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Nikolskii type inequalities in regions with piecewise Dini-smooth boundary with interior zero angles

Pelin Özkartepe

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Abstract. In this work, we investigate the Nikolskii type imequalities for arbitrary algebraic polynomials in the weighted Bergman space, where the boundary of region and the weight function have some singulariies. We obtain Nikolskii -type estimation for algebraic polynomials in the bounded regions with piecewise Dini-smooth boundary having interior zero angles.

Keywords. Algebraic polynomials, Conformal mapping, Dini-smooth curve.

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1 Introduction and Main Results

Let $\mathbb{C}$ be a complex plane, and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be the bounded Jordan region, with $0 \in G$ and the boundary $L := \partial G$ be a closed Jordan curve, $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$. $\Delta := \{w : |w| > 1\}$ (with respect to $\overline{\mathbb{C}}$). Let function $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$, and $\Psi := \Phi^{-1}$. For $R > 1$ let us set $L_R := \{z : \ |\Phi(z)| = R\}$, $G_R := intL_R$, $\Omega_R := extL_R$. For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set: $d(z, M) = dist(z, M) := \inf \{|z - \zeta| : \ \zeta \in M\}$.

Let $\{z_j\}_{j=1}^{m} \in L$ be a fixed system of distinct points. Consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$$h(z) := \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \quad z \in G_{R_0}, \ R_0 > 1, \tag{1.1}$$

where $\gamma_j > -2$, for every $j = 1, 2, ..., m$.

Denote by $\wp_n$ the class of all complex algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

For any $p > 0$ and for Jordan region G, lets define:

$$\begin{aligned} \|P_n\|_p &: = \|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) \left| P_n(z) \right|^p d\sigma_z \right)^{1/p} < \infty, 0 < p < \infty; \quad (1.2) \\ \|P_n\|_\infty &: = \|P_n\|_{A_\infty(1,G)} := \|P_n\|_{C(\overline{G})}, \ p = \infty, \end{aligned}$$

P. Özkartepe
Nizip Education Faculty, Gaziantep University, Gaziantep,Turkey
E-mail: pelinözkartepe@gmail.com

where σ_z is the two-dimensional Lebesgue measure. Clearly, $\|\cdot\|_p$ is the quasinorm (i.e. a norm for $1 \leq p \leq \infty$ and a $p-$norm for $0 < p < 1$).

In [24] we investigated following problem: find a estimate of the type

$$|P_n(z)| \leq c.\alpha_n(L, h, d(z, L), p) \|P_n\|_p \, |\Phi(z)|^{n+1}, \; z \in \Omega, \; p > 0, \tag{1.3}$$

where $c = c(L, p) > 0$ is a constant independent from n, z, P_n, and $\alpha_n(L, h, d(z, L), p) \to \infty$ (in general!) as $n \to \infty$, depending on the geometrical properties of curve L, weight function h and parameter p.

Analogous results of (1.3)-type for some norms, weight function $h(z)$ and for different unbounded regions were obtained by Lebedev, Tamrazov, Dzjadyk, Shevchuk (see, for example, [20]), Abdullayev and et all [11], [10], [14], [8], [13] and others.

The (1.3) gives estimate for $|P_n(z)|$ in the points of unbounded region Ω.Thus, in order to obtain an estimate in the whole complex plane, we need to find estimates for the finite closed region $\overline{G}$.

In this paper we study this problem and obtain a solution in the form of the following Nikol'skii-type inequality:

$$\|P_n\|_\infty \leq c\beta_n(G, h, p) \|P_n\|_p, \tag{1.4}$$

where $c = c(G, h, p) > 0$ is a constant independent of n and P_n and $\beta_n(G, h, p) \to \infty$, $n \to \infty$, depending on the geometrical properties of region G, weight function h and of p.

Note that, the estimate of (1.4)-type for some (G, p, h) was investigated in [22, pp.122-133], [19], [21, Sect.5.3], [23], [18], [2]-[9] (see, also, references therein) and others. In [9] this problem was investigated for $p > 1$ and for regions bounded by piecewise Dini-smooth boundary without cusps. For some regions and the weight function $h(z)$ analogous of (1.4) results were obtained: in [10] ($h(z) \equiv 1$) and [14] ($h(z) \neq 1$) for $p > 0$ and for regions bounded by quasiconformal curve; in [8] for $p > 1$ and for regions bounded by piecewise smooth curve without cusps; in [13] for $p > 0$ and for regions bounded by asymptotically conformal curve.

Now, we investigate this problem for $p > 0$ and for finite region bounded by piecewise Dini-smooth boundary having interior zero angles and for weight function $h(z)$, defined in (1.1).

Let us give some definitions and notations that will be used later in the text. In what follows, we always assume that $p > 0$ and the constants $c, c_0, c_1, c_2, ...$ are positive and constants $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive (generally, are different in different relations), which depends on G in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. Also note that, for any $k \geq 0$ and $m > k$, notation $j = \overline{k, m}$ denotes $j = k, k+1, ..., m$.

Let S be a rectifiable Jordan curve or arc and $z = z(s), \; s \in [0, \; |S|], \; |S| := mes \; S$, denote the natural representation of S.

Definition 1.1 *[25, p.48](see also [17, p.32])We say that a Jordan curve or arc S called Dini-smooth, if it has a parametrization $z = z(s), \; 0 \leq s \leq |S|$, such that $z'(s) \neq 0, \; 0 \leq s \leq |S|$ and $\left|z'(s_2) - z'(s_1)\right| < g(s_2 - s_1)$, $s_1 < s_2$, where g is an increasing function for which*

$$\int_0^1 \frac{g(x)}{x} dx < \infty.$$

Definition 1.2 *[9] We say that a Jordan region* $G \in PDS(\lambda_1, ..., \lambda_m)$, $0 < \lambda_j \leq 2$, $j = \overline{1, m}$, *if* $L = \partial G$ *consists of the union of finite Dini-smooth arcs* $\{L_j\}_{j=1}^{m}$, *such that* L *is locally Dini-smooth at* $z_0 \in L \setminus \{z_j\}_{j=1}^{m}$ *and have exterior (with respect to* $\overline{G}$*) angles* $\lambda_j \pi$, $0 < \lambda_j \leq 2$, *at the corner points* $\{z_j\}_{j=1}^{m} \in L$, *where two arcs meet.*

Without loss of generality, we assume that these points on the curve $L = \partial G$ are located in the positive direction such that, G has exterior $\lambda_j \pi$, $0 < \lambda_j < 2$, $j = \overline{0, m_1}$, angle at the points $\{z_j\}_{j=1}^{m_1}$, $m_1 \leq m$, and interior zero angle (i.e. $\lambda_j = 2-$interior cusps) at the points $\{z_j\}_{j=m_1+1}^{m}$.

