with the constant K at the zero point, C is a convex set, $\partial g^*_{(a,y)}(0) \neq \emptyset$ at $(a,y) \in R^{m+1} \times Y$ and $int\, T_C(x_0) \bigcap Ker\, \Lambda \neq \emptyset$, then*

$$E(h) = \sup \left\{ \sum_{i=0}^{m} \lambda_i \varphi_i^2(h) + \langle y^*, S(h) \rangle : \lambda_i \geq 0,\ \sum_{i=0}^{m} \lambda_i = 1,\ p_i \in \partial \varphi_i^1(0)\ , \right.$$

$$\left. x^* \in N_C(x_0),\ y^* \in Y^*, \sum_{i=0}^{m} \lambda_i p_i + \Lambda^* y^* + x^* = 0, \right\} \geq 0$$

at $h \in H$.

Proof. From Theorem 4.1 follows that $\max\limits_{0 \leq i \leq m} \varphi_i^1(x) \geq 0$ at $x \in Ker\, \Lambda \bigcap T_C(x_0)$. Let $h \in H$. Consider the problem

$$\max_{0 \leq i \leq m} (\varphi_i^1(x) + \varphi_i^2(h)) \to \inf, x \in T_C(x_0), \quad \Lambda x + S(h) = 0.$$

Let's put $a_i = \varphi_i^2(h),\ y = S(h)$, $e(x,z) = \max\limits_{0 \leq i \leq m} (\varphi_i^1(x) + z_i)),\ I : R^{m+1} \to R^{m+1}$, $Iz = z$, $a = (a_0, a_1, \ldots, a_m)$. Then

$$g(y,a) = \inf_{x \in T_C\,(x_0),\ \Lambda x + y = 0}\ \max_{0 \leq i \leq m} (\varphi_i^1(x) + a_i)$$

$$= \inf_{x \in T_C\,(x_0), z \in R^{m+1},\ x \in \Lambda^{-1}(-y),\ z \in I^{-1}(a)}\ \max_{0 \leq i \leq m} (\varphi_i^1(x) + z_i).$$

As $0 \in T_C(x_0)$ and $dom\ e = X \times R^{m+1}$, it is easily checked that

$$((0,0),(0,0)) \in int(\{((x,z),(-\Lambda x, Iz)) : (x,z) \in (T_C(x_0) \times R^{m+1})\} - dom\, e \times (Y \times R^{m+1}))$$

$$= int(\{((x,z),(-\Lambda x, Iz)) : (x,z) \in (T_C(x_0) \times R^{m+1})\} + (X \times R^{m+1}) \times (Y \times R^{m+1}))$$

$$= (X \times R^{m+1}) \times (Y \times R^{m+1}).$$

Let's designate $L = (-\Lambda, I)$. Then from Corollary 4.5.4 [3] follows that the following statements are equivalent:

$$(y^*, b) \in \partial g(0,0)$$

and

$$\exists\, ((p,\alpha),(0,0)) \in \partial e(x,z)_{((x,z),(y,a))=((0,0),(0,0))}; \qquad L^*(y^*,b) \in (p,\alpha) + (N_C(x_0), 0),$$

i.e. $\exists\, (p,\alpha) \in \partial e(0,0)\ ;\ \ -\Lambda^* y^* \in p + N_C(x_0), \quad b = \alpha.$

Let's designate

$$e(x,a) = \max_{0 \leq i \leq m} (\varphi_i^1(x) + a_i) = \max_{0 \leq i \leq m} (\varphi_i^1(x) + \langle (0, \ldots 0, 1, 0 \ldots, 0), (0, \ldots 0, a_i, 0 \ldots, 0) \rangle).$$

From theorem Dubovitzkii- Milyutin (see [1], p.234) follows that

$$\partial e(0,0) = co \left\{ (p_0, 1, 0, \ldots 0), (p_1, 0, 1, 0, \ldots 0), \ldots, (p_m, 0, \ldots 0, 1) : p_i \in \partial \varphi_i^1(0) \right\}$$

$$= \left\{ \left(\sum_{i=0}^{m} \lambda_i p_i, \lambda_0, \lambda_1, \ldots, \lambda_m \right) : \ \lambda_i \geq 0, \ \sum_{i=0}^{m} \lambda_i = 1, \ p_i \in \partial \varphi_i^1(0) \right\}.$$

Therefore

$$\partial g(0,0) = \left\{ (y^*, \lambda_0, \lambda_1, \ldots, \lambda_m) : \ \lambda_i \geq 0, \ \sum_{i=0}^{m} \lambda_i = 1, \ p_i \in \partial \varphi_i^1(0), \ x^* \in N_C(x_0), \right.$$

$$\left. y^* \in Y^*, \ \sum_{i=0}^{m} \lambda_i p_i + x^* + \Lambda^* y^* = 0 \right\}.$$

