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Ostrowski type inequalities for functions whose derivatives are strongly $beta$-convex

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Abstract. In this paper, we introduce the class of strongly $beta$-convex functions and establish some new Ostrowski's inequalities for functions whose first derivatives in absolute value are strongly $beta$-convex. Several results for its subclasses are also derived.

Keywords. Ostrowski inequality · Hölder inequality · power mean inequality · strongly $beta$-convex functions.

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1 Introduction

In 1938, A. M. Ostrowski proved an interesting integral inequality, given by the following theorem

Theorem 1.1 *[14] Let $f : I \rightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping in the interior $I^{\circ}$of I, and $a, b \in I^{\circ}$, with $a < b$. If $|f'| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \tfrac{1}{b-a} \int_a^b f(t)\, dt \right| \leq M(b-a) \left[\frac{1}{4} + \tfrac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right]. \tag{1.1}$$

In recent decades, inequality (1.1) has attracted much interest from many researchers, a considerable papers have been appeared on the generalizations, variants and extensions of inequality (1.1). For more details, we advise reader to [1, 10–13, 17] and references therein.

In [1] Alomari et al. established the following Ostrowski type inequalities for functions whose derivatives are convex

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Theorem 1.2 *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L([a,b])$ where $a,b \in I^{\circ}$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \left(\tfrac{1}{6} + \tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3 \right) |f'(a)| + \left(\tfrac{1}{6} + \tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3 \right) |f'(b)|$$

for each $x \in [a,b]$.

Theorem 1.3 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L([a,b])$ where $a,b \in I^{\circ}$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$, where $q \geq 1$, then the following inequality holds*

$$\begin{aligned}\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| &\leq (b-a)\left(\tfrac{b-x}{b-a}\right)^{2\left(1-\frac{1}{q}\right)} \left(\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2 |f'(a)|^q + \tfrac{(b-x)^3(b-3a+2x)}{6(b-a)^3} |f'(b)|^q \right)^{\frac{1}{q}} \\ &+ (b-a)\left(\tfrac{x-a}{b-a}\right)^{2\left(1-\frac{1}{q}\right)} \\ &\times \left(\left(\tfrac{1}{6} + \tfrac{(b-x)^3(3a-2x-b)}{6(b-a)^3} \right) |f'(a)|^q + \tfrac{1}{2}\left(\tfrac{x-a}{b-a}\right)^2 |f'(b)|^q \right)^{\frac{1}{q}}\end{aligned}$$

for each $x \in [a,b]$.

In [17] Set et al. established the following Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense

Theorem 1.4 *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L([a,b])$ where $a,b \in I^{\circ}$ with $a<b$. If $|f'|$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequality holds*

$$\begin{aligned}\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| &\leq \tfrac{b-a}{(s+1)(s+2)} \left\{ \left[2(s+1)\left(\tfrac{b-x}{b-a}\right)^{s+2} - (s+2)\left(\tfrac{b-x}{b-a}\right)^{s+1} + 1 \right] |f'(a)| \right. \\ &\left. + \left[2(s+1)\left(\tfrac{x-a}{b-a}\right)^{s+2} - (s+2)\left(\tfrac{x-a}{b-a}\right)^{s+1} + 1 \right] |f'(b)| \right\}\end{aligned}$$

for each $x \in [a,b]$.

Theorem 1.5 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L([a,b])$ where $a,b \in I^{\circ}$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$, where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right|$$

$$\leq \frac{b-a}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}$$
$$\times\left\{\left(\tfrac{b-x}{b-a}\right)^{1+\frac{1}{p}}\left(\left(\tfrac{b-x}{b-a}\right)^{s+1}\left|f'(a)\right|^{q}+\left[1-\left(\tfrac{x-a}{b-a}\right)^{s+1}\right]\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\tfrac{x-a}{b-a}\right)^{1+\frac{1}{p}}\left(\left[1-\left(\tfrac{b-x}{b-a}\right)^{s+1}\right]\left|f'(a)\right|^{q}+\left(\tfrac{x-a}{b-a}\right)^{s+1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right\}$$

for each $x \in [a,b]$.

Theorem 1.6 *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L([a,b])$ *where* $a,b \in I^{\circ}$ *with* $a < b$. *If* $|f'|^{q}$ *is s-convex on* $[a,b]$ *for some fixed* $s \in (0,1]$, *where* $q \geq 1$, *then the following inequality holds*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|$$
$$\leq (b-a)\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left(\left\{\left(\tfrac{b-x}{b-a}\right)^{2\left(1-\frac{1}{q}\right)}\left[\tfrac{1}{s+2}\left(\tfrac{b-x}{b-a}\right)^{s+2}\left|f'(a)\right|^{q}\right.\right.\right.$$
$$\left.\left.+\left(\tfrac{1}{s+2}\left(\tfrac{x-a}{b-a}\right)^{s+2}-\tfrac{1}{s+1}\left(\tfrac{x-a}{b-a}\right)^{s+1}+\tfrac{1}{(s+2)(s+1)}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right\}$$
$$+\left\{\left(\tfrac{x-a}{b-a}\right)^{2\left(1-\frac{1}{q}\right)}\left[\tfrac{1}{s+2}\left(\tfrac{x-a}{b-a}\right)^{s+2}\left|f'(b)\right|^{q}\right.\right.$$
$$\left.\left.\left.+\left(\tfrac{1}{s+2}\left(\tfrac{b-x}{b-a}\right)^{s+2}-\tfrac{1}{s+1}\left(\tfrac{b-x}{b-a}\right)^{s+1}+\tfrac{1}{(s+2)(s+1)}\right)\left|f'(a)\right|^{q}\right]^{\frac{1}{q}}\right\}\right)$$

for each $x \in [a,b]$.

Motivated by the results cited above, in this paper we introduce the class of strongly $beta$-convex functions and we establish some new Ostrowski's inequalities for functions whose first derivatives in absolute value are strongly $beta$-convex. Several results for its subclasses are also derived.

2 Preliminaries

In this section we recall some concepts of convexity that are well known in the literature.

Definition 2.1 *[9] A set* $I \subseteq \mathbb{R}^{n}$ *is said to be convex if for any* $x,y \in H$, *and* $\forall t \in [0,1]$, *we have*

$$tx+(1-t)\,y \in I.$$

Definition 2.2 *[15] A function* $f : I \to \mathbb{R}$ *is said to be convex, if*

$$f(tx+(1-t)\,y) \leq tf(x)+(1-t)\,f(y)$$

holds for all $x,y \in I$ *and* $t \in [0,1]$.

Definition 2.3 *[16] $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus c if*

$$f\left(tx + (1 - t)\, y\right) \leq tf\left(x\right) + (1 - t)\, f\left(y\right) - ct(1 - t)\left\|x - y\right\|^2$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

Definition 2.4 *[5] A nonnegative function $f : I \to \mathbb{R}$ is said to be P-convex, if*

$$f\left(tx + (1 - t)\, y\right) \leq f\left(x\right) + f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 2.5 *[2] A nonnegative function $f : I \to \mathbb{R}$ is said to be strongly P-convex, if*

$$f\left(tx + (1 - t)\, y\right) \leq f\left(x\right) + f(y) - ct(1 - t)\left\|x - y\right\|^2$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 2.6 *[6] A nonnegative function $f : I \to \mathbb{R}$ is said to be s-Godunova-Levin function, where $s \in [0, 1]$, if*

$$f\left(tx + (1 - t)\, y\right) \leq \frac{f\left(x\right)}{t^s} + \frac{f(y)}{(1 - t)^s}$$

holds for all $x, y \in I$ and all $t \in (0, 1)$.