It is clear from Definition 1.2, the each region $G \in PDS(\lambda_1, ..., \lambda_m)$, $0 < \lambda_j \leq 2$, $j = \overline{1, m}$, may have exterior nonzero $\lambda_j \pi$, $0 < \lambda_j < 2$, angles at the points $\{z_j\}_{j=1}^{m_1} \in L$, and interior zero angles ($\lambda_j = 2$) at the points $\{z_j\}_{j=m_1+1}^{m} \in L$. If $m_1 = m = 0$, then the region G doesn't have such angles, and in this case we will write: $G \in DS$; if $m_1 = m \geq 1$, then G has only $\lambda_i \pi$, $0 < \lambda_i < 2$, $i = \overline{1, m_1}$, exterior nonzero angles, and in this case we will write: $G \in PDS(\lambda_i)$; if $m_1 = 0$ and $m \geq 1$, then G has only interior zero angles, and in this case we will write: $G \in PDS(2)$.

Throughout this work, we will assume that the points $\{z_j\}_{j=1}^{m} \in L$ defined in (1.1) and Definition 1.2 are identical and $w_j := \Phi(z_j)$.

For simplicity of exposition, without loss of generality, we will take $m_1 = 1$, $m = 2$. Then, after this assumption, in the future we will have region $G \in PDS(\lambda_1, 2)$, $0 < \lambda_1 < 2$, such that at the point $z_1 \in L$ region $\overline{G}$ have exterior nonzero $\lambda_1 \pi$, $0 < \lambda_1 < 2$, and at the point $z_2 \in L$ - interior zero angle 2π, i.e. $\lambda_2 = 2$.

Now we can state our new results.

Theorem 1.1 *Let* $G \in PDS(\lambda_1, 2)$, *for some* $0 < \lambda_1 < 2$; $h(z)$ *be defined as in (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and* $\gamma_j > -2$, $j = 1, 2$, *we have:*

$$\|P_n\|_\infty \leq c_1 A_{n,1} \|P_n\|_p, \tag{1.5}$$

where $c_1 = c_1(G, \gamma_1, \gamma_2, \lambda_1, p) > 0$ *is the constant, independent from* z *and* n,

$$A_{n;1} := \begin{cases} n^{\frac{2(2+\widetilde{\gamma})}{p}}, & \text{if } \widetilde{\gamma} > -\frac{3}{2}, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} = -\frac{3}{2}, \\ n^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} < -\frac{3}{2}. \end{cases} \tag{1.6}$$

$\widetilde{\gamma} := \max\{0, \gamma_1, \gamma_2\}$.

We can take individual cases when the curve L in the both points have the same type of angle: exterior nonzero angle. In this case, from Theorem 1.1, we obtain the following:

Corollary 1.1 *Let* $G \in PDS(\lambda_1, \lambda_2)$, *for some* $0 < \lambda_1, \lambda_2 < 2$; $h(z)$ *be defined as in (1.1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and* $\gamma_j > -2$, $j = 1, 2$, *we have:*

$$|P_n(z)| \leq c_2 A_{n;2} \|P_n\|_p, \ z \in \Omega_{1+1/n}, \tag{1.7}$$

where $c_2 = c_2(G, \gamma_1, \gamma_2, p) > 0$ *is the constant, independent from* z *and* n,

$$A_{n,2} := \begin{cases} n^{\frac{(2+\widetilde{\gamma}) \cdot \widetilde{\lambda}}{p}}, & \text{if } (2 + \widetilde{\gamma}) \cdot \widetilde{\lambda} > 1, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } (2 + \widetilde{\gamma}) \cdot \widetilde{\lambda} = 1, \\ n^{\frac{1}{p}}, & \text{if } (2 + \widetilde{\gamma}) \cdot \widetilde{\lambda} < 1, \end{cases} \tag{1.8}$$

and $\widetilde{\lambda} := \max\{1; \lambda_1, \lambda_2\}$.

Corollary 1.2 *Let $G \in PDS(2,2)$; $h(z)$ be defined as in (1.1). Then, for any $P_n \in \wp_n,\ n \in \mathbb{N}$, and $\gamma_j > -2$, $j = 1, 2$, we have:*

$$|P_n(z)| \leq c_3 A_{n;3} \|P_n\|_p \ , \tag{1.9}$$

where $c_3 = c_3(G, \gamma_1, \gamma_2, p) > 0$ is the constant, independent from z and n, and

$$A_{n,3} := \begin{cases} n^{\frac{2(2+\widetilde{\gamma})}{p}}, & \text{if } \widetilde{\gamma} > -\frac{3}{2}, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} = -\frac{3}{2}, \\ n^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} < -\frac{3}{2}. \end{cases} \tag{1.10}$$

Theorem 1.2 *Let $p > 0$; $G \in PDS(\lambda_1, 2)$, for some $0 < \lambda_1 < 2$; $h(z)$ be defined as in (1.1). Then, for any $P_n \in \wp_n,\ n \in \mathbb{N},\ \gamma_j > -2$, $j = 1, 2$, we have:*

$$|P_n(z_j)| \leq c_4 A_{n,4} \|P_n\|_p \, , \tag{1.11}$$

where $c_4 = c_4(G, \gamma_1, \gamma_2, \lambda_1, p) > 0$ is the constant, independent of z and n;

$$A_{n,4} := \begin{cases} n^{\frac{(2+\gamma_1)\cdot \widetilde{\lambda}_1}{p}}, & \text{if } \gamma_1 > \frac{1}{\widetilde{\lambda}_1} - 2, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } \gamma_1 = \frac{1}{\widetilde{\lambda}_1} - 2, \\ n^{\frac{1}{p}}, & \text{if } \gamma_1 < \frac{1}{\widetilde{\lambda}_1} - 2, \end{cases}$$

for $j = 1$, $\widetilde{\lambda}_1 := \max\{1; \lambda_1\}$, and

$$A_{n,4} := \begin{cases} n^{\frac{2(2+\gamma_2)}{p}}, & \text{if } \gamma_2 > -\frac{3}{2}, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } \gamma_2 = -\frac{3}{2}, \\ n^{\frac{1}{p}}, & \text{if } \gamma_2 < -\frac{3}{2}, \end{cases}$$

for $j = 2$.

Combining Corollary 1.1 with the estimate for $|P_n(z)|$, $z \in \Omega$, in [24, Theorem 1.1], we can obtain estimation for $|P_n(z)|$ in the whole complex plane.

For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set: $d(z, M) = dist(z, M) := \inf\{|z - \zeta| : \ \zeta \in M\}$, and let $R := 1 + \frac{\varepsilon_0}{n}$.

Corollary 1.3 *Under the conditions of Theorem 1.1, the following is true:*

$$|P_n(z)| \leq c_5 \|P_n\|_p \begin{cases} A_{n,1}, & z \in \overline{G}_R, \\ \frac{|\Phi(z)|^{n+1}}{d^{\frac{2}{p}}(z, L_R)} B_{n,1}, & z \in \Omega_R, \end{cases}$$

where $c_5 = c_5(G, \gamma_1, \gamma_2, p) > 0$ is the constant, independent of z and n; $B_{n,1}$ is defined as in (1.6) and

$$B_{n;1} := \begin{cases} n^{\frac{2\widetilde{\gamma}}{p}}, & \text{if } \widetilde{\gamma} > \frac{1}{2}, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} = \frac{1}{2}, \\ n^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} < \frac{1}{2}, \end{cases} ;$$

Analogously rezults for any points $z \in \mathbb{C}$ we can write combining correspondingly results for the case $G \in PDS(\lambda_1, \lambda_2)$ and $G \in PDS(2,2)$ given above in Corollaries 1.1, 1.2 and 1.1, 1.2 from [24], respectively.

