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Central CNZ Rings

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Abstract. In this note, we introduce the concept of central CNZ rings, which is a generalization of CNZ rings [1]. A ring R is called *central CNZ* if for any $a, b \in \text{nil}(R)$, $ab = 0$ implies that ba is central, where $\text{nil}(R)$ is the set of all nilpotent elements in R. We investigate the structure of central CNZ rings and their related properties.

Keywords. CNZ ring · reversible ring · central reversible ring · Armendariz ring.

Mathematics Subject Classification (2010): 16U80, 16N40, 16U70

1 Introduction

Throughout this paper R denotes an arbitrary associative ring with identity. Given a ring R, $\text{nil}(R)$ denote the set of all nilpotent elements in R. $C(R)$ denote the center of R, that is, $C(R) = \{a \in R \mid ar = ra \text{ for all } r \in R\}$. The polynomial (resp., Laurent) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[x, x^{-1}]$). $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and modulo n, respectively. Given a ring R, $M_n(R)$ and $U_n(R)$ denote the n by n full matrix ring and upper triangular matrix ring over R, respectively. Let $D_n(R)$ be the ring of all matrices in $U_n(R)$ whose diagonal entries are all equal.

In [5], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for any $a, b \in R$. Following [8], a ring R is called *central reversible* if $ab = 0$ for any $a, b \in R$ implies ba is a central element of R. Every reversible ring is central reversible ring by a simple computation. Recently, a ring R is said to satisfy the *commutativity of nilpotent elements at zero* (simply, a *CNZ* ring)[1, Definition 2.1] if $ab = 0$ implies $ba = 0$ for $a, b \in \text{nil}(R)$. In this note, we will extend the concepts of CNZ of a ring to central CNZ. We will call a ring R *central CNZ* if $ab = 0$ implies ba is central for any $a, b \in \text{nil}(R)$.

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2 Central CNZ rings

Definition 2.1 *A ring R is called central CNZ if $ab = 0$ for any $a, b \in$ nil(R) implies ba is central in R.*

Let R be a ring and consider the ring

$$H_3(R) = \left\{ \begin{pmatrix} n & a_1 & a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & n \end{pmatrix} \mid a_1, a_2, a_3, a_4 \in R, n \in \mathbb{Z} \right\}$$

with the usual matrix adddition and multiplication. Note that the set of all nilpotent elements in $H_3(R)$ is

$$\text{nil}(H_3(R)) = \left\{ \begin{pmatrix} 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & 0 \end{pmatrix} \mid a_1, a_2, a_4 \in R, a_3 \in \text{nil}(R) \right\}.$$

Recall that a ring R is *reduced* if it has no nonzero nilpotent elements. We have the following theorem.

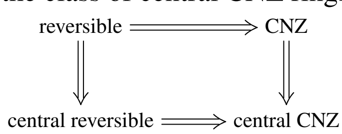
Theorem 2.1 *A ring R is reduced if and only if $H_3(R)$ is a central CNZ ring.*

Proof. Assume that R is a reduced ring. Let $X, Y \in \text{nil}(H_3(R))$ such that $XY = 0$. Write $X = \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & x_4 \\ 0 & 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & y_4 \\ 0 & 0 & 0 \end{pmatrix}$ with $XY = 0$. We have $x_1 y_4 = 0$. Then $YX = \begin{pmatrix} 0 & 0 & y_1 x_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For any $Z = \begin{pmatrix} n & a_1 & a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & n \end{pmatrix} \in H_3(R)$, we have $(YX)Z = \begin{pmatrix} 0 & 0 & n y_1 x_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Z(YX)$. That is YX is central.
Conversely, Assume that $H_3(R)$ be central CNZ and that R is not reduced. Then there exists $0 \neq a \in R$ such that $a^2 = 0$. If we take two matrices $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \text{nil}(H_3(R))$, we have $AB = 0$. Also $BA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$ is not central. This is a contradiction, so such a cannot exist. Thus R is reduced.

Clearly, every CNZ ring is central CNZ ring, but there exists a central CNZ ring which is not CNZ.

