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Calculating steady-state probabilities of the $\mathrm{G/M/n/m}$ queueing systems

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Abstract. This article proposes a method for calculating steady-state probabilities of the $G/M/n/m$ queueing systems. The approach based on the use of fictitious phases and hyper-exponential approximations with parameters of the paradoxical and complex type by method of moments. The obtained results are verified using simulation models.

Keywords. non-Markovian queueing system · hyperexponential approximation · fictitious phases · method of moments

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1 Introduction

For the study of non-Markovian process in queueing systems, phase-type distributions are used with exponential distributions of delays in the phases [2, 5, 8]. In the case of fixing the number of the phase, the states of the system has a Markov property that makes it possible to represent the transitions between them in the form of a discrete Markov process with continuous time. The order of approximation is the number of retained initial moments of the original distribution.

Recently, interest in the hyperexponential distribution has increased since its use showed its high performance in solving problems of summation of recurrent flows [4], in computing characteristics of queuing systems with impatient customers [3] and Jackson's networks of queueing [6], and also in analyzing stock management systems [1].

Article [8] shows that the use of hyperexponential approximation (H_l) makes it possible to determine with high accuracy the steady-state probabilities of non-Markovian single-channel queuing systems. These probabilities are determined using solutions of a system of linear algebraic equations obtained by the method of fictitious phases. To find parameters of the H_l-approximation of a certain distribution it is sufficient to solve the system of

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equations of the moments method. For the values $V < 1$ of the variation coefficient, roots of this system are complex-valued or paradoxical (i.e., negative or with probabilities that exceed the boundaries of the interval $[0,\ 1]$) but in most cases as a result of summation of probabilities of microstates, their complex-valued and paradoxical parts are annihilated.

The purpose of the paper is to use of the hyperexponential approximation method for calculating steady-state probabilities of the $G/M/n/m$ queueing systems. The obtained results are verified using simulation models. We also indicate ways to evaluate the accuracy of approach the obtained steady-state distribution to the true distribution without the need to use simulation models.

2 Equations for steady-state probabilities of the $H_l/M/n/m$ system

The hyperexponential distribution of order l is a phase-type distribution and provides for choosing one of l alternative phases by a random process. With probability α_s, the process is at the sth phase and is in it during an exponentially distributed time with a parameter λ_s.

Suppose that the times elapsed between two consecutive arrivals are independent random variables distributed according to the hyperexponential law H_l $(l \geq 2)$ with probabilities α_s and parameters λ_s $(1 \leq s \leq l)$ and the service time of each customer is distributed exponentially with parameter μ. Let n and m denote the number of channels in the system and limit on the queue length respectively.

Let us enumerate the $H_l/M/n/m$ system's states as follows: $x_{0(s)}$ corresponds to the empty system and the time interval until the arrival of the first customer is in the phase s $(1 \leq s \leq l)$; $x_{k(s)}$ is the state, when there are k customers in the system $(1 \leq k \leq n+m)$, the time interval until the arrival of the next customer is in the phase s $(1 \leq s \leq l)$. We denote by $p_{0(s)}$ and $p_{k(s)}$ respectively, steady-state probabilities that the system is in the each of these states. To calculate $p_{0(s)}$ and $p_{k(s)}$ we obtain the system of linear equations:

$$-\lambda_s p_{0(s)} + \mu p_{1(s)} = 0, \quad 1 \leq s \leq l;$$

$$-(\lambda_s + k\mu)p_{k(s)} + \alpha_s \sum_{u=1}^{l} \lambda_u p_{k-1(u)} + (k+1)\mu p_{k+1(s)} = 0, \ \ 1 \leq k \leq n-1, \ \ 1 \leq s \leq l;$$

$$-(\lambda_s + n\mu)p_{k(s)} + \alpha_s \sum_{u=1}^{l} \lambda_u p_{k-1(u)} + n\mu p_{k+1(s)} = 0, \ \ n \leq k \leq n+m-1, \ \ 1 \leq s \leq l;$$

$$-(\lambda_s + n\mu)p_{n+m(s)} + \alpha_s \sum_{u=1}^{l} \lambda_u \left(p_{n+m-1(u)} + p_{n+m(u)}\right) = 0, \quad 1 \leq s \leq l;$$