Definition 2.7 *[20] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. We say that $f : I \to \mathbb{R}$ is tgs-convex function on I if the inequality*

$$f\left(tx + (1 - t)\, y\right) \leq t\left(1 - t\right)\left[f\left(x\right) + f\left(y\right)\right]$$

holds for all $x, y \in I$, and $t \in (0, 1)$.

Definition 2.8 *[21] A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be beta-convex on I, if*

$$f\left(tx + (1 - t)\, y\right) \leq t^p\left(1 - t\right)^q f\left(x\right) + t^q\left(1 - t\right)^p f\left(y\right)$$

holds for all $x, y \in I$, and $t \in [0, 1]$, where $p, q > -1$.

Definition 2.9 *[3] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0, 1]$, if*

$$f(tx + (1 - t)y)\ \leq t^s f(x) + (1 - t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2.10 *[2, 7] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be strongly s-convex in the second sense for some fixed $s \in (0, 1]$, if*

$$f(tx + (1 - t)y)\ \leq t^s f(x) + (1 - t)^s f(y) - ct(1 - t)\left\|x - y\right\|^2$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2.11 *[22] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be extended s-convex for some fixed $s \in [-1, 1]$, if*

$$f(tx + (1 - t)y)\ \leq t^s f(x) + (1 - t)^s f(y)$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

Definition 2.12 *[19] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be strongly extended s-convex for some fixed $s \in [-1,1]$, if*

$$f(tx+(1-t)y) \le t^s f(x) + (1-t)^s f(y) - ct(1-t)\|x-y\|^2$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Definition 2.13 *[4] The incomplete beta function is defined by*

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt,$$

where $x \in [0,1]$ and $\alpha, \beta > 0$.

Lemma 2.1 *[1] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du = (a-b)\int_0^1 p(t)\, f'(ta+(1-t)b)\,dt$$

for each $t \in [0,1]$, where

$$p(t) = \begin{cases} t & \text{if } t \in \left[0, \frac{b-x}{b-a}\right] \\ t-1 & \text{if } t \in \left(\frac{b-x}{b-a}, 1\right] \end{cases},$$

for all $x \in [a,b]$.

3 Main results

Definition 3.1 *A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be strongly beta-convex on I, if*

$$f(tx+(1-t)y) \le t^p(1-t)^q f(x) + t^q(1-t)^p f(y) - ct(1-t)\|x-y\|^2$$

holds for all $x, y \in I$ and $t \in [0,1]$, where $p, q > -1$.

Remark 3.1 The Definition 3.1 recapture the Definition 2.2 for $c = q = 0$ and $p = 1$, Definition 2.3 for $q = 0$ and $p = 1$, Definition 2.4 for $p = q = c = 0$, Definition 2.5 for $p = q = 0$, Definition 2.6 for $p \in (-1,0]$ and $q = c = 0$, Definition 2.7 for $c = 0$ and $p = q = 1$, Definition 2.8 for $c = 0$, Definition 2.9 for $p \in (0,1]$ and $q = c = 0$, Definition 2.10 for $p \in (0,1]$ and $q = 0$, Definition 2.11 for $p \in (-1,1]$ and $q = c = 0$, Definition 2.12 for $p \in (-1,1]$ and $q = 0$.

Definition 3.2 *A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be strongly tgs-convex on I, if*

$$f(tx+(1-t)y) \le t(1-t)(f(x)+f(y)) - ct(1-t)\|x-y\|^2$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Definition 3.3 *A nonnegative function $f : I \to \mathbb{R}$ is said to be strongly s-Godunova-Levin function, where $s \in [0,1]$, if*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s} - ct(1-t)\|x-y\|^2$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Theorem 3.1 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|$ is strongly beta-convex with modulus $c > 0$, then the following inequality*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\leq (b-a)\Big(\Big(B_{\frac{b-x}{b-a}}(p+2,q+1) + B_{\frac{x-a}{b-a}}(q+2,p+1)\Big)\left|f'(a)\right| \\
&\quad + \Big(B_{\frac{b-x}{b-a}}(q+2,p+1) + B_{\frac{x-a}{b-a}}(p+2,q+1)\Big)\left|f'(b)\right| \\
&\quad - \tfrac{c}{(b-a)^3}\Big((b-x)^5\Big(\tfrac{1}{3} - \tfrac{b-x}{4(b-a)}\Big) + (x-a)^5\Big(\tfrac{1}{3} - \tfrac{x-a}{4(b-a)}\Big)\Big)\Big)
\end{aligned}$$

holds for all $x \in [a,b]$, where $\beta_x(.,.)$ is the incomplete beta function and $p, q > -1$.

Proof. From Lemma 2.1, properties of modulus, and strongly $beta$-convexity of $|f'|$, we get

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt + \int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|dt\right) \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t\Big(t^p(1-t)^q\left|f'(a)\right| + t^q(1-t)^p\left|f'(b)\right| - ct(1-t)(b-x)^2\Big)dt\right. \\
&\quad \left. + \int_{\frac{b-x}{b-a}}^1 (1-t)\Big(t^p(1-t)^q\left|f'(a)\right| + t^q(1-t)^p\left|f'(b)\right| - ct(1-t)(x-a)^2\Big)dt\right) \\
&= (b-a)\left(\left|f'(a)\right|\int_0^{\frac{b-x}{b-a}} t^{p+1}(1-t)^q\,dt + \left|f'(b)\right|\int_0^{\frac{b-x}{b-a}} t^{q+1}(1-t)^p\,dt\right. \\
&\quad - c(b-x)^2\int_0^{\frac{b-x}{b-a}} t^2(1-t)\,dt + \left|f'(a)\right|\int_{\frac{b-x}{b-a}}^1 t^p(1-t)^{q+1}\,dt
\end{aligned}$$

$$+\left|f'(b)\right|\int_{\frac{b-x}{b-a}}^{1}t^{q}(1-t)^{p+1}dt-c(x-a)^{2}\int_{\frac{b-x}{b-a}}^{1}t(1-t)^{2}dt\Bigg)$$

$$=(b-a)\left(\left|f'(a)\right|\left(\int_{0}^{\frac{b-x}{b-a}}t^{p+1}(1-t)^{q}dt+\int_{0}^{\frac{x-a}{b-a}}t^{q+1}(1-t)^{p}dt\right)\right.$$

$$+\left|f'(b)\right|\left(\int_{0}^{\frac{b-x}{b-a}}t^{q+1}(1-t)^{p}dt+\int_{0}^{\frac{x-a}{b-a}}t^{p+1}(1-t)^{q}dt\right)$$

$$\left.-c\left((b-x)^{2}\int_{0}^{\frac{b-x}{b-a}}t^{2}(1-t)dt+(x-a)^{2}\int_{0}^{\frac{x-a}{b-a}}(1-t)t^{2}dt\right)\right)$$

$$=(b-a)\Big(\Big(B_{\frac{b-x}{b-a}}(p+2,q+1)+B_{\frac{x-a}{b-a}}(q+2,p+1)\Big)\left|f'(a)\right|$$

$$+\Big(B_{\frac{b-x}{b-a}}(q+2,p+1)+B_{\frac{x-a}{b-a}}(p+2,q+1)\Big)\left|f'(b)\right|$$

$$-\tfrac{c}{(b-a)^{3}}\left((b-x)^{5}\left(\tfrac{1}{3}-\tfrac{b-x}{4(b-a)}\right)+(x-a)^{5}\left(\tfrac{1}{3}-\tfrac{x-a}{4(b-a)}\right)\right)\Big).$$

The proof is completed.