The sharpness of the estimations (1.5), (1.7) and (1.9) can be discussed by comparing them with the following result.

Remark 1.1 [10, Theorem 1.15], [2] a) For any $n \in \mathbb{N}$ there exist polynomials $Q_n^*, T_n^* \in \wp_n$ such that for unit disk B and weight function $h^*(z) = |z - z_1|^2$ the following is true:

$$|Q_n^*(z)| \geq c_6 n \left\| Q_n^* \right\|_{A_2(B)}, \ \text{ for all } \ z \in \overline{B};$$
$$|T_n^*(z_1)| \geq c_7 n^2 \left\| T_n^* \right\|_{A_2(h^*,B)};$$

b) For any $n \in \mathbb{N}$ there exists a polynomial $S_n^* \in \wp_n$, region $G^* \subset \mathbb{C}$, compact $M^* \Subset \Omega \backslash \overline{G}^*$ and constant $c_8 = c_8(G^*, M^*) > 0$ such that

$$|S_n^*(z)| \geq c_8 \frac{\sqrt{n}}{d(z, L_{1+1/n})} \left\| S_n^* \right\|_{A_2(G^*)} |\Phi(z)|^{n+1}, \qquad \text{for all } z \in M^*.$$

2 Some auxiliary results

Throughout this work, for the nonnegative functions $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b), respectively.

Lemma 2.1 *[1] Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{R_0})\}$; $w_j = \Phi(z_j),\ j = 1,2,3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So statements $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$ also equivalent;
b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^2} \preceq \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \preceq \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^{-2}},$$

where $R_0 := R_0(G)$ is a constant, depending on G.

Corollary 2.1 *Under the assumptions of Lemma 2.1, for $z_3 \in L_{R_0}$*

$$|w_1 - w_2|^{K^2} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{K^{-2}}$$

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4} \min \{ |z_i - z_j| : i, j = 1, 2, ..., m,\ i \neq j \}$, we put $\Omega(z_j,\ \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min\limits_{1 \leq j \leq m} \delta_j$, $\Omega(\delta) := \bigcup\limits_{j=1}^{m} \Omega(z_j,\ \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j,\ \delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{m} \Phi(\Omega(z_j,\ \delta))$, $\widehat{\Delta}(\delta) := \Delta \backslash \Delta(\delta)$.

The following lemma is a consequence of the results given in [25, pp.41-58], [17, pp.32-36], and estimation for the $\left| \Psi' \right|$ (see, for example, [16, Th.2.8]):

$$\left| \Psi'(\tau) \right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{2.1}$$

Lemma 2.2 *Let a Jordan region* $G \in PDS(\lambda_j;0),\ 0<\lambda_j\leq 2,\ j=\overline{1,m_1}$. *Then,*
i) for any $w\in\Delta_j$, $|\Psi(w)-\Psi(w_j)|\asymp|w-w_j|^{\lambda_j}$, $|\Psi'(w)|\asymp|w-w_j|^{\lambda_j-1}$;
ii) for any $w\in\overline{\Delta}\setminus\Delta_j$, $|\Psi(w)-\Psi(w_j)|\asymp|w-w_j|$, $|\Psi'(w)|\asymp 1$.

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h\left(z\right)$ defined as in (1.1).

Lemma 2.3 *[4] Let* L *be a* $K-$*quasiconformal curve*; $h(z)$ *is defined in (1.1). Then, for arbitrary* $P_n(z)\in\wp_n$, *any* $R>1$ *and* $n=1,2,...,$ *we have*

$$\|P_n\|_{A_p(h,G_R)}\preceq\widetilde{R}^{n+\frac{1}{p}}\|P_n\|_{A_p(h,G)}\ ,\ p>0, \tag{2.2}$$

where $\widetilde{R}=1+c(R-1)$ *and* c *is independent from* n *and* R.

Lemma 2.4 *[15] Let* $G\in PDS(\lambda_1,...,\lambda_m),\ 0<\lambda_j\leq 2,\ j=\overline{1,m}$. *Then, for arbitrary* $P_n(z)\in\wp_n$, *we have:*

$$\|P_n\|_{A_p(h,G_{1+c/n})}\preceq\|P_n\|_{A_p(h,G)}\ , \tag{2.3}$$

21 Proof of Theorem 1.1

Proof. We will use the standard scheme of proofs, wich used to proof analogously theorems from the works, for example, [15], [24], supplementing it with the corresponding estimates for the case of the $PDS(\lambda_1;2)$ Suppose that $G\in PDS(\lambda_1;2)$, for some $0<\lambda_1<2$ and $h(z)$ be defined as in (1.1). Let $\{\xi_j\}$, $1\leq j\leq m\leq n$, be the zeros (if any exist) of $P_n(z)$ lying on Ω and let

$$\widetilde{B}_j(z):=\frac{\Phi(z)-\Phi(\xi_j)}{1-\overline{\Phi(\xi_j)}\Phi(z)}\ ,\ z\in\Omega, \tag{2.4}$$

and

$$B_m(z):=\prod_{j=1}^{m}\widetilde{B}_j(z),\ z\in\Omega. \tag{2.5}$$

be the function Blaschke with respect to the points $\{\xi_j\}_{j=1}^{m}$. Well known, that

$$B_m(\xi_j)=0,\ |B_m(z)|\equiv 1,\ z\in L;\ |B_m(z)|<1,\ z\in\Omega. \tag{2.6}$$

Since $|B_m(z)|\equiv 1$, $z\in L$ and $|B_m(z)|<1$, $z\in\Omega$, then there exists circle $\left\{w:|w|=R_1:=1+\varepsilon_2, 0<\varepsilon_2<\frac{\varepsilon_1}{n},\ 0<\varepsilon_1<1,\right\}$ such that for any $j=1,2,$ the following is true:

$$\left|\widetilde{B}_j(\zeta)\right|>1-\varepsilon_2,\ \zeta\in L_{R_1}.$$

Then, we get:

$$|B_m(\zeta)|>(1-\varepsilon_2)^m\succeq 1,\ \zeta\in L_{R_1}. \tag{2.7}$$

For any $p>0$, let us set:

$$g_{n,p}\left(z\right):=\left[\frac{P_n\left(z\right)}{B_m(z)\Phi^{n+1}(z)}\right]^{p/2}\ ,\ z\in\Omega \tag{2.8}$$

