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Fractional maximal operator and its higher order commutators on generalized weighted Morrey spaces

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Abstract. *In this paper, we study the boundedness of the fractional maximal operator M_α and its kth order commutators $[b, M_\alpha]^k$ on generalized weighted Morrey spaces $M_{p,\varphi}(w)$. We find the sufficient conditions on the pair (φ_1, φ_2) with $b \in BMO(\mathbb{R}^n)$ and $w \in A_{pq}$ which ensures the boundedness of the operators M_α and $[b, M_\alpha]^k$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$. In all cases the conditions for the boundedness of the operators M_α, $[b, M_\alpha]^k$ are given in terms of supremal-type inequalities on (φ_1, φ_2) and w, which do not assume any assumption on monotonicity of $\varphi_1(x, r)$, $\varphi_2(x, r)$ in r. The main advance in comparison with the existing results is that we manage to obtain conditions for the boundedness not in integral terms but in less restrictive terms of supremal operators.*

Keywords. Generalized weighted Morrey spaces; fractional maximal operator; A_p weights; commutator; BMO

Mathematics Subject Classification (2010): 42B20, 42B35

1 Introduction

The classical Morrey spaces were originally introduced by Morrey in [26] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [5–9, 12]. Recently, Komori and Shirai [24] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [13] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$. In [13] Guliyev also studied the boundedness of the classical operators and its commutators in these spaces $M_{p,\varphi}(w)$, see also Guliyev et al. [10, 15–17].

Let $0 < \alpha < n$. The fractional maximal operator M_α and the Riesz potential operator I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)| dy, \quad I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

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If $\alpha = 0$, then $M \equiv M_0$ is the well known Hardy-Littlewood maximal operator. Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \le \upsilon_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x).$$

The commutators generated by a suitable function b and the operator M_α is formally defined by

$$[b, M_\alpha]f = M_\alpha(bf) - bM_\alpha(f).$$

Given a measurable function b the fractional maximal commutator operators $M_{b,\alpha}$ is defined by

$$M_{b,\alpha}(f)(x) := \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(x) - b(y)||f(y)|dy.$$

If $\alpha = 0$, then $M_{b,0} \equiv M_b$ is the sublinear commutator of the Hardy-Littlewood maximal operator.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\alpha]$ and $M_{b,\alpha}$ are valid:

Let b be any non-negative locally integrable function. Then

$$|[b, M_\alpha]f(x)| \le M_{b,\alpha}(f)(x), \qquad x \in \mathbb{R}^n$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M_\alpha]f(x)| \le M_{b,\alpha}(f)(x) + 2b^-(x)M_\alpha f(x), \qquad x \in \mathbb{R}^n \tag{1.1}$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ (see, for example, [29]).

The main purpose of this paper is to study Spanne-Guliyev type boundedness of the operator M_α on the generalized weighted Morrey spaces, including weak versions. Also we study Spanne-Guliyev type boundedness of the kth order commutator operator $[b, M_\alpha]^k$ on the generalized weighted Morrey spaces. We find the sufficient conditions on the pair (φ_1, φ_2) with $b \in BMO(\mathbb{R}^n)$ and $w \in A_{pq}$ which ensures the boundedness of the operators M_α, $[b, M_\alpha]^k$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < q < \infty$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Generalized weighted Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [26] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [11,22,25].

We recall that a weight function w is in the Muckenhoupt's class A_p [27], $1 < p < \infty$, if

$$\begin{aligned}[w]_{A_p} :&= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1}, \end{aligned} \tag{2.1}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls B by Hölder's inequality

$$[w]_{A_p(B)}^{1/p}=|B|^{-1}\|w\|_{L_1(B)}^{1/p}\,\|w^{-1/p}\|_{L_{p'}(B)}\geq 1. \tag{2.2}$$

For $p=1$, the class A_1 is defined by the condition $Mw(x)\leq Cw(x)$ with $[w]_{A_1}=\sup\limits_{x\in\mathbb{R}^n}\frac{Mw(x)}{w(x)}$, and for $p=\infty$ $A_\infty=\bigcup_{1\leq p<\infty}A_p$ and $[w]_{A_\infty}=\inf\limits_{1\leq p<\infty}[w]_{A_p}$.