Corollary 3.1 *In Theorem 3.1 if we put $x=\frac{a+b}{2}$, we get*

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|$$

$$\leq(b-a)\Big(\Big(B_{\frac{1}{2}}(p+2,q+1)+B_{\frac{1}{2}}(q+2,p+1)\Big)\left|f'(a)\right|$$

$$+\Big(B_{\frac{1}{2}}(q+2,p+1)+B_{\frac{1}{2}}(p+2,q+1)\Big)\left|f'(b)\right|-\tfrac{5c(b-a)^{2}}{384}\Big).$$

Corollary 3.2 *Under the assumptions of Theorem 3.1 and if $|f'|$ is strongly extended s-convex where $s\in(-1,1]$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|$$

$$\leq(b-a)\left(\left(\tfrac{1}{(s+1)(s+2)}-\tfrac{1}{s+1}\left(\tfrac{b-x}{b-a}\right)^{s+1}+\tfrac{2}{s+2}\left(\tfrac{b-x}{b-a}\right)^{s+2}\right)\left|f'(a)\right|\right.$$

$$+\left(\tfrac{1}{(s+1)(s+2)}-\tfrac{1}{s+1}\left(\tfrac{x-a}{b-a}\right)^{s+1}+\tfrac{2}{s+2}\left(\tfrac{x-a}{b-a}\right)^{s+2}\right)\left|f'(b)\right|$$

$$\left.-\tfrac{c}{(b-a)^{3}}\left((b-x)^{5}\left(\tfrac{1}{3}-\tfrac{b-x}{4(b-a)}\right)+(x-a)^{5}\left(\tfrac{1}{3}-\tfrac{x-a}{4(b-a)}\right)\right)\right).$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq (b-a)\left(\tfrac{1}{(s+1)(s+2)}\left(1-\tfrac{1}{2^{s+1}}\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right) - \tfrac{5c(b-a)^2}{384}\right).$$

Corollary 3.3 *Under the assumptions of Theorem 3.1 and if $|f'|$ is strongly tgs-convex, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq (b-a) \times \left(\left(\frac{4\left(\frac{b-x}{b-a}\right)^3 - 3\left(\frac{b-x}{b-a}\right)^4 + 4\left(\frac{x-a}{b-a}\right)^3 - 3\left(\frac{x-a}{b-a}\right)^4}{12}\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right) - \tfrac{c}{(b-a)^3}\left((b-x)^5\left(\tfrac{1}{3} - \tfrac{b-x}{4(b-a)}\right) + (x-a)^5\left(\tfrac{1}{3} - \tfrac{x-a}{4(b-a)}\right)\right)\right).$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \tfrac{5(b-a)}{96}\left(\left(\left|f'(a)\right| + \left|f'(b)\right|\right) - \tfrac{c(b-a)^2}{4}\right).$$

Corollary 3.4 *Under the assumptions of Theorem 3.1 and if $|f'|$ is strongly convex, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq (b-a)\left(\left(\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3 + \tfrac{1}{2}\left(\tfrac{x-a}{b-a}\right)^2 - \tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3\right)\left|f'(a)\right| + \left(\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2 - \tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3 + \tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3\right)\left|f'(b)\right| - \tfrac{c}{(b-a)^3}\left((b-x)^5\left(\tfrac{1}{3} - \tfrac{b-x}{4(b-a)}\right) + (x-a)^5\left(\tfrac{1}{3} - \tfrac{x-a}{4(b-a)}\right)\right)\right).$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \tfrac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right| - \tfrac{5c(b-a)^2}{48}\right).$$

Corollary 3.5 *Under the assumptions of Theorem 3.1 and if $|f'|$ is strongly P function*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \tfrac{b-a}{2}\left(\left(\left(\tfrac{b-x}{b-a}\right)^2 + \left(\tfrac{x-a}{b-a}\right)^2\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)\right.$$

$$- \tfrac{c}{(b-a)^3}\left((b-x)^5\left(\tfrac{1}{3}-\tfrac{b-x}{4(b-a)}\right)+(x-a)^5\left(\tfrac{1}{3}-\tfrac{x-a}{4(b-a)}\right)\right)\Bigg)\Bigg).$$

Moreover if we choose $x=\frac{a+b}{2}$, *we obtain*

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq \tfrac{b-a}{4}\left(\left|f'(a)\right|+\left|f'(b)\right|-\tfrac{5c(b-a)^2}{192}\right).$$

Corollary 3.6 *under the conditions of Theorem 3.1 and if* $|f'|$ *is beta-convex one has*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\leq (b-a)\left(\left(B_{\frac{b-x}{b-a}}(p+2,q+1)+B_{\frac{x-a}{b-a}}(q+2,p+1)\right)\left|f'(a)\right|\right.\\ &\quad\left.+\left(B_{\frac{b-x}{b-a}}(q+2,p+1)+B_{\frac{x-a}{b-a}}(p+2,q+1)\right)\left|f'(b)\right|\right).\end{aligned}$$

Moreover if we choose $x=\frac{a+b}{2}$, *we get*

$$\begin{aligned}&\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\leq (b-a)\left(\left(B_{\frac{1}{2}}(p+2,q+1)+B_{\frac{1}{2}}(q+2,p+1)\right)\left|f'(a)\right|\right.\\ &\quad\left.+\left(B_{\frac{1}{2}}(q+2,p+1)+B_{\frac{1}{2}}(p+2,q+1)\right)\left|f'(b)\right|\right).\end{aligned}$$

Corollary 3.7 *Under the assumptions of Theorem 3.1 and if* $|f'|$ *is extended* s*-convex, we have*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\leq (b-a)\left(\left(\tfrac{1}{(s+1)(s+2)}-\tfrac{1}{s+1}\left(\tfrac{b-x}{b-a}\right)^{s+1}+\tfrac{2}{s+2}\left(\tfrac{b-x}{b-a}\right)^{s+2}\right)\left|f'(a)\right|\right.\\ &\quad\left.+\left(\tfrac{1}{(s+1)(s+2)}-\tfrac{1}{s+1}\left(\tfrac{x-a}{b-a}\right)^{s+1}+\tfrac{2}{s+2}\left(\tfrac{x-a}{b-a}\right)^{s+2}\right)\left|f'(b)\right|\right).\end{aligned}$$

Moreover if we choose $x=\frac{a+b}{2}$, *we obtain*

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq \tfrac{b-a}{(s+1)(s+2)}\left(1-\tfrac{1}{2^{s+1}}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Remark 3.2 Corollary 3.7 will be reduced to Theorem 2.1 from [17] if we assume that $s\in(0,1]$. Moreover if we take $x=\frac{a+b}{2}$ we obtain Corollary 2.1 from [17].

Corollary 3.8 *Under the assumptions of Theorem 3.1 and if $|f'|$ is tgs-convex, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \tfrac{b-a}{12}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$
$$\times\left(4\left(\tfrac{b-x}{b-a}\right)^3 - 3\left(\tfrac{b-x}{b-a}\right)^4 + 4\left(\tfrac{x-a}{b-a}\right)^3 - 3\left(\tfrac{x-a}{b-a}\right)^4\right).$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \tfrac{5(b-a)}{96}\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Corollary 3.9 *Under the assumptions of Theorem 3.1 and if $|f'|$ is convex, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq (b-a)\left(\left(\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3 + \tfrac{1}{2}\left(\tfrac{x-a}{b-a}\right)^2 - \tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3\right)\left|f'(a)\right|\right.$$
$$\left. + \left(\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2 - \tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3 + \tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3\right)\left|f'(b)\right|\right).$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain Theorem 2.2 from [8].