Clearly, the function $g_{n,p}\left(z\right)$ is analytic in Ω, continuous on $\overline{\Omega}$, $g_{n,p}\left(\infty\right)=0$ and does not have zeros in Ω. We take an arbitrary continuous branch of the $g_{n,p}\left(z\right)$ and for this branch, we maintain the same designation. According to Cauchy integral representation for the unbounded region Ω, we have:

$$g_{n,p}(z) = -\frac{1}{2\pi i}\int\limits_{L_{R_1}} g_{n,p}(\zeta)\,\frac{d\zeta}{\zeta - z}\,,\ z\in\Omega_{R_1}. \tag{2.9}$$

According to (2.4) - (2.8), we get:

$$\begin{aligned} |P_n(z)|^{p/2} &= \frac{\left|B_m(z)\Phi^{n+1}(z)\right|^{\frac{p}{2}}}{2\pi d(z, L_{R_1})}\int\limits_{L_{R_1}}\left|\frac{P_n(\zeta)}{B_m(\zeta)\Phi^{n+1}(\zeta)}\right|^{p/2}|d\zeta| \qquad (2.10)\\ &\preceq \left|\Phi^{n+1}(z)\right|^{\frac{p}{2}}\int\limits_{L_{R_1}}|P_n(\zeta)|^{p/2}\,\frac{|d\zeta|}{|\zeta - z|}. \end{aligned}$$

Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w=\Phi(z)$ and applying the Hölder inequality, we obtain:

$$\left(\int\limits_{L_{R_1}}|P_n(\zeta)|^{\frac{p}{2}}\,\frac{|d\zeta|}{|\zeta - z|}\right)^2 \le \int\limits_{|t|=R_1} h(\Psi(t))\,|P_n(\Psi(t))|^p\left|\Psi'(t)\right|^2|dt|\cdot\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))\,|\Psi(t)-\Psi(w)|^2} \tag{2.11}$$

$$\begin{aligned} &\le \int\limits_{|t|=R_1} h(\Psi(t))\,|P_n(\Psi(t))|^p\left|\Psi'(t)\right|^2|dt|\cdot\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))\,|\Psi(t)-\Psi(w)|^2}\\ &= \int\limits_{|t|=R_1}|f_{n,p}(t)|^p\,|dt|\cdot\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))\,|\Psi(t)-\Psi(w)|^2} =: A_n\cdot F_n(w), \end{aligned}$$

where $f_{n,p}(t) := h^{\frac{1}{p}}(\Psi(t))P_n(\Psi(t))(\Psi^{'}(t))^{\frac{2}{p}}$, $|t| = R_1$.

Now, we will estimate each of the factors on the right-hand side separately. We begin from A_n. For this, by separating the circle $|t| = R_1$ to n equal parts δ_n with $mes\delta_n = \frac{2\pi R_1}{n}$ and applying the mean value theorem, we have:

$$\begin{aligned} A_n &:= \int\limits_{|t|=R_1}|f_{n,p}(t)|^p\,|dt|\\ &= \sum_{k=1}^{n}\int\limits_{\delta_k}|f_{n,p}(t)|^p\,|dt| = \sum_{k=1}^{n}\left|f_{n,p}\left(t_k^{'}\right)\right|^p mes\delta_k,\ \ t_k^{'}\in\delta_k. \end{aligned}$$

By applying mean value estimation

$$\left|f_{n,p}\left(t_k^{'}\right)\right|^p \le \frac{1}{\pi\left(\left|t_k^{'}\right|-1\right)^2}\iint\limits_{\left|\xi - t_k^{'}\right|<\left|t_k^{'}\right|-1}|f_{n,p}(\xi)|^p\,d\sigma_\xi,$$

we get:

$$(A_n)^2 \preceq \sum_{k=1}^{n}\frac{mes\ \delta_k}{\pi\left(\left|t_k^{'}\right|-1\right)^2}\iint\limits_{\left|\xi - t_k^{'}\right|<\left|t_k^{'}\right|-1}|f_{n,p}(\xi)|^p\,d\sigma_\xi,\ t_k^{'}\in\delta_k.$$

Taking into account that at most two of the discs with center t'_k are intersecting, we have:

$$A_n \preceq \frac{mes\delta_1}{\left(\left|t'_1\right|-1\right)^2} \iint\limits_{1<|\xi|<R} \left|f_{n,p}\left(\xi\right)\right|^p d\sigma_\xi \preceq n \cdot \iint\limits_{1<|\xi|<R} \left|f_{n,p}\left(\xi\right)\right|^p d\sigma_\xi .$$

Therefore, for factor A_n, according to Lemma 2.4, we get:

$$A_n \preceq n \iint\limits_{G_R\backslash G} h(\zeta) \left|P_n(\zeta)\right|^p d\sigma_\zeta \preceq n \cdot \|P_n\|_p^p . \tag{2.12}$$

To estimate the integral $F_n(w)$, denote by $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$, for any fixed $\rho > 1$,we introduce:

$$\Delta_1(\rho) : = \left\{ t = re^{i\theta} : r > \rho, \ \frac{\varphi_0+\varphi_1}{2} \ \le \theta < \frac{\varphi_1+\varphi_2}{2} \right\}, \tag{2.13}$$
$$\Delta_2(\rho) : = \left\{ t = re^{i\theta} : r > \rho, \ \frac{\varphi_1+\varphi_2}{2} \ \le \theta < \frac{\varphi_1+\varphi_0}{2} \right\};$$

$$\Delta_j : = \Delta_j(1), \ \Omega^j := \Psi(\Delta_j), \ \Omega^j_\rho := \Psi(\Delta_j(\rho));$$
$$L^j : = L \cap \overline{\Omega}^j, \ L^j_\rho := L_\rho \cap \overline{\Omega}^j_\rho, \ j = 1,2; \ L = L^1 \cup L^1, \ L_\rho = L^1_\rho \cup L^2_\rho.$$

Under these notations, from (2.11) for the $F_n(w)$, we get:

$$F_n(w) = \int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) \left|\Psi(t)-\Psi(w)\right|^2} \tag{2.14}$$

$$\preceq \sum_{j=1}^{2} \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{\prod\limits_{j=1}^{2} \left|\Psi(t)-\Psi(w_j)\right|^{\gamma_j} \left|\Psi(t)-\Psi(w)\right|^2}$$
$$\asymp \sum_{j=1}^{2} \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{\left|\Psi(t)-\Psi(w_j)\right|^{\gamma_j} \left|\Psi(t)-\Psi(w)\right|^2} =: \sum_{j=1}^{2} F_{n,j}(w),$$

since the points $\{z_j\}_{j=1}^m \in L$ are distinct. So, we need to evaluate the $F_{n,j}(w)$. For this, we take $z \in L_R$ and introduce the notations:

$$\Phi(L_{R_1}) = \Phi(\bigcup_{j=1}^{2} L^j_{R_1}) = \bigcup_{j=1}^{2} \Phi(L^j_{R_1}) = \bigcup_{j=1}^{2} \bigcup_{i=1}^{3} E^j_i(R_1), \tag{2.15}$$

where

$$E^j_1(R_1) := \ \left\{ t \in \Phi(L^j_{R_1}) : \quad |t-w_j| < \frac{c_1}{n} \right\},$$
$$E^j_2(R_1) := \ \left\{ t \in \Phi(L^j_{R_1}) : \quad \frac{c_1}{n} \le |t-w_j| < c_2 \right\}$$
$$E^j_3(R_1) := \ \left\{ t \in \Phi(L^j_{R_1}) : \quad c_2 \le |t-w_j| < c_3 < diam\, \overline{G} \right\}, \ j = 1,2.$$