$$\sum_{k=0}^{n+m} \sum_{u=1}^{l} p_{k(u)} = 1. \tag{2.1}$$

Solving the system (2.1), we find the steady-state probabilities p_k of the presence in the queueing system of k customers using the formulas

$$p_k = \sum_{u=1}^{l} p_{k(u)}, \quad 0 \leq k \leq n+m. \tag{2.2}$$

3 Features of finding probabilities p_k in the case of complex-valued or paradoxical parameters of H_l-approximation

We calculate the approximate steady-state probabilities p_k for the $G/M/n/m$ system using solutions of equations (2.1), written for the $H_l/M/n/m$ system, considering the order of approximation l from 2 to 6.

To find parameters of H_l-approximation of a certain distribution with a given coefficient of variation it is sufficient to solve the system of equations of the moments method only for the case of any one given mean value of this distribution since roots of the equations of the moments method are invariant with respect to the scale transformation.

The system of equations of the moments method for approximating the distribution of some random variable X using a random variable Y_l distributed by law of H_l is of the form

$$\sum_{s=1}^{l} \frac{\alpha_s}{\lambda_s^i} = \frac{m_i}{i!}, \qquad 0 \le i \le 2l-1; \qquad \sum_{s=1}^{l} \alpha_s = 1, \tag{3.1}$$

where $m_i = E(X^i)$ is the initial moment of order i of the random variable X. The dependence of the nature of the roots of system (3.1) on values of the variation coefficient V for the original gamma distributions and Weibull distributions is described in [8]. For the values $V < 1$ of the variation coefficient, some of the roots of system (3.1) are complex-valued but in most cases as a result of summation of probabilities of microstates the steady-state probabilities p_k are real-valued.

To illustrate this fact, we consider the solutions of system (2.1) for complex-valued parameters α_s and λ_s, limited to the case when $l = 2$, $n = 1$ and $m = 1$. In this case, using the solutions of system (2.1) and formula (2.2), we obtain

$$\begin{aligned} p_0 &= \frac{\mu^2}{\Delta}\left((\alpha_2\lambda_1 + \alpha_1\lambda_2)\mu^2 + \alpha_2\lambda_1^3 + \alpha_1\lambda_2^3 + \right.\\ &\quad \left. + \left(\alpha_2(\alpha_1 + 2\alpha_2)\lambda_1^2 + 2\alpha_1\alpha_2\lambda_1\lambda_2 + \alpha_1(2\alpha_1 + \alpha_2)\lambda_2^2\right)\mu\right), \\ p_1 &= \frac{\lambda_1\lambda_2\mu}{\Delta}\left(\mu^2 + 2(\alpha_2\lambda_1 + \alpha_1\lambda_2)\mu + \alpha_2\lambda_1^2 + \alpha_1\lambda_2^2\right), \qquad p_2 = 1 - p_0 - p_1, \\ \Delta &= (\alpha_2\lambda_1 + \alpha_1\lambda_2)\left(\mu^4 + ((\alpha_1 + 2\alpha_2)\lambda_1 + (2\alpha_1 + \alpha_2)\lambda_2)\mu^3 + (\lambda_1 + \lambda_2)^2\mu^2\right. \\ &\quad \left. + ((2\alpha_1 + \alpha_2)\lambda_1 + (\alpha_1 + 2\alpha_2)\lambda_2)\lambda_1\lambda_2\mu + \lambda_1^2\lambda_2^2\right). \end{aligned} \tag{3.2}$$

If parameters α_s and λ_s $(s = 1,\, 2)$ are complex-valued, then they can only be complex conjugate, and all possible cases of alternation of signs before the imaginary unit can be reduced to such two:

$$\begin{aligned} &1)\ \alpha_1 = a + ib, \quad \lambda_1 = c + id; \qquad \alpha_2 = a - ib, \quad \lambda_2 = c - id; \\ &2)\ \alpha_1 = a + ib, \quad \lambda_1 = c - id; \qquad \alpha_2 = a - ib, \quad \lambda_2 = c + id. \end{aligned} \tag{3.3}$$

In each of these cases, the imaginary parts in expressions (3.2) for p_k $(k = 0,\, 1,\, 2)$ are reduced, because the expressions

$$\begin{aligned} &\lambda_1 + \lambda_2, \quad \lambda_1\lambda_2, \quad \alpha_1\alpha_2, \quad \lambda_1^2 + \lambda_2^2, \quad \alpha_2\lambda_1 + \alpha_1\lambda_2, \quad \alpha_1\lambda_1 + \alpha_2\lambda_2, \\ &\alpha_2\lambda_1^2 + \alpha_1\lambda_2^2, \quad \alpha_2\lambda_1^3 + \alpha_1\lambda_2^3, \quad \alpha_2^2\lambda_1^2 + \alpha_1^2\lambda_2^2 \end{aligned}$$

of which consist p_k, are real-valued.