Example 1 Let R be a reduced ring. Consider the ring $H_3(R)$ described in the statement of Theorem 2.1. Then $H_3(R)$ is not CNZ. In fact, for $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \text{nil}(H_3(R))$, we have $AB = 0$ but $BA \neq 0$.

In the next diagram we summarize implications among aforementioned classes of rings, and we determine the position of the class of central CNZ rings.



The classes of CNZ rings and central reversible rings do not imply each other by the following examples.

Example 2 Let R be a commutative reduced ring and consider $S = U_3(R)$. In [8, Example 2.2], S is central reversible. Note that

$$\text{nil}(S)=\left\{\begin{pmatrix}0 & b & c\\ 0 & 0 & d\\ 0 & 0 & 0\end{pmatrix} \mid b, c, d \in R\right\}.$$

For $X = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ and $Y = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \in \text{nil}(S)$, we have $XY = 0$ but $YX \neq 0$. Thus S is not CNZ.

Recall that a ring R is *abelian* provided all its idempotents are central. The following example shows that there exists a central CNZ ring R such that R is not abelian.

Example 3 Consider the ring $U_2(\mathbb{Z}) = \left\{\begin{pmatrix}a & b\\ 0 & c\end{pmatrix} \mid a, b, c \in \mathbb{Z}\right\}$. Note that

$$\text{nil}(U_2(\mathbb{Z})) = \left\{\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix} \mid b \in \mathbb{Z}\right\}.$$

For any $X, Y \in \text{nil}(U_2(\mathbb{Z}))$, $XY = 0$ implies $YX = 0$. So $U_2(\mathbb{Z})$ is a CNZ ring. Therefore $U_2(\mathbb{Z})$ is central CNZ. However for $X = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $Y = \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix} \in U_2(\mathbb{Z})$, we have $XY = 0$ but $YX = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ is not central in $U_2(\mathbb{Z})$ for $Z = \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} \in U_2(\mathbb{Z})$. Thus $U_2(\mathbb{Z})$ is not central reversible.
Also all idempotent elements in $U_2(\mathbb{Z})$ are zero matrix, identity matrix and $\begin{pmatrix}1 & b\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}0 & b\\ 0 & 1\end{pmatrix}$ for any $b \in \mathbb{Z}$. For $\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}^2 = \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} \in U_2(\mathbb{Z})$, which is not central. Hence $U_2(\mathbb{Z})$ is not Abelian. Thus $U_2(\mathbb{Z})$ is not reversible.

Remark 2.1 If R is a reduced ring, then both $U_2(R)$ and $D_2(R)$ are central CNZ rings. However, if R is not a reduced ring, then $U_2(R)$ need not be central CNZ ring.

Example 4 Consider the ring $R = U_2(\mathbb{Z}_4)$. Then nil(R)$=\left\{\begin{pmatrix}a & b\\ 0 & c\end{pmatrix} \mid a, c \in \{\overline{0}, \overline{2}\},\ b \in \mathbb{Z}_4\right\}$. For $X = \begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix}, Y = \begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix} \in \text{nil}(R)$, we have $XY = 0$. But for $Z = \begin{pmatrix}0 & 2\\ 0 & 3\end{pmatrix} \in R$, $(YX)Z \neq Z(YX)$. Hence R is not a central CNZ ring.

The Dorroh extension $D(R, \mathbb{Z}) = \{(r, n) \mid r \in R,\ n \in \mathbb{Z}\}$ of a ring R is a ring with operations $(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2)$ and $(r_1, n_1)(r_2, n_2) = (r_1r_2 + n_1r_2 + n_2r_1, n_1n_2)$, where $r_i \in R$ and $n_i \in \mathbb{Z}$ for $i = 1, 2$.

Proposition 2.1 *A ring R is central CNZ if and only if the Dorroh extension $D(R, \mathbb{Z})$ of R is central CNZ.*

Proof. Note that $\text{nil}(D(R, \mathbb{Z})) = \{(r, 0) \mid r \in \text{nil}(R)\}$.
For necessity, let $(r, 0), (s, 0) \in \text{nil}(D(R, \mathbb{Z}))$ with $(r, 0)(s, 0) = 0$. Then $rs = 0$ in R.