According to Hermander's theorem (see [8]) we have that $g(y,a) = \sup_{(y^*,\lambda)\in\partial g(0,0)} (\langle y^*, y\rangle + + \sum_{i=0}^{m} \lambda_i a_i)$. Therefore by the condition there exists $x = \xi(h)$, where $x \in T_C(x_0)$, $\Lambda x + S(h) = 0$ such that

$$g(y,a) = \inf_{x \in T_C(x_0), \ \Lambda x + y = 0} \ \max_{0 \leq i \leq m} (\varphi_i^1(x) + a_i) = \max_{0 \leq i \leq m} (\varphi_i^1(\xi(h)) + \varphi_i^2(h))$$

$$= \sup_{(y^*,\lambda)\in\partial g(0,0)} \left(\langle y^*, y\rangle + \sum_{i=0}^{m} \lambda_i a_i \right) = \sup \left\{ \langle y^*, y\rangle + \sum_{i=0}^{m} \lambda_i a_i : \ \lambda_i \geq 0, \sum_{i=0}^{m} \lambda_i = 1 \ , \right.$$

$$\left. p_i \in \partial \varphi_i^1(0), \ x^* \in N_C(x_0), y^* \in Y^*, \ \sum_{i=0}^{m} \lambda_i p_i + x^* + \Lambda^* y^* = 0 \right\}.$$

Therefore

$$E(h) = g(S(h), \varphi^2(h)) = \max_{0 \leq i \leq m} (\varphi_i^1(\xi(h)) + \varphi_i^2(h)) = \sup \left\{ \langle y^*, S(h)\rangle + \sum_{i=0}^{m} \lambda_i \varphi_i^2(h) : \right.$$

$$\left. \lambda_i \geq 0, \sum_{i=0}^{m} \lambda_i = 1, \ \ p_i \in \partial \varphi_i^1(0), \ x^* \in N_C(x_0), \ y^* \in Y^*, \sum_{i=0}^{m} \lambda_i p_i + x^* + \Lambda^* y^* = 0 \right\},$$

where $\varphi^2(h) = (\varphi_0^2(h), \varphi_1^2(h), \ldots, \varphi_m^2(h))$, $\Lambda \xi(h) + S(h) = 0$.

As F satisfies $F'(x_0)x + S(x) - (\beta, \delta)$ locally Lipschitz condition with the constant K at the point x_0, where $\beta > 2$, then

$$\left\| F(x_0 + th + t^2\xi) - F(x_0) - \Lambda(th + t^2\xi) - S(th + t^2\xi) \right\| \leq K \left\| th + t^2\xi \right\|^\beta$$

at $t > 0$. Therefore from the ratio $\Lambda h = 0$ and $\Lambda \xi + S(h) = 0$ it follows that

$$\left\| F(x_0 + th + t^2\xi) - F(x_0) + S(th) - S(th + t^2\xi) \right\| \leq K \left\| th + t^2\xi \right\|^\beta$$

at $t > 0$. From here we have that

$$\left\| F(x_0 + th + t^2\xi) \right\| \leq \left\| S(th + t^2\xi) - S(th) \right\| + K \left\| th + t^2\xi \right\|^\beta$$

$$\leq K \left\| t^2\xi \right\| (\|th\| + \left\| t^2\xi \right\|) + \tilde{o}(\|th\|^2) + K \left\| th + t^2\xi \right\|^\beta = o_1(t^2).$$

at $t > 0$, where $\frac{o_1(t)}{t} \to 0$ at $t \downarrow 0$. According to the generalized Lyusternik's theorem there exists $r : R_+ \to X$ and a number $m > 0$ such that

$$F(x_0 + th + t^2\xi + r(t)) = 0, \ \|r(t)\| \leq m \left\| F(x_0 + th + t^2\xi) \right\| = o_2(t^2)$$

at $t>0$, where $r(t)=s(x_0+th+t^2\xi)$ (see [1], p. 173), $o_2(t)=mo_1(t)$, $\frac{o_2(t)}{t}\to 0$ at $t\downarrow 0$.

Besides we have that

$$f_i(x_0+th+t^2\xi+r(t))-f_i(x_0)-\varphi_i^1(th+t^2\xi+r(t))-\varphi_i^2(th+t^2\xi+r(t))$$
$$\leq K\left\|th+t^2\xi+r(t)\right\|^\beta$$

at $t>0$. As there exists $c>0$ such that $\left|\varphi_i^1(x)\right|\leq c\left\|x\right\|$ at $x\in X$, then from here it follows

$$f_i(x_0+th+t^2\xi+r(t))-f_i(x_0)\leq\varphi_i^1(th+t^2\xi+r(t))+\varphi_i^2(th+t^2\xi+r(t))-\varphi_i^2(th)$$
$$+\varphi_i^2(th)+K\left\|th+t^2\xi+r(t)\right\|^\beta\leq\varphi_i^1(th)+\varphi_i^1(t^2\xi)+\varphi_i^1(r(t))$$
$$+K\left\|t^2\xi+r(t)\right\|(\|th\|+\left\|t^2\xi+r(t)\right\|)+o(\|th\|^2)+\varphi_i^2(th)$$
$$+K\left\|th+t^2\xi+r(t)\right\|^\beta\leq t^2\varphi_i^1(\xi)+t^2\varphi_i^2(h)+o_3(t^2)$$

at $t>0$, where $\frac{o_3(t)}{t}\to 0$ at $t\downarrow 0$. Having put $f(x)=\max\limits_{0\leq i\leq m}f_i(x)$ we have that

$$f(x_0+th+t^2\xi+r(t))=\max_{0\leq i\leq m}f_i(x_0+th+t^2\xi+r(t))\leq t^2\max_{0\leq i\leq m}(\varphi_i^1(\xi)+\varphi_i^2(h))+o_3(t^2).$$