In the case of complex-valued or paradoxical roots α_s and λ_s of system (3.1), let us name the function $F_{H_l}(t) = 1 - \sum_{s=1}^{l} \alpha_s e^{-\lambda_s t}$ $(t \ge 0)$ the distribution pseudo-function

Table 1 Values of the absolute deviation $\Delta_l(F)$ for different distributions

Distribution name	$\Delta_2(F)$	$\Delta_3(F)$	$\Delta_4(F)$	$\Delta_5(F)$	$\Delta_6(F)$
$\Gamma(0.001)$	0.3629	0.2605	0.2092	0.1773	0.1549
$U[0,\ 2]$	0.1139	0.0632	0.0411	0.0295	0.0224
$\Gamma(0.7)$	0.0007	$7.2 \cdot 10^{-5}$	$1.4 \cdot 10^{-5}$	$3.7 \cdot 10^{-6}$	$1.2 \cdot 10^{-6}$
$W(0.7)$	0.0071	0.0026	0.0006	∞	$6.1 \cdot 10^{-5}$
$W(0.8)$	0.0043	∞	0.0004	0.0001	∞
$W(0.9)$	0.0049	0.0005	∞	0.0001	$4.8 \cdot 10^{-5}$
$W(0.95)$	0.0031	0.0005	0.0001	∞	$3.5 \cdot 10^{-5}$
$\Gamma(4)$	0.3146	0.1412	0.0787	0.0497	0.0340
$W(3)$	0.3973	0.2790	0.2170	0.1786	0.1524

by law of H_l. Let us show that the function $F_{H_l}(t)$ is a real-valued function if α_s and λ_s $(1 \le s \le l)$ are roots of system (3.1).

In fact, if some of the roots of system (3.1) are complex-valued, then they can only be complex conjugate, and all possible cases of alternation of signs before the imaginary unit can be reduced to two cases presented in (3.3). In each of these cases, the imaginary parts in the expression for $F_{H_l}(t)$ are reduced, so the result is the real-valued function:

$$1)\quad \alpha_1 e^{-\lambda_1 t} + \alpha_2 e^{-\lambda_2 t} = 2\, e^{-ct}\left(a \cdot \cos(dt) + b \cdot \sin(dt)\right);$$
$$2)\quad \alpha_1 e^{-\lambda_1 t} + \alpha_2 e^{-\lambda_2 t} = 2\, e^{-ct}\left(a \cdot \cos(dt) - b \cdot \sin(dt)\right).$$

The absolute deviation of the function of distribution by law G from a function $F_{H_l}(t)$ which parameters are roots of system (3.1), we will evaluate with the help of integral

$$\Delta_l(F) = \int_0^\infty |F_{H_l}(t) - F_G(t)|dt,$$

where $F_G(t)$ is the probability distribution function by law G.

Let $\Gamma(V),\ W(V)$ and $U[a,\ b]$ denote the gamma distribution, Weibull distribution with coefficients of variation V, and uniform distribution on the interval $[a,\ b]$ respectively.

Table 1 gives deviation values of $\Delta_l(F)$ for $l = 2, \ldots, 6$, calculated by results of approximation of different distributions with means 1. With increasing order of H_l-distribution, the value of deviation $\Delta_l(F)$ decreases, and with the increase of the coefficient of variation for $V > 1$ the deviation increases, much faster for the Weibull distribution compared with the gamma distribution. For distributions $W(0.7)$, $W(0.8)$, $W(0.9)$ and $W(0.95)$ for some values of l the deviation $\Delta_l(F) = \infty$. In each of these cases, one of roots λ_s of system (3.1) is real, but negative. Therefore, for the corresponding distribution pseudo-function, the limit relation $\lim\limits_{t\to\infty} F_{H_l}(t) = \infty$ is valid. For these values of l, the steady-state probabilities p_k, obtained using solutions of equations (2.1), written for the $H_l/M/n/m$ system, can be paradoxical.