Analogously,

$$\Phi(L_R) = \Phi(\bigcup_{j=1}^{2} L_R^j) = \bigcup_{j=1}^{2} \Phi(L_R^j) = \bigcup_{j=1}^{2}\bigcup_{i=1}^{3} E_i^j(R),$$

where

$$\begin{aligned} E_1^j(R) &:= \left\{\tau \in \Phi(L_R^j): \quad |\tau - w_j| < \frac{2c_1}{n}\right\}, \\ E_2^j(R) &:= \left\{\tau \in \Phi(L_R^j): \quad \frac{2c_1}{n} \le |\tau - w_j| < c_2\right\}, \\ E_3^j(R) &:= \left\{\tau \in \Phi(L_R^j): \quad c_2 \le |\tau - w_j| < c_3 < diam\, \overline{G}\right\},\ j = 1, 2. \end{aligned}$$

Then, after these definitions, taking arbitrary fixed $w = \Phi(z) \in \Phi(L_R)$, the quantity $F_{n,j}(w)$ can be written as follows:

$$F_{n,j}(w) = \sum_{i=1}^{3} \int\limits_{E_i^j(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j} |\Psi(t) - \Psi(w)|^2} =: \sum_{i=1}^{3} F_{n,j}^i(w) \tag{2.16}$$

The quantity $F_{n,j}^i(w)$ we will estimate for each $i = 1, 2, 3$ and $j = 1, 2$ separately, depending of location of the $w \in \Phi(L_R)$.

Case 1. Let $w \in \Phi(L_R^1)$.

According to the above notations, we will make evaluations for case $w \in E_i^1(R)$ for each $i = 1, 2, 3$.

1.1) Let $w \in E_1^1(R)$. In this case, we will estimate the quantity

$$F_{n,1}(w) = \sum_{i=1}^{3} \int\limits_{E_i^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{\gamma_1} |\Psi(t) - \Psi(w)|^2} =: \sum_{i=1}^{3} F_{n,1}^i(w) \tag{2.17}$$

for $\gamma_1 \ge 0$ and $\gamma_1 < 0$ separately.

For each $i = 1, 2, 3$ and $j = 1, 2$ we put: $E_{i,1}^j(R_1) := \left\{t \in \Phi(L_{R_1}^j) : |t - w_j| \ge |t - w|\right\}$, $E_{i,2}^j(R_1) := E_i^j(R_1) \setminus E_{i,1}^j(R_1)$.

1.1.1) If $\gamma_1 \ge 0$, then

$$\begin{aligned} F_{n,1}^1(w) &= \int\limits_{E_1^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{\gamma_1} |\Psi(t) - \Psi(w)|^2} \\ &= \int\limits_{E_{1,1}^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w)|^{2+\gamma_1}} + \int\limits_{E_{1,2}^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} \\ &=: F_{n,1}^{1,1}(w) + F_{n,1}^{1,2}(w), \end{aligned} \tag{2.18}$$

and, so Lemma 2.2 yields:

$$F_{n,1}^{1,1}(w) \preceq \int\limits_{E_{1,1}^1(R_1)} \frac{|dt|}{|t - w|^{(2+\gamma_1)\lambda_1}} \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1 - 1}, & \text{if } (2+\gamma_1)\lambda_1 > 1, \\ \ln n, & \text{if } (2+\gamma_1)\lambda_1 > 1, \\ 1, & \text{if } (2+\gamma_1)\lambda_1 < 1, \end{cases} \tag{2.19}$$

and

$$F_{n,1}^{1,2}(w) \preceq \int\limits_{E_{1,2}^{1}(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)\lambda_1}} \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1}, & \text{if } (2+\gamma_1)\lambda_1>1, \\ \ln n, & \text{if } (2+\gamma_1)\lambda_1=1, \\ 1, & \text{if } (2+\gamma_1)\lambda_1<1. \end{cases} \tag{2.20}$$

If $\gamma_1 < 0$, then

$$F_{n,1}^{1}(w) = \int\limits_{E_1^1(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{(-\gamma_1)}|dt|}{|\Psi(t)-\Psi(w)|^2} \tag{2.21}$$

$$\preceq \int\limits_{E_1^1(R_1)} \frac{|t-w_1|^{(-\gamma_1)\lambda_1}|dt|}{|t-w|^{2\lambda_1}} \preceq \left(\frac{1}{n}\right)^{(-\gamma_1)\lambda_1} \int\limits_{E_1^1(R_1)} \frac{|dt|}{|t-w|^{2\lambda_1}}$$

$$\preceq \left(\frac{1}{n}\right)^{(-\gamma_1)\lambda_1} \begin{cases} n^{2\lambda_1-1}, & \text{if } 2\lambda_1>1, \\ \ln n, & \text{if } 2\lambda_1>1, \\ 1, & \text{if } 2\lambda_1<1 \end{cases}$$

$$\preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1}, & \text{if } \lambda_1>\frac{1}{2}, \\ 1, & \text{if } \lambda_1\leq\frac{1}{2}. \end{cases}$$

1.1.2) If $\gamma_1 \geq 0$, then

$$F_{n,1}^{2}(w) = \int\limits_{E_2^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}|\Psi(t)-\Psi(w)|^2} \tag{2.22}$$

$$= \int\limits_{E_{2,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{E_{2,2}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}}$$

$$=: F_{n,1}^{2,1}(w) + F_{n,1}^{2,2}(w).$$

and, so from Lemma 2.2, we get:

$$F_{n,1}^{2,1}(w) \preceq \int\limits_{E_{2,1}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)\lambda_1}} \preceq n^{(2+\gamma_1)\lambda_1} mes E_{2,1}^1(R_1) \preceq n^{(2+\gamma_1)\lambda_1-1}, \tag{2.23}$$

and

$$F_{n,1}^{2,2}(w) \preceq \int\limits_{E_{2,2}^{1}(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)\lambda_1}} \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1}, & \text{if } (2+\gamma_1)\lambda_1>1, \\ \ln n, & \text{if } (2+\gamma_1)\lambda_1=1, \\ 1, & \text{if } (2+\gamma_1)\lambda_1<1. \end{cases} \tag{2.24}$$

Therefore, from (2.22)-(2.24) for $\gamma_1 \geq 0$, we have:

$$F_{n,1}^{2}(w) \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1}, & \text{if } (2+\gamma_1)\lambda_1>1, \\ \ln n, & \text{if } (2+\gamma_1)\lambda_1=1, \\ 1, & \text{if } (2+\gamma_1)\lambda_1<1. \end{cases} \tag{2.25}$$