Since R is central CNZ, sr is a central element of R. For any $(t, m) \in D(R, \mathbb{Z})$, we have $(s, 0)(r, 0)(t, m) = ((sr)t + msr, 0) = (t(sr) + msr, 0) = (t, m)(s, 0)(r, 0)$. So $D(R, \mathbb{Z})$ is central CNZ.
For sufficiency, let $r, s \in \text{nil}(R)$ with $rs = 0$. This implies $(r, 0)(s, 0) = 0$ such that $(r, 0)$ and $(s, 0)$ are nilpotents in $D(R, \mathbb{Z})$. By hypothesis, $(s, 0)(r, 0)$ is central. So sr is central. Therefore R is a central CNZ ring.

Remark 2.2 The property of central CNZ is preserved under isomorphisms and subrings.

The ring R is called *Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i and j (See [6]). For example every reduced ring is Armendariz.

Theorem 2.2 *For an Armendariz ring R, R is central CNZ if and only if $R[x]$ is central CNZ.*

Proof. Note that since R is Armendariz, we have nil$(R[x])$ =nil$(R)[x]$ by [2, Corollary 5.2].

Assume that R is a central CNZ ring. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in \text{nil}(R[x])$ with $f(x)g(x) = 0$. Since R is Armendariz, for all i and j, $a_i b_j = 0$ with $a_i, b_j \in \text{nil}(R)$. By hypothesis, we have $b_j a_i$ is central for all i and j. Thus for any $h(x) = \sum_{k=0}^{t} c_k x^k \in R[x]$, we have $(g(x)f(x))h(x) = h(x)(g(x)f(x))$. So $R[x]$ is central CNZ.

Conversely, assume that $R[x]$ is a central CNZ ring. Clearly, the central CNZ property of rings are preserved under subrings. So R is a central CNZ ring.

One may ask whether the polynomial rings over central CNZ rings are Armendariz. However the answer is negative by the following example.

Example 5 Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}.$$

Then R is CNZ by [1, Example 2.5(2)] and so central CNZ. Consider the polynomial ring over R. For $f(x) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} x$, $g(x) = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} x \in R[x]$. We have $f(x)g(x) = 0$ but $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \neq 0$. Hence R is not Armendariz.

A ring R is called *right (left) principally quasi-Baer* [3] if the right (left) annihilator of a principal right ideal of R is generated by an idempotent. A ring R is called *right (left) principally projective* [4] if the right (left) annihilator of an element of R is generated by an idempotent.

Theorem 2.3 *If R is a CNZ ring, then R is central CNZ. The converse holds if R satisfies any of the following conditions:*

(1) R is a semiprime ring.
(2) R is a right (left) principally projective ring.

(3) R is a right (left) principally quasi-Baer ring.
(4) R is a Von Neumann regular ring.

Proof. First statement is clear. Conversely assume that R is a central CNZ ring. Now consider the following cases.

(1) Assume that R is a semiprime ring. Let $a, b \in \text{nil}(R)$ with $ab = 0$. Since R is central CNZ, we have ba is central. Then $baRba = b(ab)aR = 0$. By hypothesis, we get $ba = 0$. So R is CNZ.

(2) Assume that R is a right principally projective ring and $ab = 0$ for any $a, b \in \text{nil}(R)$. Then there exists an idempotent $e \in R$ such that $r_R(a) = eR$. Thus we have $b = eb$ also $ae = 0$. Hence $ba = (eb)a = (ba)e = b(ae) = 0$, since R is central CNZ. So R is a CNZ ring.

(3) It is similar to the proof of (2).

(4) Assume that R is a Von Neumann regular ring. Let $a, b \in \text{nil}(R)$ such that $ab = 0$. So there exists an $x \in R$ such that $a = axa$. Multiply by b from left, we get $ba = (ba)xa = x(ab)a = 0$ since R is central CNZ. Hence $ba = 0$. So R is CNZ.