If $h\in S_{\tilde C}(x_0)=\bigcup_{\lambda>0}\frac{\tilde C-x_0}{\lambda}$, then there exists $\lambda_0>0$ such that $h\in\frac{\tilde C-x_0}{\lambda_0}$, i.e. $x_0+\lambda_0 h\in\tilde C$. Therefore $x_0+\lambda_0 h\in int\ C$. Then there exists $\alpha_0>0$, where $\lambda_0>\alpha_0$, such that $x_0+\lambda_0 h+t(\lambda_0\xi+{}+\lambda_0\frac{r(t)}{t^2})\in int\ C$ at $t\in[0,\alpha_0]$. Therefore

$$x_0+t\left(h+t\xi+\frac{r(t)}{t}\right)=\left(1-\frac{t}{\lambda_0}\right)x_0+\frac{t}{\lambda_0}\left(x_0+\lambda_0 h+\lambda_0 t\left(\xi+\frac{r(t)}{t^2}\right)\right)\in C,$$

i.e. we have that $x_0+th+t^2\xi+r(t)\in C$ at $t\in[0,\alpha_0]$.

If we assume that $E(h)<0$, we have $f(x_0+th+t^2\xi+r(t))<0$ at small $t>0$. As x_0 is the local minimum point in problem (4.2), we receive the contradiction. The theorem is proved.

Note that, if C is a convex set and $int\,T_C(x_0)\neq\emptyset$, then from the Theorem 1.1.4[3] it follows that $int\,C\neq\emptyset$.

Let's note that, if the condition of Theorem 4.2 is satisfied, $H\neq\{0\}$ and $E(h)$ is continuous, then

$$E(h)=\sup\Bigg\{\sum_{i=0}^m\lambda_i\varphi_i^2(h)+\langle y^*,S(h)\rangle:\lambda_i\geq 0,\ \sum_{i=0}^m\lambda_i=1,\ p_i\in\partial\varphi_i^1(0)\ ,$$
$$x^*\in N_C(x_0),\ y^*\in Y^*,\ \sum_{i=0}^m\lambda_i p_i+\Lambda^*y^*+x^*=0\Bigg\}\geq 0$$

at $h\in\{x\in X:\ \varphi_i^1(x)\leq 0,\ \ i\in I,\ \ F'(x_0)x=0,\ \ x\in T_C(x_0)\}$.

If $V:X\to Y$ satisfy $(1,2,1,\delta,o(2))$ locally Lipschitz condition with the constant K at the point x_0, then from definition of $(1,2,1,\delta,o(2))$ locally Lipschitz mappings, we have that V is continuous at the point x_0. If $V:X\to Y$ satisfy $(1,2,1,\delta)$ locally Lipschitz condition with the constant K at the point x_0, then from definition of $(1,2,1,\delta)$ locally Lipschitz mappings we have that V is continuous on the set $x_0+\delta B$.

Let's note that in Theorem 4.2 it is possible to replace the condition: the function f_i, $i \in I$, satisfies $\varphi_i^1 + \varphi_i^2 - (\beta, \delta)$ locally semi-Lipschitz condition with the constant K at the point x_0 and F satisfies $F'(x_0)x + S(x) - (\beta, \delta)$ locally Lipschitz condition with the constant K at the point x_0, where $\beta > 2$, with the condition: the function f_i,$i \in I$, satisfies $\varphi_i^1 + \varphi_i^2 - (o(2), \delta)$ locally semi-Lipschitz condition at the point x_0 and F satisfies $F'(x_0)x + S(x) - (o(2), \delta)$ locally Lipschitz condition at the point x_0.

If in Theorem 4.2 to replace the condition : functions $\varphi_i^2 : X \to R$, $i \in I$, and operator $S : X \to Y$ satisfy $(1, 2, 1, \delta, o(2))$ and $(1, 2, 1, \delta, \tilde{o}(2))$ locally Lipschitz condition with the constant K at the zero point respectively with the condition: the functions $\varphi_i^2 : X \to R$, $i \in I$, and operator $S : X \to Y$ satisfy $(1, \mu, 1, \delta)$, where $\mu > 1$, locally Lipschitz condition with the constant K at the zero point, Theorem 4.2 also remains true.

For example, the following theorem is true.