According to well known inequality

$$(a+b)^{\epsilon} \leq c(\epsilon)(a^{\epsilon}+b^{\epsilon}), a,b>0,\ \epsilon>0, \tag{2.26}$$

and using estimations

$$|t-w_1| \le |t-w| + |w-w_1| \preceq |t-w| + \frac{1}{n}$$

and consequently,

$$|t-w_1|^{(-\gamma_1)\lambda_1} \preceq |t-w|^{(-\gamma_1)\lambda_1} + \left(\frac{1}{n}\right)^{(-\gamma_1)\lambda_1},$$

for $\gamma_1 < 0$, from (2.17), we have:

$$F_{n,1}^2(w) = \int\limits_{E_2^1(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{(-\gamma_1)}|dt|}{|\Psi(t)-\Psi(w)|^2} \tag{2.27}$$

$$\preceq \int\limits_{E_2^1(R_1)} \frac{|t-w_1|^{(-\gamma_1)\lambda_1}|dt|}{|t-w|^{2\lambda_1}} \preceq n^{\gamma_1\lambda_1} \int\limits_{E_2^1(R_1)} \frac{|dt|}{|t-w|^{2\lambda_1}} + \int\limits_{E_2^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)\lambda_1}}$$
$$\preceq n^{(2+\gamma_1)\lambda_1 - 1}.$$

1.1.3) If $\gamma_1 \ge 0$, then Lemma 2.2 implies:

$$F_{n,1}^3(w) = \int\limits_{E_3^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}|\Psi(t)-\Psi(w)|^2} \tag{2.28}$$
$$\preceq c_2^{-\gamma_1} \int\limits_{E_3^1(R_1)} \frac{|dt|}{|t-w|^{2\lambda_1}} \preceq n^{2\lambda_1 - 1},$$

and for $\gamma_1 < 0$, also Lemma 2.4 yields:

$$F_{n,1}^3(w) \preceq c_3^{-\gamma_1} \int\limits_{E_3^1(R_1)} \frac{|dt|}{|t-w|^{2\lambda_1}} \preceq n^{2\lambda_1 - 1}. \tag{2.29}$$

1.2) Let $w \in E_2^1(R)$.
1.2.1) For any $\gamma_1 > -2$

$$F_{n,1}^1(w) = \int\limits_{E_{1,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{E_{1,2}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \tag{2.30}$$
$$=: F_{n,1}^{1,1}(w) + F_{n,1}^{1,2}(w),$$

and so, according to Lemmas 2.1 and 2.2, we obtain:

$$F_{n,1}^{1,1}(w) \preceq \int\limits_{E_{1,1}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)\lambda_1}} \preceq \int\limits_{1/n}^{c} \frac{ds}{s^{(2+\gamma_1)\lambda_1}} \tag{2.31}$$
$$\preceq \begin{cases} n^{(2+\gamma_1)\lambda_1 - 1}, & \text{if } (2+\gamma_1)\lambda_1 > 1, \\ \ln n, & \text{if } (2+\gamma_1)\lambda_1 = 1, \\ 1, & \text{if } (2+\gamma_1)\lambda_1 < 1, \end{cases}$$

and

$$F_{n,1}^{1,2}(w) \preceq \int\limits_{E_{1,2}^1(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)\lambda_1}} \preceq n^{(2+\gamma_1)\lambda_1} mesE_{1,2}^1(R_1) \preceq n^{(2+\gamma_1)\lambda_1-1}. \quad (2.32)$$

1.2.2) For any $\gamma_1 > -2$, according to Lemmas 2.1 and 2.2, we have:

$$\begin{aligned} F_{n,1}^2(w) &\preceq \int\limits_{E_{2,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{E_{2,2}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \quad (2.33) \\ &\preceq \int\limits_{E_{2,1}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)\lambda_1}} + \int\limits_{E_{2,2}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)\lambda_1}} \\ &\preceq \int\limits_{1/n}^{c_1} \frac{ds}{s^{(2+\gamma_1)\lambda_1}} + \int\limits_{1/n}^{c_2} \frac{ds}{s^{(2+\gamma_1)\lambda_1}} \\ &\preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1}, & \text{if } (2+\gamma_1)\lambda_1 > 1, \\ \ln n, & \text{if } (2+\gamma_1)\lambda_1 > 1, \\ 1, & \text{if } (2+\gamma_1)\lambda_1 < 1. \end{cases} \end{aligned}$$

1.2.3) For any $\gamma_1 > -2$, according to Lemmas 2.1 and 2.2, we get:

$$\begin{aligned} F_{n,1}^3(w) &\preceq \int\limits_{E_3^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{E_3^1(R_1)} \frac{|dt|}{|t-w|^{2\lambda_1}} \\ &\preceq \int\limits_{1/n}^{c_3} \frac{ds}{s^{2\lambda_1}} \preceq \begin{cases} n^{2\lambda_1-1}, & \text{if } 2\lambda_1 > 1, \\ \ln n, & \text{if } 2\lambda_1 = 1, \\ 1, & \text{if } 2\lambda_1 < 1, \end{cases} \end{aligned}$$

1.3) Let $w \in E_3^1(R)$.
If $\gamma_1 \geq 0$, from Lemmas 2.1 and 2.2, we get:

$$\begin{aligned} F_{n,1}^1(w) &\preceq \int\limits_{E_1^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}} \preceq \int\limits_{E_1^1(R_1)} \frac{|dt|}{|t-w_1|^{\gamma_1\lambda_1}} \quad (2.34) \\ &\preceq n^{\gamma_1\lambda_1} \cdot mesE_1^1(R_1) \preceq n^{\gamma_1\lambda_1-1}, \end{aligned}$$

and for $\gamma_1 < 0$,

$$\begin{aligned} F_{n,1}^1(w) &\preceq \int\limits_{E_1^1(R_1)} |t-w_1|^{(-\gamma_1)\lambda_1} |dt| \preceq \left(\frac{1}{n}\right)^{(-\gamma_1)\lambda_1} \cdot mesE_1^1(R_1) \quad (2.35) \\ &\preceq \left(\frac{1}{n}\right)^{(-\gamma_1)\lambda_1+1} \preceq 1. \end{aligned}$$

1.3.2) In this case for any $\gamma_1 > -2$, according to Lemmas 2.1 and 2.2, we obtain:

$$F_{n,1}^2(w) \preceq \int\limits_{E_{2,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{E_{2,2}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \tag{2.36}$$
$$\preceq \int\limits_{E_{2,1}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)\lambda_1}} + \int\limits_{E_{2,2}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)\lambda_1}}$$
$$\preceq \int\limits_{1/n}^{c_1} \frac{ds}{s^{(2+\gamma_1)\lambda_1}} + \int\limits_{1/n}^{c_2} \frac{ds}{s^{(2+\gamma_1)\lambda_1}}$$
$$\preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1}, & \text{if } (2+\gamma_1)\lambda_1 > 1, \\ \ln n, & \text{if } (2+\gamma_1)\lambda_1 > 1, \\ 1, & \text{if } (2+\gamma_1)\lambda_1 < 1. \end{cases}$$

1.3.3) Analogously, for any $\gamma_1 > -2$,

$$F_{n,1}^3(w) \preceq \int\limits_{E_3^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{E_3^1(R_1)} \frac{|dt|}{|t-w|^{2\lambda_1}} \preceq n^{2\lambda_1-1}. \tag{2.37}$$

Combining estimates (2.17)-(2.37), for $w \in \Phi(L_R)$, we have:

$$F_{n,1} \preceq \begin{cases} n^{(2+\widetilde{\gamma}_1)\lambda_1-1}, & \text{if } (2+\widetilde{\gamma}_1)\widetilde{\lambda}_1 > 1, \\ \ln n, & \text{if } (2+\widetilde{\gamma}_1)\widetilde{\lambda}_1 > 1, \\ 1, & \text{if } (2+\widetilde{\gamma}_1)\widetilde{\lambda}_1 < 1, \end{cases} \tag{2.38}$$

where $\widetilde{\gamma}_1 := \max\{0;\gamma_1\}$, $\widetilde{\lambda}_1 := \max\{1;\lambda_1\}$.