Proposition 2.2 *Let R be a Von Neumann regular abelian ring. Then R is central CNZ.*

Proof. Let $a, b \in \text{nil}(R)$ with $ab = 0$. Since R is Von Neumann regular abelian, there exists an $x \in R$ such that $a = axa$. Multiply this equality by b from left, we have $ba = b(axa)$. Also, xa is idempotent. Since R is abelian, we have $ba = (xa)ba = 0$. So R is CNZ, therefore R is central CNZ.

Proposition 2.3 *For a ring R, suppose that R/I is a central CNZ ring, for some ideal I of R. If I is reduced then R is central CNZ.*

Proof. Assume that R/I is a central CNZ ring. Let $a, b \in \text{nil}(R)$ with $ab = 0$. As $a, b \in \text{nil}(R)$, we have $\overline{a}, \overline{b}$ are nilpotents in R/I and $\overline{a}\overline{b} = \overline{0}$, where $\overline{a} = a + I$ and $\overline{b} = b + I$. Since R/I is central CNZ, we have $\overline{b}\overline{a}$ is central in R/I. That is, $bar - rba \in I$ for all $r \in R$. On the other hand, if $ab = 0$ with $a, b \in R$ then we have $bIa = 0$ since $bIa \subseteq I$, $(bIa)^2 = 0$ and I is reduced. Also $(bar - rba)^3 = 0$ then we get $bar - rba = 0$. Thus ba is central in R. This implies that R is central CNZ.

Remark 2.3 Let R_i be rings for each i. Note that $\text{nil}(\prod_{i=1}^{n} R_i) = \prod_{i=1}^{n} \text{nil}(R_i)$.

Theorem 2.4 *Let $R_1, R_2, \ldots, R_n$ be rings. Then $\prod_{i=1}^{n} R_i$ is central CNZ if and only if R_i is central CNZ for each i.*

Proof. Assume that $\prod_{i=1}^{n} R_i$ is central CNZ. Since the central CNZ property of rings are preserved under subrings and isomorphisms, we have R_i is central CNZ for each i. Conversely, assume that R_i is central CNZ for each i. Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in \text{nil}(\prod_{i=1}^{n} R_i)$ with $ab = 0$. Then we have $a_i b_i = 0$ for $a_i, b_i \in \text{nil}(R_i)$ for each i. Since R_i is central CNZ, we get $b_i a_i$ is central in R_i for each i. So ba is central in $\prod_{i=1}^{n} R_i$.

Proposition 2.4 *Let R be a ring and e an idempotent of R. If e is central in R, then the following statements are equivalent:*

(1) R is central CNZ.
(2) eR and $(1-e)R$ are central CNZ.

Proof. $(1) \Rightarrow (2)$ It is obvious since eR and $(1-e)R$ are subrings of R.
$(2) \Rightarrow (1)$ Since $R \cong eR \oplus (1-e)R$, it follows from Theorem 2.4.

Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements. Let $S^{-1}R$ be the localization of R at S.

Proposition 2.5 *A ring R is central CNZ if and only if $S^{-1}R$ is central CNZ.*

Proof. It suffices to prove the necessary condition since the subrings of central CNZ rings are also central CNZ. To prove this let $x = u^{-1}a$, $y = v^{-1}b \in \text{nil}(S^{-1}R)$ with $xy = 0$. Then $u, v \in S$, $a, b \in \text{nil}(R)$. Since S consists of central regular elements, we have $0 = xy = (u^{-1}a)(v^{-1}b) = (uv)^{-1}(ab)$ and so $ab = 0$. Since R is central CNZ, it follows that ba is central. For any $z = w^{-1}c \in S^{-1}R$, we have $(yx)z = [(v^{-1}b)(u^{-1}a)](w^{-1}c) = (vuw)^{-1}[(ba)c] = (vuw)^{-1}[c(ba)] = z(yx)$. Hence $S^{-1}R$ is central CNZ.