Theorem 4.3 *If X and Y are Banach spaces, x_0 is the local minimum point in problem (4.2), $f_i(x_0) = 0$ at $i \in I$, the functions f_i, $i \in I$, satisfy $\varphi_i^1 + \varphi_i^2 - (o(2), \delta)$ locally semi-Lipschitz condition at the point x_0, $\varphi_i^1 : X \to R$ are sublinear continuous functions at $i \in I$, $\varphi_i^2 : X \to R$ are positive homogeneous functions of degree 2 and satisfy $(1, \mu, 1, \delta)$, where $\mu > 1$, locally Lipschitz condition with the constant K at the zero point at $i \in I$, the mapping $F : X \to Y$ is strictly differentiable at the point x_0, $F'(x_0)X = Y$ and F satisfies $F'(x_0)x + S(x) - (\tilde{o}(2), \delta)$ locally Lipschitz condition at the point x_0, $S : X \to Y$ is a positive homogeneous operator of degree 2 and satisfies $(1, \mu, 1, \delta)$, where $\mu > 1$, locally Lipschitz condition with the constant K at the zero point, C is the convex set, $\partial g^*_{(a,y)}(0) \neq \emptyset$ at $(a, y) \in R^{m+1} \times Y$ and $int\, T_C(x_0) \bigcap Ker\, \Lambda \neq \emptyset$, then*

$$E(h) = \sup \left\{ \sum_{i=0}^{m} \lambda_i \varphi_i^2(h) + \langle y^*, S(h) \rangle : \lambda_i \geq 0,\ \sum_{i=0}^{m} \lambda_i = 1,\ p_i \in \partial \varphi_i^1(0)\ , \right.$$

$$\left. x^* \in N_C(x_0),\ y^* \in Y^*,\ \sum_{i=0}^{m} \lambda_i p_i + \Lambda^* y^* + x^* = 0 \right\} \geq 0$$

at $h \in H$.

Let's consider the problem

$$f_0(x) \to \min, f_j(x) \leq 0,\ j = 1, 2, \ldots, m,\ \ \Lambda x + y_0 = 0,\ \ x \in C, \tag{4.3}$$

where $f_i : X \to R$, $i \in I$, $\Lambda : X \to Y$, Y is Banach space, $y_0 \in Y$, $C \subset X$.

Corollary 4.1 *If X and Y are Banach spaces, x_0 is a local minimum point in problem (4.3), the functions f_i, $i \in I$, satisfy $\varphi_i^1 + \varphi_i^2 - (\beta, \delta)$ locally semi-Lipschitz condition with the constant K at the point x_0, where $\beta > 2$, $\varphi_i^1 : X \to R$ are sublinear continuous functions and $f_i(x_0) = 0$ at $i \in I$, $\varphi_i^2 : X \to R$ are positive homogeneous functions of degree 2 and satisfy $(1, 2, 1, \delta, o(2))$ locally Lipschitz condition with the constant K at the zero point at $i \in I$, $\Lambda : X \to Y$ is a linear continuous surjective operator, C is a convex set, $int\, T_C(x_0) \bigcap Ker\, \Lambda \neq \emptyset$ and $\partial g^*_{(a,y)}(0) \neq \emptyset$ at $(a, y) \in R^{m+1} \times Y$, then there exist $\alpha_0 \geq 0$, $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$, where $\sum_{i=0}^m \alpha_i = 1$, $x_i^* \in \partial \varphi_i^1(0)$, $i \in I$, $y^* \in Y^*$ and $x^* \in N_C(x_0)$ such that $\sum_{i=0}^m \alpha_i x_i^* + \Lambda^* y^* + x^* = 0$ and*

$$E(h) = \sup \left\{ \sum_{i=0}^{m} \lambda_i \varphi_i^2(h) :\ \lambda_i \geq 0,\ \sum_{i=0}^{m} \lambda_i = 1,\ x_i^* \in \partial \varphi_i^1(0),\ y^* \in Y^*\ , \right.$$

$$\left. x^* \in N_C(x_0),\ \sum_{i=0}^{m} \lambda_i x_i^* + \Lambda^* y^* + x^* = 0 \right\} \geq 0$$

at $h \in \{x \in S_{\tilde{C}}(x_0) : \ \varphi_i^1(x) \leq 0, \ i = 0, 1, \ldots m, \ \Lambda x = 0\}$.

Using Proposition 4.5.1 and 4.3.8 [3] and Hermander's theorems and with $\Lambda \equiv 0$, $y_0 = 0$, $C = X$ using Proposition 4.5.1, 4.3.8 and 6.2.7 [3] it is also possible to prove Corollary 4.1. It is clear that in problem (4.3) it is possible to replace the set C by the set $C \bigcap B(x_0, 2\delta)$.