Calculations show that properties of the solutions of system (2.1) almost repeats the form of the roots α_s $(1 \le s \le l)$ of system (3.1). Let's show it on examples of $U[0, 0.25]/M/n/m$ and $\Gamma(0.7)/M/n/m$ queueing systems.

For the order of approximation l from 2 to 6 the roots of system (3.1) for uniform distribution on the interval $[0,\ 0.25]$ are as follows:

$$l = 2:\quad \alpha_{1,2} = 0.5 \pm 0.86603i, \quad \lambda_{1,2} = 12 \pm 6.92820i;$$
$$l = 3:\quad \alpha_1 = 2.65193, \quad \alpha_{2,3} = -0.82596 \pm 0.60435i,$$
$$\lambda_1 = 18.57748, \quad \lambda_{2,3} = 14.7113 \pm 14.03505i;$$
$$l = 4:\quad \alpha_{1,2} = -0.58906 \mp 0.89679i, \quad \alpha_{3,4} = 1.08906 \pm 4.95602i,$$
$$\lambda_{1,2} = 16.83032 \pm 21.25934i, \quad \lambda_{3,4} = 23.16968 \pm 6.93787i;$$

$$
\begin{aligned}
l = 5: \quad & \alpha_1 = 15.24547, \quad \alpha_{2,3} = 1.02783 \mp 0.49426i, \quad \alpha_{4,5} = -8.15056 \pm 2.37119i, \\
& \lambda_1 = 29.17391, \quad \lambda_{2,3} = 18.59739 \pm 28.56818i, \quad \lambda_{4,5} = 26.81565 \pm 13.94129i; \\
l = 6: \quad & \alpha_{1,2} = 0.31983 \pm 1.17903i, \quad \alpha_{3,4} = -3.40926 \mp 12.71978i, \\
& \alpha_{5,6} = 3.58943 \pm 36.22605i, \\
& \lambda_{1,2} = 20.12746 \pm 35.94138i, \quad \lambda_{3,4} = 29.88567 \pm 21.01018i; \\
& \lambda_{5,6} = 33.98688 \pm 6.94008i.
\end{aligned}
$$

For $l = 2$ solutions of the corresponding system (2.1) are complex conjugate with positive real parts; for $l = 3$ $p_{k(1)} > 0 \; \forall \; k$, $p_{k(2)}$ and $p_{k(3)}$ are complex conjugate with negative real parts for most values of k. For $l = 4$ we have two pairs of complex conjugate solutions with negative real parts for most values of k in the first pair and with positive real parts $\forall$ k in the second pair. For $l = 5$ $p_{k(1)} > 0 \; \forall \; k$, and for $s = 2,\, 3$ and $s = 4,\, 5$ we have two pairs of complex conjugate solutions $p_{k(s)}$ with positive real parts in the first pair and with negative real parts in the second pair for most values of k. For $l = 6$ we have three pairs of complex conjugate solutions $p_{k(s)}$ with negative real parts in the second pair and with positive real parts in the first and third pairs.

For the order of approximation l from 2 to 6 the roots of system (3.1) for $\Gamma(0.7)$ distribution with mean 0.125 are as follows:

$$
\begin{aligned}
l = 2: \quad & \alpha_{1,2} = 0.5 \pm 6.18520i, \quad \lambda_{1,2} = 16 \pm 1.31077i; \\
l = 3: \quad & \alpha_1 = 0.02814, \quad \alpha_{2,3} = 0.48592 \pm 15.70863i, \\
& \lambda_1 = 39.57700, \quad \lambda_{2,3} = 16.21150 \pm 0.53937i; \\
l = 4: \quad & \alpha_1 = 0.00548, \quad \alpha_2 = 0.08597, \quad \alpha_{3,4} = 0.45428 \pm 29.37436i, \\
& \lambda_1 = 69.97401, \quad \lambda_2 = 25.49286, \quad \lambda_{3,4} = 16.26656 \pm 0.29607i; \\
l = 5: \quad & \alpha_1 = 0.00186, \quad \alpha_2 = 0.01855, \quad \alpha_3 = 0.16685, \quad \alpha_{4,5} = 0.40637 \pm 47.30434i, \\
& \lambda_1 = 108.92822, \quad \lambda_2 = 37.04067, \quad \lambda_3 = 21.45221, \\
& \lambda_{4,5} = 16.28945 \pm 0.18743i; \\
l = 6: \quad & \alpha_1 = 0.00081, \quad \alpha_2 = 0.00679, \quad \alpha_3 = 0.03738, \quad \alpha_4 = 0.26943, \\
& \alpha_{5,6} = 0.34279 \pm 69.58145i, \\
& \lambda_1 = 156.50494, \quad \lambda_2 = 51.28750, \quad \lambda_3 = 27.95129, \quad \lambda_4 = 19.65369, \\
& \lambda_{5,6} = 16.30125 \pm 0.12940i.
\end{aligned}
$$

For l from 2 to 6 properties of solutions $p_{k(s)}$ of system (2.1) in the sense of their signs and whether they are real or complex, completely coincide with the properties of the roots α_s $(1 \le s \le l)$ of system (3.1) with the same numbers.

4 Numerical results

Let us present the results of calculating steady-state probabilities on examples of the $U[0, 0.25]/M/10/15$, $U[0, 0.125]/M/20/15$ systems and $\Gamma(V)/M/n/15$, $W(0.9)/M/n/15$ systems for $n = 10,\, 20$ and $V = 0.001,\, 0.7,\, 4$.

Let $E(T_\lambda)$ denote the mean of the times elapsed between two consecutive arrivals. We take $E(T_\lambda) = 0.125$ and $E(T_\lambda) = 0.0625$ for $n = 10$ and $n = 20$ respectively, and $\mu = 1$ is the parameter of exponential distribution of service times.

The obtained results are verified using simulation models constructed with the help of the GPSS World tools [7]. The results obtained using GPSS World slightly differ from one another for different numbers of library random-number generators used for simulating

Table 2 Results of the calculation of steady-state characteristics of the $G/M/10/15$ and $G/M/20/15$ systems with different G-distributions