Corollary 2.1 *For a ring R, $R[x]$ is central CNZ if and only if $R[x, x^{-1}]$ is central CNZ.*

Proof. Assume that $R[x]$ is central CNZ and let $S = \{1, x, x^2, \ldots\}$. Since $R[x, x^{-1}] = S^{-1}R[x]$, it follows that $R[x, x^{-1}]$ is central CNZ by Proposition 2.5. The sufficient is clear, since the subrings of central CNZ rings are also central CNZ.

Remark 2.4 For any positive integer n, $M_n(R)$ need not be central CNZ. For example, $M_2(\mathbb{Z}_4)$ is not central CNZ.

Proposition 2.6 *Let R be a ring. If R satisfies one of the following conditions, then R is central CNZ:*

(1) nil$(R) \subseteq C(R)$.
(2) $U(R) \subseteq C(R)$.

Proof. Let $a, b \in$nil(R) with $ab = 0$. Then ba is nilpotent.

(1) By hypothesis, ba is central. So R is central CNZ.
(2) By hypothesis, $1 - ba \in U(R) \subseteq C(R)$, we have $1 - ba \in C(R)$. Hence ba is central. So R is central CNZ.

The trivial extension of a ring R is the ring

$$T(R, R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

with the usual matrix operations. Note that nil$(T(R, R)) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in nil(R), b \in R \right\}$. One may ask whether the trivial extension over central CNZ rings are central CNZ. However the answer is negative by the following example.

Example 6 Let $R = U_2(\mathbb{Z})$. Then R is central CNZ by Example 3. However, $T(R, R)$ is not central CNZ. For (a, b) and (c, d) are nilpotents, we have $(a, b)(c, d) = 0$. But $(c, d)(a, b)$ is not central for $(e, f) \in T(R, R)$. Where $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R$.

3 Some matrix subrings

In this section, we study the extension rings of central CNZ rings. Let S and T be any rings, M an S-T-bimodule and R the formal triangular matrix ring $\begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$. Note that $\text{nil}(R) = \begin{pmatrix} nil(S) & M \\ 0 & nil(T) \end{pmatrix}$.

Proposition 3.1 *Let S, T, M and R be as above. If R is central CNZ, then S and T are central CNZ.*

Proof. Assume that $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ is central CNZ. For the idempotent $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, eRe is central CNZ, since the subrings of central CNZ rings are central CNZ. As eRe is isomorphic to S, so S is central CNZ. Replacing e by $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, a similar discussion reveals that T is central CNZ.

The converse statement of Proposition 3.1 need not be true in general.

Example 7 Let $R = \begin{pmatrix} U_2(\mathbb{Z}) & M_2(\mathbb{Z}) \\ 0 & U_2(\mathbb{Z}) \end{pmatrix}$. Let e_{ij} denote the 4×4 matrix unit having (i, j) entry is 1 if $i = j$ elsewhere 0. Let $a = e_{12} + e_{43}$ and $b = e_{12} + e_{14}$. Then a and b are nilpotents and $ab = 0$. $ba = e_{13}$ is not central. For if $c = e_{34}$, then $(ba)c = e_{13}e_{34} = e_{14}$, $c(ba) = e_{34}e_{13} = 0$. Hence ba is not central.

For any ring R and $n \geq 2$, $D_n(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = a_{22} = \ldots = a_{nn}\}$. Note that $D_2(R) = T(R, R)$. Example 6 shows that there exists a central CNZ ring such that $D_2(R)$ need not be central CNZ and consequently $D_n(R)$ need not be central CNZ for $n \geq 2$ by Remark 2.2.

For any ring R and $n \geq 2$, $V_n(R)$ is the subring of $M_n(R)$,

$$V_n(R) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \ldots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 & a_2 \\ 0 & 0 & 0 & \ldots & 0 & a_1 \end{bmatrix} \mid a_i \in R, 1 \leq i \leq n \right\}.$$

The center of $V_n(R)$ is given by

$$C(V_n(R)) = \{(a_i) \in V_n(R) \mid a_i \in C(R)\}.$$

By Example 6, in case R is central CNZ, $V_n(R)$ need not be central CNZ. However, there are rings R for which $V_n(R)$ is central CNZ.