5 On a necessary and sufficient conditions of the weak minimum

Let X be a Banach space (or vector space) and let $\{E_\alpha : \alpha \in A\}$ be the family of finite-dimensional subspaces of space X directed on increase and satisfying the condition $\bigcup_{\alpha \in A} E_\alpha = X$, where $E_\alpha \neq E_\beta$ at $\alpha \neq \beta$. The set A of indexes directed (reflexive, transitive, antisymmetric) by the relation $\leq$. Let's note that A is directed on increase $\alpha \leq \beta$, if $E_\alpha \subset E_\beta$. As any linear system has algebraic basis, existence of this family of finite-dimensional subspaces $E_\alpha, \alpha \in A$, in X follows from Zorn's lemma. It is clear, that $E_\alpha, \alpha \in A$, is a Banach space relative to the induced topology from the Banach space X (If X is a vector space, we consider that $E_\alpha, \alpha \in A$ is allocated with the only separable vector topology).

Let g_α designate the initial mapping E_α in X. It is known that (see [14]) inductive topology in X for the family $(E_\alpha, g_\alpha, \alpha \in A)$ is a local convex space. By $(X)_s$ let's designate space X, supplied with the induced topology. Similarly, from 2.6.4 and 2.6.5 [14] we have that $\{x_k\} \subset X$ converges to x with respect to topology in $(X)_s$ if and only if for some $\alpha \in A$ $\{x_k\}$ converges to x in E_α. Therefore topology in $(X)_s$ is stronger than topology in X. Then we have that $(X)_s$ is a Hausdorff space.

Let's put $T_s(x_0; C) = \{x \in X : \exists x_k \to x \ in \ (X)_s, \ \exists t_k \downarrow 0, \ that \ x_0 + t_k x_k \in C\}$ and designate $C_\alpha = C \bigcap (x_0 + E_\alpha)$. If x_0 is the local minimum of the function $f_0(x)$ on the set

$$P_\alpha = \{x \in C_\alpha : f_j(x) \leq 0, \ j \in J, \ F(x) = 0\}$$

at $\alpha \in A$, let's call x_0 weak local minimum of the function $f_0(x)$ on the set P.

It follows from definition of the weak local minimum and Theorem 4.2 that the following Theorem 5.1 is true.

Let's put $H_\alpha = \{x \in S_{\tilde{C_\alpha}}(x_0) : \varphi_i^1(x) \leq 0, \ i \in I, \ \Lambda x = 0\}$ at $\alpha \in A$, where $\tilde{C_\alpha} = int C_\alpha \bigcup \{x_0\}$, $S_{\tilde{C_\alpha}}(x_0) = \bigcup_{\lambda > 0} \frac{\tilde{C_\alpha} - x_0}{\lambda}$, $\varphi_i^1 : X \to R$ and $\Lambda x = F'(x_0)x$.

Theorem 5.1 *If X and Y are Banach spaces, x_0 is the weak local minimum point in problem (4.2), $f_i(x_0) = 0$ at $i \in I$, $\beta > 2$, the functions f_i, $i \in I$, satisfy $\varphi_i^1 + \varphi_i^2 - (\beta, \delta)$ locally semi-Lipschitz condition with the constant K at the point x_0, $\varphi_i^1 : X \to R$ are sublinear continuous functions at $i \in I$, $\varphi_i^2 : X \to R$ are continuous positive homogeneous functions of degree 2 at $i \in I$ and satisfy $(1, 2, 1, \delta, o(2))$ locally Lipschitz condition with the constant K at the zero point, the mapping $F : X \to Y$ is strictly differentiable at the point x_0, $F'(x_0)X = Y$ and F satisfies $F'(x_0)x + S(x) - (\beta, \delta)$ locally Lipschitz condition with the constant K at the point x_0, $S : X \to Y$ is a continuous positive homogeneous operator of degree 2 and satisfies $(1, 2, 1, \delta, \tilde{o}(2))$ locally Lipschitz condition with the constant K at the zero point, C is a convex set, in $g_\alpha(y, a) = \inf_{x \in T_{C_\alpha}(x_0), \ \Lambda x + y = 0} \ \max_{0 \leq i \leq m} (\varphi_i^1(x) + a_i)$ the infimum is reached and $int \, T_{C_\alpha}(x_0) \bigcap Ker \, \Lambda \neq \emptyset$, $\alpha \in A$, then*

$$E(h) = \sup \left\{ \sum_{i=0}^{m} \lambda_i \varphi_i^2(h) + \langle y^*, S(h) \rangle : \lambda_i \geq 0, \ \sum_{i=0}^{m} \lambda_i = 1, \ p_i \in \partial_{E_\alpha} \varphi_i^1(0) \ , \right.$$

$$x^* \in N_{C_\alpha}(x_0),\ y^* \in Y^*,\ \sum_{i=0}^{m} \lambda_i p_i + \Lambda^* y^* + x^* = 0 \Bigg\} \geq 0$$

at $h \in H_\alpha$, $\alpha \in A$.

Let's note that if $\max\limits_{0\leq i\leq m} \varphi_i^1(x) > 0$ at $x \in T_{C_\alpha}(x_0)$, $x \neq 0$, then in $g_\alpha(y,a) = \inf\limits_{x\in T_{C_\alpha}(x_0),\ \Lambda x+y=0} \times \max\limits_{0\leq i\leq m} (\varphi_i^1(x) + a_i)$ the infimum is reached at $\alpha \in A$.