Case 2. Let $w \in \Phi(L_R^2)$.

Analogously to the Case 1, in this case we will obtain estimates for $w \in E_1^2(R)$, $w \in E_2^2(R)$ and $w \in E_3^2(R)$.

2.1) Let $w \in E_1^2(R) \cup E_2^2(R)$. We will estimate the quantity

$$F_{n,2}(w) = \sum_{i=1}^{3} \int\limits_{E_i^2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}\,|\Psi(t)-\Psi(w)|^2} =: \sum_{i=1}^{3} F_{n,2}^i(w) \tag{2.39}$$

for $\gamma_1 \geq 0$ and $\gamma_1 < 0$ separately.

According to the estimation [26, p.181] (see, also [16]) for arbitrary continuum with simple connected complementary, the following holds:

$$|\Psi(t)-\Psi(w_2)| \succeq |t-w_2|^2. \tag{2.40}$$

We will use this fact in evaluations in this section instead of Lemma 2.2

2.1.1) For each $i = 1, 2$, we obtain:

$$\sum_{i=1}^{2} F_{n,2}^i(w) = \sum_{i=1}^{2} \int\limits_{E_i^2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}\,|\Psi(t)-\Psi(w)|^2} \tag{2.41}$$

$$\preceq \left(\int\limits_{E_{1,1}^2(R_1)} + \int\limits_{E_{2,1}^2(R_1)} \right) \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_2}} + \left(\int\limits_{E_{1,2}^2(R_1)} + \int\limits_{E_{2,2}^2(R_1)} \right) \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{2+\gamma_2}}$$

$$\preceq \left(\int\limits_{E_{1,1}^2(R_1)} + \int\limits_{E_{2,1}^2(R_1)} \right) \frac{|dt|}{|t-w|^{2(2+\gamma_2)}} \preceq n^{2(2+\gamma_2)-1},$$

if $\gamma_2 \geq 0$, and

$$\sum_{i=1}^{2} F_{n,2}^i(w) = \sum_{i=1}^{2} \int\limits_{E_i^2(R_1)} \frac{|\Psi(t)-\Psi(w_2)|^{(-\gamma_2)}\,|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq n^3. \tag{2.42}$$

if $\gamma_2 < 0$.

2.1.2) For $i=3$ we get:

$$F_{n,2}^3(w) = \int\limits_{E_3^2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}\,|\Psi(t)-\Psi(w)|^2} \tag{2.43}$$

$$\preceq c_2^{-\gamma_2} \int\limits_{E_3^2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{E_3^2(R_1)} \frac{|dt|}{|t-w|^2} \preceq n,$$

if $\gamma_2 \geq 0$, and

$$F_{n,2}^3(w) \preceq n, \tag{2.44}$$

if $\gamma_2 < 0$.

2.2) Let $w \in E_3^2(R)$. For each $\gamma_2 > -2$, analogously to case 2.1.1, we obtain:

$$\sum_{i=1}^{2} F_{n,2}^i(w) = \sum_{i=1}^{2} \int\limits_{E_i^2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}\,|\Psi(t)-\Psi(w)|^2} \tag{2.45}$$

$$\preceq \left(\int\limits_{E_{1,1}^2(R_1)} + \int\limits_{E_{2,1}^2(R_1)} \right) \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_2}} + \left(\int\limits_{E_{1,2}^2(R_1)} + \int\limits_{E_{2,2}^2(R_1)} \right) \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{2+\gamma_2}}$$

$$\preceq \left(\int\limits_{E_{1,1}^2(R_1)} + \int\limits_{E_{2,1}^2(R_1)} \right) \frac{|dt|}{|t-w|^{2(2+\gamma_2)}} + \left(\int\limits_{E_{1,2}^2(R_1)} + \int\limits_{E_{2,2}^2(R_1)} \right) \frac{|dt|}{|t-w_2|^{2(2+\gamma_2)}}$$

$$\preceq \begin{cases} n^{2(2+\gamma_2)-1}, & \text{if } 2\,(2+\gamma_2) > 1, \\ \ln n, & \text{if } 2\,(2+\gamma_2) = 1, \\ 1, & \text{if } 2\,(2+\gamma_2) < 1. \end{cases}$$

2.2.2) For each $\gamma_2 > -2$, we have:

$$F_{n,2}^3(w) = \int\limits_{E_3^2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}\,|\Psi(t)-\Psi(w)|^2} \tag{2.46}$$

$$\preceq \int\limits_{E_3^2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{E_3^2(R_1)} \frac{|dt|}{|t-w|^2} \preceq n.$$

Combining (2.39)-(2.46), we obtain:

$$F_{n,2}(w) \preceq \begin{cases} n^{2(2+\widetilde{\gamma}_2)-1}, & \text{if } 2\,(2+\gamma_2) > 1, \\ \ln n, & \text{if } 2\,(2+\gamma_2) = 1, \\ 1, & \text{if } 2\,(2+\gamma_2) < 1, \end{cases} \tag{2.47}$$

where $\widetilde{\gamma}_2 := \max\{0;\gamma_2\}$.

Therefore, comparing relations (2.14), (2.16), (2.38) and (2.47), we have:

$$F_n(w) \preceq \begin{cases} n^{(2+\widetilde{\gamma}_1)\widetilde{\lambda}_1-1}, & \text{if } (2+\widetilde{\gamma}_1)\,\widetilde{\lambda}_1 > 1, \\ \ln n, & \text{if } (2+\widetilde{\gamma}_1)\,\widetilde{\lambda}_1 > 1, \\ 1, & \text{if } (2+\widetilde{\gamma}_1)\,\widetilde{\lambda}_1 < 1, \end{cases} + \begin{cases} n^{2(2+\widetilde{\gamma}_2)-1}, & \text{if } 2\,(2+\gamma_2) > 1, \\ \ln n, & \text{if } 2\,(2+\gamma_2) = 1, \\ 1, & \text{if } 2\,(2+\gamma_2) < 1. \end{cases},$$

and consequently, from (2.10), (2.11) and (2.12), we completed the proof for any $z \in L_R$. So, it also true for $z \in \overline{G}$, and we completed the proofs.