Theorem 3.1 *Let R be a reduced ring. Then the following statements are equivalent.*

(1) R is central CNZ.
(2) For any positive integer n, $V_n(R)$ is central CNZ.

Proof. (1) ⇒ (2) Let R be a reduced ring and $n \geq 2$. Then $V_n(R)$ is reversible by [7, Theorem 2.5] and so it is central CNZ.
(2) ⇒ (1) It is obvious Remark 2.2.

Let R be a ring and S a subring of R and

$$T[R,S]=\{(r_1,r_2,\cdots,r_n,s,s,\cdots):r_i\in R,s\in S,n\geq 1,1\leq i\leq n\}.$$

Then $T[R,S]$ is a ring under the componentwise addition and multiplication. In the following we give necessary and sufficient conditions for $T[R,S]$ to be central CNZ.

Proposition 3.2 *Let R be a ring and S a subring of R. Then the following are equivalent.*

(1) $T[R,S]$ is central CNZ.
(2) R and S are central CNZ.

Proof. (1) $\Rightarrow$ (2) Let a, $b\in R$ be nilpotents with $ab=0$. Let $A=(a,0,0,0,...)$, $B=(b,0,0,0,...)$. Then A and B are nilpotents in $T[R,S]$ and $AB=0$. By (1), BA is central in $T[R,S]$. Hence ba is central in R and R is central CNZ. Let s, $t\in S$ be nilpotents with $st=0$. Let $X=(0,s,s,s,...)$, $Y=(0,t,t,t,.....)\in T[R,S]$. Then X and Y are nilpotents in $T[R,S]$ and $XY=0$. By (1), YX is central in $T[R,S]$. So ts is central in S. Hence S is central CNZ.

(2) $\Rightarrow$ (1) Let $a=(a_1,a_2,\cdots,a_n,b,b,\cdots)$, $c=(c_1,c_2,\cdots,c_m,d,d,\cdots)\in T[R,S]$ be nilpotents with $ac=0$. Then all components of a and c are nilpotents. We divide the proof into some cases:
Case I: $n<m$. Then $a_ic_i=0$ for $1\leq i\leq n$, $bc_j=0$ for $n+1\leq j\leq m$ and $bd=0$. By (2), c_ia_i are central in R for $1\leq i\leq n$, c_jb are central in R for $n+1\leq j\leq m$ and db is central in S. So ca is central in $T[R,S]$.
A similar discussion reveals that for the cases $n=m$ and $n>m$. It follows that $T[R,S]$ is central CNZ.

The rings $H_{(s,t)}(R)$: Let R be a ring, and let $s,t\in C(R)$. Let

$$H_{(s,t)}(R)=\left\{\begin{pmatrix}a&0&0\\c&d&e\\0&0&f\end{pmatrix}\in M_3(R)\mid a,c,d,e,f\in R, a-d=sc, d-f=te\right\}.$$

Then $H_{(s,t)}(R)$ is a subring of $M_3(R)$. Note that $A=\begin{pmatrix}a&0&0\\c&d&e\\0&0&f\end{pmatrix}\in H_{(s,t)}(R)$ if and only if $A\in M_3(R)$ and $a-d=sc, d-f=te$ if and only if $A=\begin{pmatrix}sc+te+f&0&0\\c&te+f&e\\0&0&f\end{pmatrix}$.