If in Theorem 5.1 to replace the condition: functions $\varphi_i^2 : X \to R$, $i \in I$, and operator $S : X \to Y$ satisfy $(1,2,1,\delta,o(2))$ and $(1,2,1,\delta,\tilde{o}(2))$ locally Lipschitz condition with the constant K at the zero point respectively by the condition: functions $\varphi_i^2 : X \to R$, $i \in I$, and operator $S : X \to Y$ satisfy $(1,\mu,1,\delta)$, where $\mu > 1$, locally Lipschitz condition with the constant K at the zero point , Theorem 5.1 remains also true.

Let's put $H_\tau = \{x \in T_{C_\tau}(x_0) : \varphi_i^1(x) \leq 0,\ i = 0,1,\ldots,m,\ \Lambda x = 0\ \}$ at $\tau \in A$, where $\Lambda x = F'(x_0)x$, $\varphi_i^1 : X \to R$.

Theorem 5.2 *If X and Y are Banach spaces, $f_i(x_0) = 0$ at $i \in I$, $\beta > 2$, the functions f_i, $i \in I$, satisfy $\varphi_i^1 + \varphi_i^2 - (\beta,\delta)$ locally Lipschitz condition with the constant K at the point x_0, $\varphi_i^1 : X \to R$ the sublinear continuous functions at $i \in I$, $\varphi_i^2 : X \to R$ are continuous positive homogeneous functions of degree 2 at $i \in I$, the mapping $F : X \to Y$ is strictly differentiable at the point x_0, $F'(x_0)X = Y$ and F satisfies $F'(x_0)x + S(x) - (\beta,\delta)$ locally Lipschitz condition with the constant K at the point x_0, $S : X \to Y$ is a continuous positive homogeneous operator of degree 2, C is the convex set, $\max\limits_{0\leq i\leq m} \varphi_i^1(x) \geq 0$ at $x \in Ker\,\Lambda \bigcap T_{C_\tau}(x_0)$, $int\,T_{C_\tau}(x_0) \bigcap Ker\,\Lambda \neq \emptyset$ at $\tau \in A$ and*

$$E(h) = \sup\left\{ \sum_{i=0}^{m} \lambda_i \varphi_i^2(h) + \langle y^*, S(h)\rangle : \lambda_i \geq 0,\ \sum_{i=0}^{m} \lambda_i = 1,\ p_i \in \partial_{E_\tau} \varphi_i^1(0)\ , \right.$$

$$\left. x^* \in N_{C_\tau}(x_0),\ y^* \in Y^*,\ \sum_{i=0}^{m} \lambda_i p_i + \Lambda^* y^* + x^* = 0 \right\} > 0$$

at $h \in H_\tau$, then x_0 is the strict weak local minimum point in problem (4.2).

Proof. Let's assume opposite. Then there is the index $\tau \in A$ such that x_0 is not the strict local minimum point of the function f_0 on the set $P_\tau = \{x \in C_\tau : f_j(x) \leq 0,\ j \in J,\ F(x) = 0\}$. Therefore for any $\delta_k > 0$ there will be $h_k \neq 0$ such that $x_0 + h_k \in C_\tau$, $\|h_k\| \leq \delta_k$, $f_i(x_0 + h_k) \leq f_i(x_0) = 0$ at $i \in I$ and $F(x_0 + h_k) = F(x_0) = 0$. Let's put $t_k = \|h_k\|$, $x_k = \frac{h_k}{\|h_k\|}$. Without loss of generality it is possible to consider that $x_k \to h$, $h \neq 0$. If $t_k \downarrow 0$, we have that $h \in T(x_0; C_\tau)$ and $h \in T_S(x_0; P_\tau)$. As $x_0 + h_k \in C_\tau$, we have that $h_k \in T(x_0; C_\tau)$.

As functions f_i satisfy $\varphi_i^1 + \varphi_i^2 - (\beta,\delta)$ locally Lipschitz condition with the constant K at the point x_0, where $\beta > 2$, then

$$\left| f_i(x_0 + y) - f_i(x_0) - \varphi_i^1(y) - \varphi_i^2(y) \right| \leq K \|y\|^\beta$$

at $y \in \delta B$. Therefore there exists $\lambda > 0$ such that

$$\left| f_i(x_0 + t_k x_k) - f_i(x_0) - \varphi_i^1(t_k x_k) - \varphi_i^2(t_k x_k) \right| \leq K \|t_k x_k\|^\beta$$

at $t_k \in [0,\lambda]$. From here we receive that

$$\varphi_i^1(t_k x_k) \leq f_i(x_0 + t_k x_k) - \varphi_i^2(t_k x_k) + K \|t_k x_k\|^\beta$$

at $t_k \in [0,\lambda]$. As φ_i^2 continuous positively homogeneous function of degree 2 , then $t_k\varphi_i^1(x_k) \le K\,\|t_kx_k\|^\beta - t_k^2\varphi_i^2(x_k)$ at $t_k \in [0,\lambda]$. From here we receive that $\varphi_i^1(h) \le 0$ at $i \in I$.