Corollary 3.10 *Under the assumptions of Theorem 3.1 and if $|f'|$ is P function*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \tfrac{b-a}{2}\left(\left(\tfrac{b-x}{b-a}\right)^2 + \left(\tfrac{x-a}{b-a}\right)^2\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \tfrac{b-a}{4}\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Theorem 3.2 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$, and $f' \in L[a,b]$, and let $\mu > 1$ with $\frac{1}{\lambda} + \frac{1}{\mu} = 1$. If $|f'|^\mu$ strongly $beta$-convex with modulus $c > 0$, then the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right|$$
$$\leq \tfrac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}}\left(\left(\tfrac{b-x}{b-a}\right)^{1+\frac{1}{\lambda}}\left(\left|f'(a)\right|^\mu \beta_{\frac{b-x}{b-a}}(p+1,q+1) + \left|f'(b)\right|^\mu \beta_{\frac{b-x}{b-a}}(q+1,p+1)\right.\right.$$

$$- c\tfrac{(b-x)^4}{(b-a)^2}\left(\tfrac{1}{2}-\tfrac{b-x}{3(b-a)}\right)\Big)^{\frac{1}{\mu}}$$
$$+\left(\tfrac{x-a}{b-a}\right)^{1+\frac{1}{\lambda}}\Big(\left|f'(a)\right|^{\mu}\beta_{\frac{x-a}{b-a}}(q+1,p+1)+\left|f'(b)\right|^{\mu}\beta_{\frac{x-a}{b-a}}(p+1,q+1)$$
$$- c\tfrac{(x-a)^4}{(b-a)^2}\left(\tfrac{1}{2}-\tfrac{x-a}{3(b-a)}\right)\Big)^{\frac{1}{\mu}}\Bigg)$$

holds for all $x\in[a,b]$, where $\beta_x(.,.)$ is the incomplete beta function and $p,q>-1$.

Proof. From Lemma 2.1, properties of modulus, and Hölder inequality, we get

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq (b-a)\left(\left(\int_0^{\frac{b-x}{b-a}} t^{\lambda}dt\right)^{\frac{1}{\lambda}}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right. \\
&\quad\left.+\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda}dt\right)^{\frac{1}{\lambda}}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right) \\
&=\tfrac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}}\left(\left(\frac{b-x}{b-a}\right)^{1+\frac{1}{\lambda}}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right. \\
&\quad\left.+\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{\lambda}}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right). \qquad (3.1)
\end{aligned}$$

Since $|f'|^{\mu}$ is strongly $beta$-convex, we deduce

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq\tfrac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}}\left(\left(\tfrac{b-x}{b-a}\right)^{1+\frac{1}{\lambda}}\right. \\
&\quad\times\left(\int_0^{\frac{b-x}{b-a}}\left(t^p(1-t)^q\left|f'(a)\right|^{\mu}+t^q(1-t)^p\left|f'(b)\right|^{\mu}-ct(1-t)(b-x)^2\right)dt\right)^{\frac{1}{\mu}} \\
&\quad+\left(\tfrac{x-a}{b-a}\right)^{1+\frac{1}{\lambda}}
\end{aligned}$$

$$\times \left(\int_0^{\frac{x-a}{b-a}} \left(t^q (1-t)^p \left|f'(a)\right|^\mu + t^p (1-t)^q \left|f'(b)\right|^\mu - ct(1-t)(x-a)^2 \right) dt \right)^{\frac{1}{\mu}} \Bigg)$$

$$= \frac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}} \left(\left(\frac{b-x}{b-a} \right)^{1+\frac{1}{\lambda}} \left(\left|f'(a)\right|^\mu \beta_{\frac{b-x}{b-a}}(p+1,q+1) + \left|f'(b)\right|^\mu \beta_{\frac{b-x}{b-a}}(q+1,p+1) \right.\right.$$

$$\left. - c\frac{(b-x)^4}{(b-a)^2}\left(\frac{1}{2} - \frac{b-x}{3(b-a)}\right)\right)^{\frac{1}{\mu}}$$

$$+ \left(\frac{x-a}{b-a}\right)^{1+\frac{1}{\lambda}} \left(\left|f'(a)\right|^\mu \beta_{\frac{x-a}{b-a}}(q+1,p+1) + \left|f'(b)\right|^\mu \beta_{\frac{x-a}{b-a}}(p+1,q+1) \right.$$

$$\left.\left. - c\frac{(x-a)^4}{(b-a)^2}\left(\frac{1}{2} - \frac{x-a}{3(b-a)}\right)\right)^{\frac{1}{\mu}}\right).$$

The proof is completed.

Corollary 3.11 *In Theorem 3.2 if we put $x = \frac{a+b}{2}$, we get*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le \frac{b-a}{2^{\frac{\lambda+1}{\lambda}}(\lambda+1)^{\frac{1}{\lambda}}}$$

$$\times \left(\left(\left|f'(a)\right|^\mu \beta_{\frac{1}{2}}(p+1,q+1) + \left|f'(b)\right|^\mu \beta_{\frac{1}{2}}(q+1,p+1) - c\frac{(b-a)^2}{48} \right)^{\frac{1}{\mu}} \right.$$

$$\left. + \left(\left|f'(a)\right|^\mu \beta_{\frac{1}{2}}(q+1,p+1) + \left|f'(b)\right|^\mu \beta_{\frac{1}{2}}(p+1,q+1) - c\frac{(b-a)^2}{48} \right)^{\frac{1}{\mu}} \right).$$

Corollary 3.12 *Under the assumptions of Theorem 3.2 and if $|f'|$ is strongly extended s-convex where $s \in (-1,1]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right|$$

$$\le \frac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}} \left(\left(\frac{b-x}{b-a}\right)^{\frac{\lambda+1}{\lambda}} \left(\frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1}\left|f'(a)\right|^\mu + \frac{1}{s+1}\left(1 - \left(\frac{x-a}{b-a}\right)^{1+s}\right)\left|f'(b)\right|^\mu \right.\right.$$

$$\left. - c\frac{(b-x)^4}{(b-a)^2}\left(\frac{1}{2} - \frac{b-x}{3(b-a)}\right)\right)^{\frac{1}{\mu}}$$

$$+ \left(\frac{x-a}{b-a}\right)^{\frac{\lambda+1}{\lambda}} \left(\frac{1}{s+1}\left(1 - \left(\frac{b-x}{b-a}\right)^{1+s}\right)\left|f'(a)\right|^\mu + \frac{1}{s+1}\left(\frac{x-a}{b-a}\right)^{s+1}\left|f'(b)\right|^\mu \right.$$

$$\left.\left. - c\frac{(x-a)^4}{(b-a)^2}\left(\frac{1}{2} - \frac{x-a}{3(b-a)}\right)\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}2^{\frac{\lambda+1}{\lambda}}}$$

$$\times \left(\left(\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\left|f'(a)\right|^\mu + \frac{1}{s+1}\left(1 - \left(\frac{1}{2}\right)^{1+s}\right)\left|f'(b)\right|^\mu - c\frac{(b-a)^2}{48} \right)^{\frac{1}{\mu}} \right.$$

$$+\left(\tfrac{1}{s+1}\left(1-\left(\tfrac{1}{2}\right)^{1+s}\right)\left|f'(a)\right|^{\mu}+\tfrac{1}{s+1}\left(\tfrac{1}{2}\right)^{s+1}\left|f'(b)\right|^{\mu}-c\tfrac{(b-a)^{2}}{48}\right)^{\frac{1}{\mu}}\Bigg).$$

Corollary 3.13 *Under the assumptions of Theorem 3.2 and if $|f'|$ is strongly tgs-convex, we have*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|$$
$$\leq\tfrac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}}\left(\left(\tfrac{b-x}{b-a}\right)^{2+\frac{1}{\mu}}\left(\tfrac{1}{2}-\tfrac{b-x}{3(b-a)}\right)^{\frac{1}{\mu}}\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}-c\,(b-x)^{2}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\tfrac{x-a}{b-a}\right)^{2+\frac{1}{\mu}}\left(\tfrac{1}{2}-\tfrac{x-a}{3(b-a)}\right)^{\frac{1}{\mu}}\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}-c\,(x-a)^{2}\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\leq\tfrac{b-a}{2^{1+\frac{1}{\mu}}(\lambda+1)^{\frac{1}{\lambda}}}\left(\tfrac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{3}-\tfrac{c(b-a)^{2}}{12}\right)^{\frac{1}{\mu}}.$$