G-distribuion name, value of n	Characteristic name	Method of calculation and values of characteristics					
		H_2	H_3	H_4	H_5	H_6	GPSS World
$\Gamma(0.001)$, $n=10$	N	8.4783	8.4809	8.4809	8.4809	8.4809	8.4759
	$\Delta_{l(sim)}$	0.0036	0.0026	0.0026	0.0026	0.0026	–
	$\Delta_{l,l-1}$	–	$3.38 \cdot 10^{-3}$	$3.02 \cdot 10^{-5}$	$3.64 \cdot 10^{-7}$	$3.49 \cdot 10^{-9}$	–
$\Gamma(0.001)$, $n=20$	N	16.2230	16.2245	16.2245	16.2245	16.2245	16.1875
	$\Delta_{l(sim)}$	0.0102	0.0095	0.0095	0.0095	0.0095	–
	$\Delta_{l,l-1}$	–	$2.21 \cdot 10^{-3}$	$1.07 \cdot 10^{-5}$	$9.55 \cdot 10^{-8}$	$1.24 \cdot 10^{-9}$	–
$U[0,\ 1/4]$, $n=10$	N	8.8186	8.8206	8.8206	8.8206	8.8206	8.8138
	$\Delta_{l(sim)}$	0.0034	0.0023	0.0023	0.0023	0.0023	–
	$\Delta_{l,l-1}$	–	$2.73 \cdot 10^{-3}$	$4.64 \cdot 10^{-5}$	$1.16 \cdot 10^{-6}$	$3.57 \cdot 10^{-8}$	–
$U[0,\ 1/8]$, $n=20$	N	16.4409	16.4422	16.4422	16.4422	16.4422	16.4356
	$\Delta_{l(sim)}$	0.0038	0.0038	0.0038	0.0038	0.0038	–
	$\Delta_{l,l-1}$	–	$1.81 \cdot 10^{-3}$	$1.93 \cdot 10^{-5}$	$4.05 \cdot 10^{-7}$	$1.36 \cdot 10^{-8}$	–
$\Gamma(0.7)$, $n=10$	N	8.9531	8.9531	8.9531	8.9531	8.9531	8.9605
	$\Delta_{l(sim)}$	0.0017	0.0017	0.0017	0.0017	0.0017	–
	$\Delta_{l,l-1}$	–	$2.11 \cdot 10^{-5}$	$9.39 \cdot 10^{-8}$	$1.21 \cdot 10^{-9}$	$2.55 \cdot 10^{-11}$	–
$\Gamma(0.7)$, $n=20$	N	16.5339	16.5339	16.5339	16.5339	16.5339	16.5306
	$\Delta_{l(sim)}$	0.0027	0.0027	0.0027	0.0027	0.0027	–
	$\Delta_{l,l-1}$	–	$1.41 \cdot 10^{-5}$	$4.20 \cdot 10^{-8}$	$4.46 \cdot 10^{-10}$	$1.08 \cdot 10^{-11}$	–
$W(0.9)$, $n=10$	N	9.2464	9.2463	–	9.2463	9.2463	9.2428
	$\Delta_{l(sim)}$	0.0019	0.0019	–	0.0019	0.0019	–
	$\Delta_{l,l-1}$	–	$1.97 \cdot 10^{-4}$	–	–	$4.55 \cdot 10^{-9}$	–
$W(0.9)$, $n=20$	N	16.7440	16.7440	–	16.7440	16.7440	16.7426
	$\Delta_{l(sim)}$	0.0027	0.0027	–	0.0027	0.0027	–
	$\Delta_{l,l-1}$	–	$1.40 \cdot 10^{-4}$	–	–	$2.04 \cdot 10^{-9}$	–
$\Gamma(4)$, $n=10$	N	10.1667	9.7778	9.7522	9.7532	9.7536	9.7521
	$\Delta_{l(sim)}$	0.0784	0.0216	0.0094	0.0052	0.0032	–
	$\Delta_{l,l-1}$	–	0.0680	0.0146	$5.41 \cdot 10^{-3}$	$2.25 \cdot 10^{-3}$	–
$\Gamma(4)$, $n=20$	N	16.5677	16.2055	16.1864	16.1874	16.1876	16.1773
	$\Delta_{l(sim)}$	0.0652	0.0183	0.0092	0.0062	0.0048	–
	$\Delta_{l,l-1}$	–	0.0576	0.0117	$4.41 \cdot 10^{-3}$	$1.83 \cdot 10^{-3}$	–

random variables, i.e., times elapsed between two consecutive arrivals and service times. Therefore, we use averaged results obtained using simulation models with different values of random-numbers generators that take on values of natural numbers from 6 to 10.

Let us introduce the designation: N is the average number of customers in a queueing system, and

$$\Delta_{(l,l-1)} = \sum_{k=0}^{n+15} |p_{k(l)} - p_{k(l-1)}|, \quad \Delta_{l(sim)} = \sum_{k=0}^{n+15} |p_{k(l)} - p_{k(sim)}|,$$

$$p_{k(sim)} = \frac{1}{5}\sum_{i=6}^{10} p_{k(sim,i)}, \qquad 0 \le k \le n+15, \quad 2 \le l \le 6.$$

Here $p_{k(l)}$ are values of probabilities p_k obtained using the H_l-approximation, $p_{k(sim)}$ is the average value of probabilities $p_{k(sim,i)}$, obtained by means of the simulation model using the number i of random-numbers generator for $6 \leq i \leq 10$. Thus, the quantities $\Delta_{l(sim)}$ are measures of deviations of the distributions $\{p_{k(l)}\}$ from distribution $\{p_{k(sim)}\}$, and the quantities $\Delta_{(l,l-1)}$ give an opportunity to estimate the deviation of distributions $\{p_{k(l)}\}$ from distributions $\{p_{k(l-1)}\}$.

In Table 2 we present the results of calculation of steady-state characteristics of the $G/M/10/15$ and $G/M/20/15$ systems with the considered gamma, Weibull and uniform distributions. The values of deviations $\Delta_{l(sim)}$ and $\Delta_{(l,l-1)}$ decrease with increasing order of H_l-distributions in approximations, and it means that the values of distribution $\{p_{k(l)}\}$ with each step getting closer to a true distribution $\{p_k\}$. With the growth of the variation coefficient of distributions after the value of $V > 1$, as expected taking into account the behavior of deviations $\Delta_l(F)$, the values of the absolute deviations $\Delta_{l(sim)}$ and $\Delta_{(l,l-1)}$ also increase. For the distribution $W(0.9)$ the deviation $\Delta_4(F) = \infty$ and, consequently, some values of "probabilities" of the distribution $\{p_{k(4)}\}$ go beyond the interval $[0,\ 1]$.