By the condition F satisfies $F'(x_0)x + S(x) - (\beta,\delta)$ locally Lipschitz condition with the constant K at the point x_0, $S : X \to Y$ is a continuous positive homogeneous operator of degree 2. Therefore

$$\left\|F(x_0+t_kx_k) - F'(x_0)t_kx_k - S(t_kx_k) - F(x_0)\right\| \le K\,\|t_kx_k\|^\beta$$

at rather large k. From here we have that $\|F'(x_0)t_kx_k\| \le K\,\|t_kx_k\|^\beta + \|S(t_kx_k)\|$ at rather large k. Then we will receive that $\|F'(x_0)h\| \le 0$, i.e. $F'(x_0)h = 0$.

From the inequality $\|F(x_0+t_kx_k) - F'(x_0)t_kx_k - S(t_kx_k) - F(x_0)\| \le K\,\|t_kx_k\|^\beta$ we have that $\|F'(x_0)h_k + S(h_k)\| \le K\,\|h_k\|^\beta$. Having designated $F'(x_0)h_k + S(h_k) = q_k$, we have that $\|q_k\| \le K\,\|t_kx_k\|^\beta$.

By the condition we have that $\left|f_i(x_0+h_k) - \varphi_i^1(h_k) - \varphi_i^2(h_k) - f_i(x_0)\right| \le K\,\|h_k\|^\beta$ at $i \in I$.

Therefore $f_i(x_0+h_k) \ge \varphi_i^1(h_k) + \varphi_i^2(h_k) - K\,\|h_k\|^\beta$ at $i \in I$. Then we will receive that

$$\max_{0\le i\le m} f_i(x_0+h_k) \ge \max_{0\le i\le m}(\varphi_i^1(h_k) + \varphi_i^2(h_k) - K\,\|h_k\|^\beta).$$

Let's consider the problem

$$\max_{0\le i\le m}(\varphi_i^1(x) + \varphi_i^2(h_k) - K\,\|h_k\|^\beta) \to \inf,\ \ x \in T_{C_\tau}(x_0),\quad \Lambda x + S(h_k) - q_k = 0.$$

Let's put $a_i^k = \varphi_i^2(h_k) - K\,\|h_k\|^\beta$, $y_k = S(h_k) - q_k$, $a^k = (a_0^k, a_1^k, \dots, a_m^k)$ and

$$g_\tau(y_k, a^k) = \inf_{x\in T_{C_\tau}(x_0),\ \Lambda x + y_k = 0}\ \max_{0\le i\le m}(\varphi_i^1(x) + a_i^k).$$

From the proof of Theorem 4.2 we have that

$$g_\tau(y,a) = \inf_{x\in T_{C_\tau}(x_0),\ \Lambda x + y = 0}\ \max_{0\le i\le m}(\varphi_i^1(x) + a_i)$$

$$= \sup_{(y^*,\lambda)\in\partial_{E_\tau} g_\tau(0,0)}(\langle y^*, y\rangle + \sum_{i=0}^m \lambda_i a_i) = \sup\{\langle y^*, y\rangle + \sum_{i=0}^m \lambda_i a_i :\ \lambda_i \ge 0,\ \sum_{i=0}^m \lambda_i = 1,$$

$$p_i \in \partial_{E_\tau}\varphi_i^1(0),\ x^* \in N_{C_\tau}(x_0), y^* \in Y^*,\ \sum_{i=0}^m \lambda_i p_i + x^* + \Lambda^* y^* = 0\}.$$

Therefore $(y,a) \to g_\tau(y,a)$ is a lower semi-continuous function and

$$g_\tau(y_k, a^k) = \sup\{\langle y^*, S(h_k) - q_k\rangle + \sum_{i=0}^m \lambda_i(\varphi_i^2(h_k) - K\,\|h_k\|^\beta) :\ \lambda_i \ge 0, \sum_{i=0}^m \lambda_i = 1,$$

$$p_i \in \partial_{E_\tau}\varphi_i^1(0),\ x^* \in N_{C_\tau}(x_0),\ y^* \in Y^*, \sum_{i=0}^m \lambda_i p_i + x^* + \Lambda^* y^* = 0\}.$$