Corollary 3.14 *Under the assumptions of Theorem 3.2 and if $|f'|$ is strongly convex, we have*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|$$
$$\leq\tfrac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}}\left(\left(\tfrac{b-x}{b-a}\right)^{1+\frac{1}{\lambda}}\left(\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^{2}\left|f'(a)\right|^{\mu}+\tfrac{1}{2}\left(\tfrac{x-a}{b-a}\right)^{2}\left|f'(b)\right|^{\mu}\right.\right.$$
$$\left.-\,c\tfrac{(b-x)^{4}}{(b-a)^{2}}\left(\tfrac{1}{2}-\tfrac{b-x}{3(b-a)}\right)\right)^{\frac{1}{\mu}}$$
$$+\left(\tfrac{x-a}{b-a}\right)^{1+\frac{1}{\lambda}}\left(\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^{2}\left|f'(a)\right|^{\mu}+\tfrac{1}{2}\left(\tfrac{x-a}{b-a}\right)^{2}\left|f'(b)\right|^{\mu}\right.$$
$$\left.\left.-\,c\tfrac{(x-a)^{4}}{(b-a)^{2}}\left(\tfrac{1}{2}-\tfrac{x-a}{3(b-a)}\right)\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|$$
$$\leq\tfrac{b-a}{2^{1+\frac{2}{\mu}}(\lambda+1)^{\frac{1}{\lambda}}}\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}-c\tfrac{(b-a)^{2}}{3}\right)^{\frac{1}{\mu}}.$$

Corollary 3.15 *Under the assumptions of Theorem 3.2 and if $|f'|$ is strongly P function*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|$$

$$\leq \frac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}}\left(\left(\frac{b-x}{b-a}\right)^{2}\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}-c\frac{(b-x)^{3}}{b-a}\left(\frac{1}{2}-\frac{b-x}{3(b-a)}\right)\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\frac{x-a}{b-a}\right)^{2}\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}-c\frac{(x-a)^{3}}{b-a}\left(\frac{1}{2}-\frac{x-a}{3(b-a)}\right)\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|$$
$$\leq \frac{b-a}{2(\lambda+1)^{\frac{1}{\lambda}}}\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}-c\frac{(b-a)^{3}}{24}\right)^{\frac{1}{\mu}}.$$

Corollary 3.16 *under the conditions of Theorem 3.2 and if $|f'|$ is $beta$-convex one has*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|\leq \frac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}}$$
$$\times\left(\left(\frac{b-x}{b-a}\right)^{1+\frac{1}{\lambda}}\left(\left|f'(a)\right|^{\mu}\beta_{\frac{b-x}{b-a}}(p+1,q+1)+\left|f'(b)\right|^{\mu}\beta_{\frac{b-x}{b-a}}(q+1,p+1)\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{\lambda}}\left(\left|f'(a)\right|^{\mu}\beta_{\frac{x-a}{b-a}}(q+1,p+1)+\left|f'(b)\right|^{\mu}\beta_{\frac{x-a}{b-a}}(p+1,q+1)\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x=\frac{a+b}{2}$, we get

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|\leq \frac{b-a}{2^{\frac{\lambda+1}{\lambda}}(\lambda+1)^{\frac{1}{\lambda}}}$$
$$\times\left(\left(\left|f'(a)\right|^{\mu}\beta_{\frac{1}{2}}(p+1,q+1)+\left|f'(b)\right|^{\mu}\beta_{\frac{1}{2}}(q+1,p+1)\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\left|f'(a)\right|^{\mu}\beta_{\frac{1}{2}}(q+1,p+1)+\left|f'(b)\right|^{\mu}\beta_{\frac{1}{2}}(p+1,q+1)\right)^{\frac{1}{\mu}}\right).$$

Corollary 3.17 *Under the assumptions of Theorem 3.2 and if $|f'|$ is extended s-convex where $s\in(-1,1]$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)\,du\right|$$
$$\leq \frac{b-a}{(s+1)^{\frac{1}{\mu}}(\lambda+1)^{\frac{1}{\lambda}}}\left(\left(\frac{b-x}{b-a}\right)^{1+\frac{1}{\lambda}}\left(\left(\frac{b-x}{b-a}\right)^{s+1}\left|f'(a)\right|^{\mu}+\left(1-\left(\frac{x-a}{b-a}\right)^{1+s}\right)\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{\lambda}}\left(\left(1-\left(\frac{b-x}{b-a}\right)^{1+s}\right)\left|f'(a)\right|^{\mu}+\left(\frac{x-a}{b-a}\right)^{s+1}\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\leq \frac{b-a}{(s+1)^{\frac{1}{\mu}}(\lambda+1)^{\frac{1}{\lambda}}2^{1+\frac{1}{\lambda}}}$$

$$\times\left(\left(\left(\tfrac{1}{2}\right)^{s+1}\left|f'(a)\right|^{\mu}+\left(1-\left(\tfrac{1}{2}\right)^{1+s}\right)\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\left(1-\left(\tfrac{1}{2}\right)^{1+s}\right)\left|f'(a)\right|^{\mu}+\left(\tfrac{1}{2}\right)^{s+1}\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}\right).$$

Remark 3.3 Corollary 3.17 will be reduced to Theorem 2.2 from [17] if we assume that $s\in(0,1]$.

Corollary 3.18 *Under the assumptions of Theorem 3.2 and if $|f'|$ is tgs-convex, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\tfrac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}}\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}$$
$$\times\left(\left(\tfrac{b-x}{b-a}\right)^{2+\frac{1}{\mu}}\left(\tfrac{1}{2}-\tfrac{b-x}{3(b-a)}\right)^{\frac{1}{\mu}}+\left(\tfrac{x-a}{b-a}\right)^{2+\frac{1}{\mu}}\left(\tfrac{1}{2}-\tfrac{x-a}{3(b-a)}\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\tfrac{b-a}{2(\lambda+1)^{\frac{1}{\lambda}}}\left(\tfrac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{6}\right)^{\frac{1}{\mu}}.$$

Corollary 3.19 *Under the assumptions of Theorem 3.2 and if $|f'|$ is convex, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\tfrac{b-a}{2^{\frac{1}{\mu}}(\lambda+1)^{\frac{1}{\lambda}}}\left(\left(\tfrac{b-x}{b-a}\right)^{1+\frac{1}{\lambda}}+\left(\tfrac{x-a}{b-a}\right)^{1+\frac{1}{\lambda}}\right)$$
$$\left(\left(\tfrac{b-x}{b-a}\right)^{2}\left|f'(a)\right|^{\mu}+\left(\tfrac{x-a}{b-a}\right)^{2}\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}.$$

Moreover if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\tfrac{b-a}{2^{1+\frac{2}{\mu}}(\lambda+1)^{\frac{1}{\lambda}}}\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}.$$

Corollary 3.20 *Under the assumptions of Theorem 3.2 and if $|f'|$ is P function*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq\tfrac{b-a}{(\lambda+1)^{\frac{1}{\lambda}}}\left(\left(\tfrac{b-x}{b-a}\right)^{2}+\left(\tfrac{x-a}{b-a}\right)^{2}\right)\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}.$$

Moreover if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\tfrac{b-a}{2(\lambda+1)^{\frac{1}{\lambda}}}\left(\left|f'(a)\right|^{\mu}+\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}.$$