22 Proof of Theorem 1.2.

Proof. Suppose that $G \in PDS(\lambda_1; 2)$, for some $0 < \lambda_1 < 2$; $h(z)$ be defined as in (1.1). For each $R > 1$, let $w = \varphi_R(z)$ denotes be a univalent conformal mapping G_R onto the B, normalized by $\varphi_R(0) = 0,\ \varphi_R'(0) > 0$, and let $\{\zeta_j\}$, $1 \le j \le m \le n$, be a zeros of $P_n(z)$ (if any exist) lying on G_R. Let

$$b_{m,R}(z) := \prod_{j=1}^{m} \widetilde{b}_{j,R}(z) =: \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \overline{\varphi_R(\zeta_j)}\varphi_R(z)}, \tag{2.48}$$

denotes a Blaschke function with respect to zeros $\{\zeta_j\}$, $1 \le j \le m \le n$, of $P_n(z)$ ([27]). Clearly,

$$|b_{m,R}(z)| \equiv 1,\ z \in L_R, \text{ and } |b_{m,R}(z)| < 1,\ z \in G_R. \tag{2.49}$$

For any $p > 0$ and $z \in G_R$, let us set:

$$T_{n.p}(z) := \left[\frac{P_n(z)}{b_{m,R}(z)}\right]^{p/2} . \tag{2.50}$$

The function $T_{n,p}(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_{n,p}(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $T_{n,p}(z)$ at the $z = z_j$, $j = 1, 2$, gives:

$$T_{n,p}(z_1) = \frac{1}{2\pi i} \int_{L_R} T_{n,p}(\zeta) \frac{d\zeta}{\zeta - z_1}.$$

Then, according to (2.49), we obtain:

$$\begin{aligned} |P_n(z_j)|^{p/2} &\le \frac{|b_{m,R}(z_1)|^{p/2}}{2\pi} \int_{L_R} \left|\frac{P_n(\zeta)}{b_{m,R}(\zeta)}\right|^{p/2} \frac{|d\zeta|}{|\zeta - z_j|} \\ &\preceq \int_{L_R} |P_n(\zeta)|^{p/2} \frac{|d\zeta|}{|\zeta - z_j|} . \end{aligned} \tag{2.51}$$

Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w = \Phi(z)$ and applying the Hölder inequality, we obtain:

$$\left(\int\limits_{L_R} |P_n(\zeta)|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z_j|} \right)^2 \tag{2.52}$$

$$\leq \int\limits_{|t|=R} h(\Psi(t)) \left|P_n(\Psi(t))\right|^p \left|\Psi'(t)\right|^2 |dt| \cdot \int\limits_{|t|=R} \frac{|dt|}{h(\Psi(t)) \left|\Psi(t) - \Psi(w_j)\right|^2}$$

$$= \int\limits_{|t|=R} |f_{n,p}(t)|^p |dt| \cdot \int\limits_{|t|=R} \frac{|dt|}{h(\Psi(t)) \left|\Psi(t) - \Psi(w_j)\right|^2},$$

where $f_{n,p}(t)$ has been defined as in (2.11). Since $R > 1$ is arbitrary, then (2.52) holds also for $R = R_1 := 1 + \frac{\varepsilon_1}{n}$, $0 < \varepsilon_1 < 1$. So, we have:

$$\left(\int\limits_{L_{R_1}} |P_n(\zeta)|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z_j|} \right)^2 \tag{2.53}$$

$$\leq \left(\int\limits_{|t|=R_1} |f_{n,p}(t)|^p |dt| \right) \cdot \left(\int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) \left|\Psi(t) - \Psi(w_j)\right|^2} \right)$$

$$=: A_n \cdot F_n(w_j),$$

and, A_n and $F_n(w_j)$ has been defined as in (2.11) for $R = R_1$. Therefore, from (2.51) and (2.53), we have:

$$|P_n(z_1)| \preceq A_n \cdot F_n(w_j), \tag{2.54}$$

where, according to (2.12), the estimate

$$A_n \preceq n \cdot \|P_n\|_p^p$$

is satisfied. For the estimate of the quantity $F_n(w_j)$ we use the notations at the estimation of the $F_n(w)$ as in (2.14)-(2.16). Therefore, under these notations, for the $F_n(w_j)$, we get:

$$F_n(w_j) \preceq \sum_{j=1}^{2} \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{\left|\Psi(t) - \Psi(w_j)\right|^{2+\gamma_j}} \tag{2.55}$$

$$\preceq \sum_{j=1}^{2} \sum_{i=1}^{3} \int\limits_{E^j_i(L_{R_1})} \frac{|dt|}{\left|\Psi(t) - \Psi(w_j)\right|^{2+\gamma_j}} =: \sum_{j=1}^{2} \sum_{i=1}^{3} F^i_{n,j}(w_j),$$

since the points $\{z_j\}_{j=1}^m \in L$ are distinct. So, we need to evaluate the $F^i_{n,j}(w_j)$ for each $j = 1, 2$ and $i = 1, 2, 3$.

Case 1: $j = 1$.

$$F^1_{n,1}(w_1) + F^2_{n,1}(w_1) = \int\limits_{E^1_1(L_{R_1}) \cup E^2_1(L_{R_1})} \frac{|dt|}{\left|\Psi(t) - \Psi(w_1)\right|^{2+\gamma_1}} \tag{2.56}$$

$$\preceq \int\limits_{E_1^1(L_{R_1})\cup E_1^2(L_{R_1})} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)\lambda_1}} \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1}, & \text{if } (2+\gamma_1)\lambda_1>1, \\ \ln n, & \text{if } (2+\gamma_1)\lambda_1=1, \\ 1, & \text{if } (2+\gamma_1)\lambda_1<1, \end{cases}$$

and

$$F_{n,1}^3(w_1) = \int\limits_{E_1^3(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \preceq \frac{1}{c_2^{2+\gamma_1}} \int\limits_{E_1^3(L_{R_1})} |dt| \preceq 1. \tag{2.57}$$

Case 2: $j=2$.

$$F_{n,2}^1(w_2)+F_{n,2}^2(w_2) = \int\limits_{E_2^1(L_{R_1})\cup E_2^2(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{2+\gamma_2}} \tag{2.58}$$

$$\preceq \int\limits_{E_2^1(L_{R_1})\cup E_2^2(L_{R_1})} \frac{|dt|}{|t-w_2|^{2(2+\gamma_2)}} \preceq \begin{cases} n^{2(2+\gamma_2)-1}, & \text{if } 2(2+\gamma_2)>1, \\ \ln n, & \text{if } 2(2+\gamma_2)=1, \\ 1, & \text{if } 2(2+\gamma_2)<1, \end{cases}$$

and

$$F_{n,2}^3(w_2) = \int\limits_{E_2^3(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{2+\gamma_2}} \preceq \frac{1}{c_2^{2+\gamma_2}} \int\limits_{E_2^3(L_{R_1})} |dt| \preceq 1. \tag{2.59}$$

Combining relations (2.54) - (2.59), we complete the proof.

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