As

$$\max_{0\le i\le m} f_i(x_0+h_k) \ge \sup\{\langle y^*, S(h_k) - q_k\rangle + \sum_{i=0}^m \lambda_i(\varphi_i^2(h_k) - K\,\|h_k\|^\beta) :\ \lambda_i \ge 0, \sum_{i=0}^m \lambda_i = 1,$$

$$p_i \in \partial_{E_\tau}\varphi_i^1(0),\ x^* \in N_{C_\tau}(x_0),\ y^* \in Y^*, \sum_{i=0}^{m} \lambda_i p_i + x^* + \Lambda^* y^* = 0\},$$

then

$$0 \geq \sup\{\langle y^*, S(t_k x_k) - q_k\rangle + \sum_{i=0}^{m} \lambda_i(\varphi_i^2(t_k x_k) - K\,\|t_k x_k\|^\beta) : \lambda_i \geq 0, \sum_{i=0}^{m} \lambda_i = 1,$$

$$p_i \in \partial_{E_\tau}\varphi_i^1(0),\ x^* \in N_{C_\tau}(x_0),\ y^* \in Y^*, \sum_{i=0}^{m} \lambda_i p_i + x^* + \Lambda^* y^* = 0\},$$

or

$$0 \geq \sup\{\left\langle y^*, S(x_k) - \frac{q_k}{t_k^2}\right\rangle + \sum_{i=0}^{m} \lambda_i(\varphi_i^2(x_k) - K t_k^{\beta-2}\,\|x_k\|^\beta) : \lambda_i \geq 0, \sum_{i=0}^{m} \lambda_i = 1,$$

$$p_i \in \partial_{E_\tau}\varphi_i^1(0), x^* \in N_{C_\tau}(x_0),\ y^* \in Y^*, \sum_{i=0}^{m} \lambda_i p_i + x^* + \Lambda^* y^* = 0\}.$$

From here at $k \to \infty$ we have that

$$0 \geq \sup\{\langle y^*, S(h)\rangle + \sum_{i=0}^{m} \lambda_i \varphi_i^2(h) :\ \ \lambda_i \geq 0, \sum_{i=0}^{m} \lambda_i = 1,\ p_i \in \partial_{E_\tau}\varphi_i^1(0),$$

$$x^* \in N_{C_\tau}(x_0),\ y^* \in Y^*, \sum_{i=0}^{m} \lambda_i p_i + x^* + \Lambda^* y^* = 0\}.$$

As $h \in H_\tau$, then we receive the contradiction. The theorem is proved.

If in Theorem 5.2 to replace C_τ by C and E_τ by X, Theorem 5.2 remains also true.

Let's put $H_\tau = \{x \in T_{C_\tau}(x_0) : \ \langle p_i, x\rangle \leq 0,\ i \in I,\ \Lambda x = 0\}$ at $\tau \in A$, where $p_i \in X^*$, $\Lambda x = F'(x_0)x$.

Corollary 5.1 *If X and Y are Banach spaces , $f_i(x_0) = 0$ at $i \in I$, $\beta > 2$, the functions f_i, $i \in I$, satisfy $\langle p_i, x\rangle + \varphi_i^2 - (\beta, \delta)$ locally Lipschitz condition with the constant K at the point x_0, $\varphi_i^2 : X \to R$ are continuous positive homogeneous functions of degree 2 at $i \in I$, the mapping $F : X \to Y$ is strictly differentiable at the point x_0, $F'(x_0)X = Y$ and F satisfies $F'(x_0)x + S(x) - (\beta, \delta)$ locally Lipschitz condition with the constant K at the point x_0, $S : X \to Y$ is a continuous positive homogeneous operator of degree 2, C is the convex set, $\max\limits_{0 \leq i \leq m} \langle p_i, x\rangle \geq 0$ at $x \in Ker\,\Lambda \bigcap T_{C_\tau}(x_0)$ and $int\,T_{C_\tau}(x_0) \bigcap Ker\,\Lambda \neq \emptyset$ and*

$$E(h) = \sup\{\sum_{i=0}^{m} \lambda_i \varphi_i^2(h) + \langle y^*, S(h)\rangle : \lambda_i \geq 0,\ \sum_{i=0}^{m} \lambda_i = 1,\ x^* \in N_{C_\tau}(x_0),$$

$$y^* \in Y^*, \sum_{i=0}^{m} \lambda_i p_i + \Lambda^*\, y^* + x^* = 0\} > 0$$

at $h \in H_\tau$, then x_0 is the strict weak local minimum point in problem (4.2).

Let's note that in Theorem 5.2 it is possible to replace the condition: the functions f_i, $i \in I$, satisfy $\varphi_i^1 + \varphi_i^2 - (\beta, \delta)$ locally Lipschitz condition the constant K at the point x_0, the mapping $F : X \to Y$ satisfies $F'(x_0)x + S(x) - (\beta, \delta)$ locally Lipschitz condition with the constant K at the point x_0, where $\beta > 2$, by the condition: the function f_i, $i \in I$, satisfies $\varphi_i^1 + \varphi_i^2 - (o(2), \delta)$ locally Lipschitz condition at the point x_0 and F satisfies $F'(x_0)x + S(x) - (\tilde{o}(2), \delta)$ locally Lipschitz condition at the point x_0.

If $C \subset X$ is any set and there exists the hypertangent vector to the set C at the point $x_0 \in C$ and $T_C(x_0)$ is the Clark tangent cone of the set C at the point x_0, then Theorems 4.2, 4.3 and 5.1 remain also true.

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