Theorem 3.3 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$, and $f' \in L[a,b]$, and let $\mu \geq 1$. If $|f'|^q$ strongly $beta$-convex with modulus $c > 0$, then the following inequality holds :*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right|$$
$$\leq \frac{b-a}{2^{1-\frac{1}{\mu}}} \left(\left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(B_{\frac{b-x}{b-a}}(p+2,q+1)\left|f'(a)\right|^{\mu} \right.\right.$$
$$\left. + B_{\frac{b-x}{b-a}}(q+2,p+1)\left|f'(b)\right|^{\mu} - c(b-x)^2 \left(\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3 - \frac{1}{4}\left(\frac{b-x}{b-a}\right)^4 \right)\right)^{\frac{1}{\mu}}$$
$$+ \left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(B_{\frac{x-a}{b-a}}(q+2,p+1)\left|f'(a)\right|^{\mu} \right.$$
$$\left.\left. + B_{\frac{x-a}{b-a}}(p+2,q+1)\left|f'(b)\right|^{\mu} - c(x-a)^2 \left(\frac{1}{3}\left(\frac{x-a}{b-a}\right)^3 - \frac{1}{4}\left(\frac{x-a}{b-a}\right)^4 \right)\right)^{\frac{1}{\mu}} \right)$$

holds for all $x \in [a,b]$, where $\beta_x(.,.)$ is the incomplete beta function and $p, q > -1$.

Proof. From Lemma 2.1, properties of modulus, and power mean inequality, we get

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right|$$
$$\leq (b-a)\left(\left(\int_0^{\frac{b-x}{b-a}} t\,dt \right)^{1-\frac{1}{\mu}} \left(\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^{\mu} dt \right)^{\frac{1}{\mu}} \right.$$
$$\left. + \left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt \right)^{1-\frac{1}{\mu}} \left(\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^{\mu} dt \right)^{\frac{1}{\mu}} \right)$$
$$\leq \frac{b-a}{2^{1-\frac{1}{\mu}}} \left(\left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^{\mu} dt \right)^{\frac{1}{\mu}} \right.$$
$$\left. + \left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^{\mu} dt \right)^{\frac{1}{\mu}} \right).$$

Since $|f'|^{\mu}$ is strongly $beta$-convex, we deduce

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right|$$

$$\leq \frac{b-a}{2^{1-\frac{1}{\mu}}} \left(\left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(\int_{0}^{\frac{b-x}{b-a}} t \left(t^{p} (1-t)^{q} \left|f'(a)\right|^{\mu} \right.\right.\right.$$
$$\left.\left. + \ t^{q} (1-t)^{p} \left|f'(b)\right|^{\mu} - ct(1-t)(b-x)^{2} \right) dt \right)^{\frac{1}{\mu}}$$
$$+ \left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(\int_{\frac{b-x}{b-a}}^{1} (1-t) \left(t^{p} (1-t)^{q} \left|f'(a)\right|^{\mu} \right.\right.$$
$$\left.\left.\left. + \ t^{q} (1-t)^{p} \left|f'(b)\right|^{\mu} - ct(1-t)(x-a)^{2} \right) dt \right)^{\frac{1}{\mu}} \right)$$
$$= \frac{b-a}{2^{1-\frac{1}{\mu}}} \left(\left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(\left|f'(a)\right|^{\mu} \int_{0}^{\frac{b-x}{b-a}} t^{p+1} (1-t)^{q} dt \right.\right.$$
$$\left. + \ \left|f'(b)\right|^{\mu} \int_{0}^{\frac{b-x}{b-a}} t^{q+1} (1-t)^{p} dt - c(b-x)^{2} \int_{0}^{\frac{b-x}{b-a}} t^{2} (1-t) dt \right)^{\frac{1}{q}}$$
$$+ \left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(\left|f'(a)\right|^{\mu} \int_{0}^{\frac{x-a}{b-a}} t^{q+1} (1-t)^{p} dt \right.$$
$$\left.\left. + \ \left|f'(b)\right|^{\mu} \int_{0}^{\frac{x-a}{b-a}} t^{p+1} (1-t)^{q} dt - c(x-a)^{2} \int_{0}^{\frac{x-a}{b-a}} t^{2} (1-t) dt \right)^{\frac{1}{\mu}} \right)$$
$$= \frac{b-a}{2^{1-\frac{1}{\mu}}} \left(\left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(B_{\frac{b-x}{b-a}} (p+2, q+1) \left|f'(a)\right|^{\mu} + B_{\frac{b-x}{b-a}} (q+2, p+1) \left|f'(b)\right|^{\mu} \right.\right.$$
$$\left. - \ c(b-x)^{2} \left(\frac{1}{3} \left(\frac{b-x}{b-a}\right)^{3} - \frac{1}{4} \left(\frac{b-x}{b-a}\right)^{4} \right) \right)^{\frac{1}{\mu}}$$
$$+ \left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)} \left(B_{\frac{x-a}{b-a}} (q+2, p+1) \left|f'(a)\right|^{\mu} + B_{\frac{x-a}{b-a}} (p+2, q+1) \left|f'(b)\right|^{\mu} \right.$$
$$\left.\left. - \ c(x-a)^{2} \left(\frac{1}{3} \left(\frac{x-a}{b-a}\right)^{3} - \frac{1}{4} \left(\frac{x-a}{b-a}\right)^{4} \right) \right)^{\frac{1}{\mu}} \right).$$

The proof is completed.

Corollary 3.21 *In Theorem 3.3 if we put* $x = \frac{a+b}{2}$*, we get*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_{a}^{b} f(u)\, du \right|$$

$$\leq \frac{b-a}{2^{3-\frac{3}{\mu}}}\left(\left(B_{\frac{1}{2}}\left(p+2,q+1\right)\left|f'\left(a\right)\right|^{\mu}+B_{\frac{1}{2}}\left(q+2,p+1\right)\left|f'\left(b\right)\right|^{\mu}-\frac{5c(b-a)^{2}}{768}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(B_{\frac{1}{2}}\left(q+2,p+1\right)\left|f'\left(a\right)\right|^{\mu}+B_{\frac{1}{2}}\left(p+2,q+1\right)\left|f'\left(b\right)\right|^{\mu}-\frac{5c(b-a)^{2}}{768}\right)^{\frac{1}{\mu}}\right).$$

Corollary 3.22 *Under the assumptions of Theorem 3.3 and if $|f'|$ is strongly extended s-convex where $s\in(-1,1]$, we have*

$$\left|f\left(x\right)-\frac{1}{b-a}\int_{a}^{b}f\left(u\right)du\right|$$
$$\leq \frac{b-a}{2^{1-\frac{1}{\mu}}}\left(\left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)}\left(\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\left|f'\left(a\right)\right|^{\mu}+\left(\frac{x-a}{b-a}\right)^{s+1}\left(\frac{1}{s+1}-\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)\right)\right.\right.$$
$$\left.\times\left|f'\left(b\right)\right|^{\mu}-c\left(b-x\right)^{2}\left(\frac{1}{3}\left(\frac{b-x}{b-a}\right)^{3}-\frac{1}{4}\left(\frac{b-x}{b-a}\right)^{4}\right)\right)^{\frac{1}{\mu}}$$
$$+\left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)}\left(\left(\frac{b-x}{b-a}\right)^{s+1}\left(\frac{1}{s+1}-\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)\right)\left|f'\left(a\right)\right|^{\mu}\right.$$
$$\left.\left.+\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}\left|f'\left(b\right)\right|^{\mu}-c\left(x-a\right)^{2}\left(\frac{1}{3}\left(\frac{x-a}{b-a}\right)^{3}-\frac{1}{4}\left(\frac{x-a}{b-a}\right)^{4}\right)\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f\left(u\right)du\right|\leq\frac{b-a}{2^{3\left(1-\frac{1}{\mu}\right)}}$$
$$\times\left(\left(\frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}\left|f'\left(a\right)\right|^{\mu}+\frac{s+3}{(s+1)(s+2)}\left(\frac{1}{2}\right)^{s+2}\left|f'\left(b\right)\right|^{\mu}-\frac{c(b-a)^{2}}{768}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\frac{s+3}{(s+1)(s+2)}\left(\frac{1}{2}\right)^{s+2}\left|f'\left(a\right)\right|^{\mu}+\frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}\left|f'\left(b\right)\right|^{\mu}-\frac{c(b-a)^{2}}{768}\right)^{\frac{1}{\mu}}\right).$$