Lemma 3.1 *Let R be a ring, and let s and t be central invertible in R. Then*
$$C(H_{(s,t)}(R))=\left\{\begin{pmatrix}sc+te+f&0&0\\c&te+f&e\\0&0&f\end{pmatrix}\in H_{(s,t)}(R)\mid c,e,f\in C(R)\right\}.$$

Proof. Let $A=\begin{pmatrix}sc+te+f&0&0\\c&te+f&e\\0&0&f\end{pmatrix}\in C(H_{(s,t)}(R))$. Let e_{ij} denote the matrix units in $M_3(R)$. Let $x\in R$ and $B=(1+t)xe_{11}+(1+t)xe_{22}+xe_{23}+xe_{33}\in H_{(s,t)}(R)$. Then $AB=BA$ implies $xf=fx$ and $f\in C(R)$. Let $B=txe_{11}+txe_{22}+xe_{23}\in H_{(s,t)}(R)$. Then $AB=BA$ implies, among others, $ctx=txc$, $tex+fx=txe+xf$. These imply $cx=xc$ and $ex=xe$ since s and t are central invertible. Hence all components of A are central.

Conversely, let $A=\begin{pmatrix}sc+te+f&0&0\\c&te+f&e\\0&0&f\end{pmatrix}\in H_{(s,t)}(R)$ and assume that all components of A are central. Let $B = \begin{pmatrix} sy+tu+v & 0 & 0 \\ y & tu+v & u \\ 0 & 0 & v \end{pmatrix} \in H_{(s,t)}(R)$. We show that $AB = BA$. In fact $(3,3)$ component of AB is $fv = vf$ is the $(3,3)$ component of BA since f is central in R. $(2,3)$ component of AB is $(te+f)u+ev$. Since $ev = ve$, $(te+f)u = u(te+f)$, hence $u(te+f)+ve$ is the $(2,3)$ component of BA. $(2,2)$ component of AB is $(te+f)(tu+v)$. Since $te+f$ is central and $(te+f)u = u(te+f)$, $(tu+v)(te+f)$ is the $(2,2)$ component of BA. $(2,1)$ component of AB is $c(sy+tu+v)+(te+f)y$, and then $c(sy+tu+v)+(te+f)y = y(sc+te+f)+(tu+v)c$ is $(2,1)$ component of BA. $(1,1)$ component of AB is $(sc+te+f)(sy+tu+v)$. Since $sc+te+f$ is central in R, $(sc+te+f)(sy+tu+v) = (sy+tu+v)(sc+te+f)$ is the $(1,1)$ component of BA. Hence $AB = BA$ for all $B \in H_{(s,t)}(R)$. Thus A is central in $H_{(s,t)}(R)$.

Theorem 3.2 *Let R be a ring, and let s and t be central invertible in R. Then R is central CNZ if and only if $H_{(s,t)}(R)$ is central CNZ.*

Proof. Assume that R is central CNZ. Let $A = \begin{pmatrix} sc+te+f & 0 & 0 \\ c & te+f & e \\ 0 & 0 & f \end{pmatrix} \in H_{(s,t)}(R)$ and $B = \begin{pmatrix} sy+tu+v & 0 & 0 \\ y & tu+v & u \\ 0 & 0 & v \end{pmatrix} \in H_{(s,t)}(R)$ be nilpotents with $AB = 0$. Then $sc+te+f$, $te+f$, f and $sy+tu+v$, $tu+v$, v are nilpotents and $(sc+te+f)(sy+tu+v) = 0$, $c(sy+tu+v)+(te+f)y = 0$, $(te+f)(tu+v) = 0$, $(te+f)u+ev = 0$, $fv = 0$. By assumption vf, $(tu+v)(te+f)$ and $(sy+tu+v)(sc+te+f)$ are central in R. Since $(sy+tu+v)(sc+te+f) = sy(sc+te+f)+(tu+v)sc+(tu+v)(te+f)$ and $(tu+v)(te+f)$ are central, $sy(sc+te+f)+(tu+v)sc$ is central. Since $(tu+v)(te+f)$ is central, we have $(tu+v)e+uf$ is central in R. Hence all entries of BA are central. By Lemma 3.1, BA is central in $H_{(s,t)}(R)$.
Suppose that $H_{(s,t)}(R)$ is central CNZ. Let $a, b \in R$ be nilpotents such that $ab = 0$. Then $A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$, $B = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \in H_{(s,t)}(R)$ are nilpotents with $AB = 0$. By supposition BA is central in $H_{(s,t)}(R)$. By Lemma 3.1, ba is central in R.

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