Presented results show that increasing the number of channels of the $G/M/n/m$ system has no significant effect on accuracy of calculating the steady-state probabilities.

Testing the proposed method on the $M/G/1/m$ systems, for which we can find exact values of the steady-state distribution $\{p_k\}$, shows that in cases where the deviation $\Delta_{(6,5)}$ is less than 10^{-2}, the deviation of the distribution $\{p_{k(l)}\}$ from the true distribution $\{p_k\}$ and deviation $\Delta_{(l+1,l)}$ are numbers of the same order, and at the same time the deviations of distribution $\{p_{k(sim)}\}$ from the distribution $\{p_k\}$ usually no less than 10^{-4}. Thus, in most cases we can use values $\Delta_{(l,l-1)}$ to evaluate accuracy of the approximation of the distribution $\{p_{k(l-1)}\}$ to the true $\{p_k\}$ for $3 \leq k \leq 6$. In cases where $\Delta_{(l,l-1)} < 10^{-4}$, we can argue that the distribution $\{p_{k(l-1)}\}$ is more accurate approximation than $\{p_{k(sim)}\}$.

5 Conclusions

This paper shows that the application of hyperexponential approximation of distributions the times elapsed between two consecutive arrivals allows us to calculate steady-state probabilities of the $G/M/n/m$ queueing systems with high accuracy (higher than in the case of using simulation models). We find these probabilities using solutions of a system of linear algebraic equations obtained by the method of fictitious phases.

To obtain parameters of H_l-approximation of a certain distribution it is necessary to solve the system of equations of the moments method. For the values $V < 1$ of the variation coefficient, some of the roots of this system are complex-valued or, having a sense of probabilities, go beyond the interval $[0,\ 1]$, but in most cases the final result is close to the desired distribution $\{p_k\}$.

Computing deviations $\Delta_{(l,l-1)}$ allows us to track the accuracy of approaching distributions $\{p_{k(l-1)}\}$ to the true distribution $\{p_k\}$ without the need to use simulation models.

References

1. Nazarov, A. A., Broner, V. I.: *Inventory management system with hyperexponential distribution of resources consumption*. Vestnik of Tomsk State Univ. Management, Comp. Eng. and Inform., **34** (1), 43–49 (2016) (in Russian).
2. Neuts, M. F.: Matrix-geometric Solutions in Stochastic Models, *Baltimore: The John's Hopkins University Press* (1981).

3. Ryzhikov, Yu. I., Ulanov, A. V.: *Calculation of the $M/H_2/n - H_2$ hyperexponential system with customers impatient in the queue*. Vestnik of Tomsk State Univ. Management, Comp. Eng. and Inform., **27** (2), 47–53 (2014) (in Russian).
4. Ryzhikov, Yu. I., Ulanov, A. V.: *Application of hyperexponential approximation in problems of summation of flows*. Intelligent Technologies in Transport, **4**, 34–39 (2015) (in Russian).
5. Ryzhikov, Yu. I., Ulanov, A. V.: *Application of hyperexponential approximation in the problems of calculating non-Markovian queuing systems*. Vestnik of Tomsk State Univ. Management, Comp. Eng. and Inform., **36** (3), 60–65 (2016) (in Russian).
6. Tsitsiashvili, G. Sh.: *Synergistic effect in a network with hyperexponential distributions of service times*. Vestnik of Tomsk State Univ. Management, Comp. Eng. and Inform., **34** (1), 65–68 (2016) (in Russian).
7. Zhernovyi, Yu.: Creating Models of Queueing Systems Using GPSS World, *Saarbrücken: LAP Lambert Academic Publishing* (2015).
8. Zhernovyi, Yu. V.: *Calculating steady-state characteristics of single-channel queuing systems using phase-type distributions*. Cybernetics and Systems Analysis, **54** (5), 824–832 (2018).