Corollary 3.23 *Under the assumptions of Theorem 3.3 and if $|f'|$ is strongly tgs-convex, we have*

$$\left|f\left(x\right)-\frac{1}{b-a}\int_{a}^{b}f\left(u\right)du\right|$$
$$\leq \frac{b-a}{2^{1-\frac{1}{\mu}}}\left(\left(\frac{b-x}{b-a}\right)^{\left(2-\frac{2}{\mu}\right)}\left(\left|f'\left(a\right)\right|^{\mu}+\left|f'\left(b\right)\right|^{\mu}-c\left(b-x\right)^{2}\right)^{\frac{1}{\mu}}\right.$$
$$\times\left(\frac{1}{3}\left(\frac{b-x}{b-a}\right)^{3}-\frac{1}{4}\left(\frac{b-x}{b-a}\right)^{4}\right)^{\frac{1}{\mu}}+\left(\frac{x-a}{b-a}\right)^{\left(2-\frac{2}{\mu}\right)}$$
$$\left.\times\left(\left|f'\left(a\right)\right|^{\mu}+\left|f'\left(b\right)\right|^{\mu}-c\left(x-a\right)^{2}\right)^{\frac{1}{\mu}}\left(\frac{1}{3}\left(\frac{x-a}{b-a}\right)^{3}-\frac{1}{4}\left(\frac{x-a}{b-a}\right)^{4}\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \tfrac{b-a}{4}\left(\tfrac{5}{24}\right)^{\frac{1}{\mu}}\left(\left|f'(a)\right|^{\mu} + \left|f'(b)\right|^{\mu} - \tfrac{c(b-a)^2}{4}\right)^{\frac{1}{\mu}}.$$

Corollary 3.24 *Under the assumptions of Theorem 3.3 and if $|f'|$ is strongly convex, we have*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
\leq& \tfrac{b-a}{2^{1-\frac{1}{\mu}}}\Bigg(\left(\tfrac{b-x}{b-a}\right)^2\Bigg(\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)\left|f'(a)\right|^{\mu} + \left(\tfrac{1}{2} - \tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)\right)\left|f'(b)\right|^{\mu} \\
&- c(b-x)^2\left(\tfrac{b-x}{b-a}\right)\left(\tfrac{1}{3} - \tfrac{1}{4}\left(\tfrac{b-x}{b-a}\right)\right)\Bigg)^{\frac{1}{\mu}} \\
&+ \left(\tfrac{x-a}{b-a}\right)^2\Bigg(\left(\tfrac{1}{2} - \tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)\right)\left|f'(a)\right|^{\mu} + \tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)\left|f'(b)\right|^{\mu} \\
&- c(x-a)^2\left(\tfrac{x-a}{b-a}\right)\left(\tfrac{1}{3} - \tfrac{1}{4}\left(\tfrac{x-a}{b-a}\right)\right)\Bigg)^{\frac{1}{\mu}}\Bigg).
\end{aligned}$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain

$$\begin{aligned}
&\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \tfrac{b-a}{8} \\
&\times\left(\left(\tfrac{|f'(a)|^{\mu}+2|f'(b)|^{\mu}}{3} - \tfrac{c5(b-a)^2}{96}\right)^{\frac{1}{\mu}} + \left(\tfrac{2|f'(a)|^{\mu}+|f'(b)|^{\mu}}{3} - \tfrac{c5(b-a)^2}{96}\right)^{\frac{1}{\mu}}\right).
\end{aligned}$$

Corollary 3.25 *Under the assumptions of Theorem 3.3 and if $|f'|$ is strongly P function*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
\leq& \tfrac{b-a}{2^{1-\frac{1}{\mu}}}\Bigg(\left(\tfrac{b-x}{b-a}\right)^2\left(\tfrac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2} - \tfrac{c(b-x)^3}{b-a}\left(\tfrac{1}{3} - \tfrac{1}{4}\left(\tfrac{b-x}{b-a}\right)\right)\right)^{\frac{1}{\mu}} \\
&+ \left(\tfrac{x-a}{b-a}\right)^2\left(\tfrac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2} - \tfrac{c(x-a)^3}{b-a}\left(\tfrac{1}{3} - \tfrac{1}{4}\left(\tfrac{x-a}{b-a}\right)\right)\right)^{\frac{1}{\mu}}\Bigg).
\end{aligned}$$

Moreover if we choose $x = \frac{a+b}{2}$, we obtain

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \tfrac{b-a}{4}\left(\left|f'(a)\right|^{\mu} + \left|f'(b)\right|^{\mu} - \tfrac{c5(b-a)^2}{96}\right)^{\frac{1}{\mu}}.$$

Corollary 3.26 *under the conditions of Theorem 3.3 and if $|f'|$ is* beta*-convex one has*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \tfrac{b-a}{2^{1-\frac{1}{\mu}}}$$

$$\times\left(\left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)}\left(B_{\frac{b-x}{b-a}}\left(p+2,q+1\right)\left|f'\left(a\right)\right|^{\mu}+B_{\frac{b-x}{b-a}}\left(q+2,p+1\right)\left|f'\left(b\right)\right|^{\mu}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)}\left(B_{\frac{x-a}{b-a}}\left(q+2,p+1\right)\left|f'\left(a\right)\right|^{\mu}+B_{\frac{x-a}{b-a}}\left(p+2,q+1\right)\left|f'\left(b\right)\right|^{\mu}\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x=\frac{a+b}{2}$, we get

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f\left(u\right)du\right|$$
$$\leq\frac{b-a}{2^{3-\frac{3}{\mu}}}\left(\left(B_{\frac{1}{2}}\left(p+2,q+1\right)\left|f'\left(a\right)\right|^{\mu}+B_{\frac{1}{2}}\left(q+2,p+1\right)\left|f'\left(b\right)\right|^{\mu}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(B_{\frac{1}{2}}\left(q+2,p+1\right)\left|f'\left(a\right)\right|^{\mu}+B_{\frac{1}{2}}\left(p+2,q+1\right)\left|f'\left(b\right)\right|^{\mu}\right)^{\frac{1}{\mu}}\right).$$

Corollary 3.27 *Under the assumptions of Theorem 3.3 and if $|f'|$ is extended s-convex where $s\in(-1,1]$, we have*

$$\left|f\left(x\right)-\frac{1}{b-a}\int_a^b f\left(u\right)du\right|$$
$$\leq\frac{b-a}{2^{1-\frac{1}{\mu}}}\left(\left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)}\right.$$
$$\times\left(\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\left|f'\left(a\right)\right|^{\mu}+\left(\frac{x-a}{b-a}\right)^{s+1}\left(\frac{1}{s+1}-\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)\right)\left|f'\left(b\right)\right|^{\mu}\right)^{\frac{1}{\mu}}$$
$$+\left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{\mu}\right)}$$
$$\left.\times\left(\left(\frac{b-x}{b-a}\right)^{s+1}\left(\frac{1}{s+1}-\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)\right)\left|f'\left(a\right)\right|^{\mu}+\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}\left|f'\left(b\right)\right|^{\mu}\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|f\left(\tfrac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f\left(u\right)du\right|\leq\frac{b-a}{2^{3-\frac{1}{\mu}+\frac{s}{\mu}}}$$
$$\times\left(\left(\frac{(s+1)|f'(a)|^{\mu}+(s+3)|f'(b)|^{\mu}}{(s+1)(s+2)}\right)^{\frac{1}{\mu}}+\left(\frac{(s+3)|f'(a)|^{\mu}+(s+1)|f'(b)|^{\mu}}{(s+1)(s+2)}\right)^{\frac{1}{\mu}}\right).$$

Corollary 3.28 *Under the assumptions of Theorem 3.3 and if $|f'|$ is tgs-convex, we have*

$$\left|f\left(x\right)-\frac{1}{b-a}\int_a^b f\left(u\right)du\right|\leq\frac{b-a}{2^{1-\frac{1}{\mu}}}\left(\left|f'\left(a\right)\right|^{\mu}+\left|f'\left(b\right)\right|^{\mu}\right)^{\frac{1}{\mu}}$$
$$\times\left(\left(\frac{b-x}{b-a}\right)^{2+\frac{1}{\mu}}\left(\frac{1}{3}-\frac{1}{4}\left(\frac{b-x}{b-a}\right)\right)^{\frac{1}{\mu}}+\left(\frac{x-a}{b-a}\right)^{2+\frac{1}{\mu}}\left(\frac{1}{3}-\frac{1}{4}\left(\frac{x-a}{b-a}\right)\right)^{\frac{1}{\mu}}\right).$$

Moreover if we choose $x = \frac{a+b}{2}$, *we obtain*

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \tfrac{b-a}{4}\left(\tfrac{5}{24}\right)^{\frac{1}{\mu}}\left(\left|f'(a)\right|^{\mu} + \left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}.$$

Corollary 3.29 *Under the assumptions of Theorem 3.3 and if* $|f'|$ *is convex, we have*

$$\begin{aligned}&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ &\leq \tfrac{b-a}{2^{1-\frac{1}{\mu}}}\left(\left(\tfrac{b-x}{b-a}\right)^2\left(\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)\left|f'(a)\right|^{\mu} + \left(\tfrac{1}{2} - \tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)\right)\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}\right. \\ &\quad \left. + \left(\tfrac{x-a}{b-a}\right)^2\left(\left(\left(\tfrac{1}{2} - \tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)\right)\left|f'(a)\right|^{\mu} + \tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)\left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}\right)\right).\end{aligned}$$

Moreover if we choose $x = \frac{a+b}{2}$, *we obtain*

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \tfrac{b-a}{8}\left(\left(\tfrac{|f'(a)|^{\mu}+2|f'(b)|^{\mu}}{3}\right)^{\frac{1}{\mu}} + \left(\tfrac{2|f'(a)|^{\mu}+|f'(b)|^{\mu}}{3}\right)^{\frac{1}{\mu}}\right).$$

Remark 3.4 The second inequality of the corollary 29 is the correct estimate of Corollary 2.6 from [17].

Corollary 3.30 *Under the assumptions of Theorem 3.3 and if* $|f'|$ *is* P *function*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \tfrac{b-a}{2^{1-\frac{1}{\mu}}}\left(\left(\tfrac{b-x}{b-a}\right)^2 + \left(\tfrac{x-a}{b-a}\right)^2\right)\left(\tfrac{|f'(a)|^{\mu}+|f'(b)|^{\mu}}{2}\right)^{\frac{1}{\mu}}.$$

Moreover if we choose $x = \frac{a+b}{2}$, *we obtain*

$$\left| f\left(\tfrac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \tfrac{b-a}{4}\left(\left|f'(a)\right|^{\mu} + \left|f'(b)\right|^{\mu}\right)^{\frac{1}{\mu}}.$$

References

1. Alomari M., Darus, M.: Some Ostrowski's type inequalities for convex functions with applications, RGMIA, 13(1) (2010), Article 3.
2. Angulo, H., Gimenez, J., Moros, A.M., Nikodem, K.: On strongly h-convex functions. Ann. Funct. Anal. 2 (2011), no. 2, 85–91.
3. Breckner, W.W.: Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) Publ. Inst. Math. (Beograd) (N.S.) 23(37) (1978), 13–20.
4. DiDonato, A.R., Jarnagin, M.P.: The efficient calculation of the incomplete beta-function ratio for half-integer values of the parameters a, b. Math. Comp. 21 (1967), no. 100, 652–662.
5. Dragomir, S.S., Pečarić, J.E., Persson, L.E.: Some inequalities of Hadamard type. Soochow J. Math. 21 (1995), no. 3, 335–341.

6. Dragomir, S.S.: nequalities of Hermite-Hadamard type for h-convex functions on linear spaces. Proyecciones 34 (2015), no. 4, 323–341.
7. Hua, J., Xi, B.-Y., Qi, F.: Some new inequalities of Simpson type for strongly s-convex functions. Afr. Mat. 26 (2015), no. 5-6, 741–752.
8. Kirmaci, U.S.: Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl. Math. Comput. 147 (2004), no. 1, 137–146.
9. Mangasarian, O.L.: Nonlinear programming. McGraw-Hill Book Co., New York-London-Sydney 1969.
10. Meftah, B.: Some new Ostrowski's inequalities for functions whose n^{th} derivatives are r-convex. Int. J. Anal. 2016, Art. ID 6749213, 7 pp.
11. Meftah, B.: Some new Ostrowski's inequalities for n-times differentiable mappings which are quasi-convex. Facta Univ. Ser. Math. Inform. 32 (2017), no. 3, 319–327.
12. Meftah, B.: Some new Ostrowski's inequalities for functions whose n^{th} derivatives are logarithmically convex. Ann. Math. Sil. 32 (2018), no. 1, 275–284.
13. Meftah, B.: Some Ostrowski's inequalities for functions whose nth derivatives are s-convex. An Univ Oradea Fasc. Mat. 25 (2018), no. 2, 185–212.
14. Ostrowski, A.M.: Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert. (German) Comment. Math. Helv. 10 (1937), no. 1, 226–227.
15. Pečarić, J.E., Proschan, F., Tong, Y.L.: Convex functions, partial orderings, and statistical applications. Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.
16. Polyak, B.T.: Existence theorems and convergence of minimizing sequences in extremum problems with restictions, Soviet Math. Dokl. 7 (1966), 72-75.
17. Set, E., Özdemir, M.E., Sarikaya, M.Z.: New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. Facta Univ. Ser. Math. Inform. 27 (2012), no. 1, 67–82.
18. Set, E., Özdemir, M.E., Sarıkaya, M.Z., Akdemir, A.O.: Ostrowski-type inequalities for strongly convex functions. Georgian Math. J. 25 (2018), no. 1, 109–115.
19. Sun, Y., Yin, H.: Some Integral Inequalities of Simpson Type for Strongly Extended s-Convex Functions. Advances in Pure Mathematics, 6 (2016), no.11, 745-753.
20. Tunç, M., Göv, E., Şanal, Ü.: On tgs-convex function and their inequalities. Facta Univ. Ser. Math. Inform. 30 (2015), no. 5, 679–691.
21. Tunç, M., Sanal, U., Gov, E.: Some Hermite-Hadamard inequalities for $beta$-convex and its fractional applications. New Trends in Mathematical Sciences, 3 (2015), no 4, p. 18.
22. Xi, B.-Y., Qi, F.: Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means. J. Nonlinear Convex Anal. 16 (2015), no. 5, 873–890.