A weight function w belongs to the Muckenhoupt-Wheeden class $A_{p,q}$ [28] for $1<p,q<\infty$ if

$$\begin{aligned}[w]_{A_{p,q}}:&=\sup_B[w]_{A_{p,q}(B)}\\&=\sup_B\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{1/q}\left(\frac{1}{|B|}\int_B w(x)^{-p'}dx\right)^{1/p'}<\infty,\end{aligned}$$

where the sup is taken with respect to all balls B. Note that, for all balls B by Hölder's inequality

$$[w]_{A_{p,q}(B)}=|B|^{\frac{1}{p}-\frac{1}{q}-1}\|w\|_{L_q(B)}\,\|w^{-1}\|_{L_{p'}(B)}\geq 1. \tag{2.3}$$

If $p=1$, w is in $A_{1,q}$ with $1<q<\infty$ if

$$\begin{aligned}[w]_{A_{1,q}}:&=\sup_B[w]_{A_{1,q}(B)}\\&=\sup_B\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{1/q}\left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{w(x)}\right)<\infty,\end{aligned}$$

where the sup is taken with respect to all balls B.

We define the generalized weighed Morrey spaces as follows.

Definition 2.1 *Let $1\leq p<\infty$, φ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f\in L_{p,w}^{\rm loc}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)}=\sup_{x\in\mathbb{R}^n,r>0}\varphi(x,r)^{-1}\,w(B(x,r))^{-\frac{1}{p}}\,\|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))}\equiv\|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)}=\left(\int_{B(x,r)}|f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f\in WL_{p,w}^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)}=\sup_{x\in\mathbb{R}^n,r>0}\varphi(x,r)^{-1}\,w(B(x,r))^{-\frac{1}{p}}\,\|f\|_{WL_{p,w}(B(x,r))}<\infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))}\equiv\|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)}=\sup_{t>0}t\left(\int_{\{y\in B(x,r):\,|f(y)|>t\}}w(y)dy\right)^{\frac{1}{p}}.$$

Remark 2.1 (1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

In the sequel $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively. We also denote

$$\mathbb{A} = \{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0^+} \varphi = 0\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|ug\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty),$$

The following theorem was proved in [4].

Theorem 2.1 *[4] Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,\cdot)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(\mathbb{R}_+)$ to $L_{\infty,v_2}(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u (\|v_1\|^{-1}_{L_\infty(\cdot,\infty)}) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

3 Fractional maximal operator in the spaces $M_{p,\varphi}(\mathbb{R}^n, w)$

The following Guliyev weighted local estimates are valid (see [13]).

Theorem 3.1 *Let $1 \le p < q < \infty$, $0 \le \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $\omega \in A_{p,q}(\mathbb{R}^n)$. Then, for $p > 1$ the inequality*

$$\|M_\alpha f\|_{L_{q,w^q}(B(x,r))} \lesssim (w^q(B(x,r)))^{\frac{1}{q}} \sup_{t\ge r} \|f\|_{L_{p,w^p}(B(x,t)} (w^q(B(x,t)))^{-\frac{1}{q}} \tag{3.1}$$

holds for any ball $B(x,r)$ and for all $f \in L^{loc}_{p,w}(\mathbb{R}^n)$.

Moreover, for $p = 1$ the inequality

$$\|M_\alpha f\|_{WL_{q,w^q}(B(x,r))} \lesssim (w^q(B(x,r)))^{\frac{1}{q}} \sup_{t\ge r} \|f\|_{L_{1,w}(B(x,t)} (w^q(B(x,t)))^{-\frac{1}{q}} \tag{3.2}$$

holds for any ball $B(x,r)$ and for all $f \in L^{loc}_{1,w}(\mathbb{R}^n)$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $w \in A_{p,q}(\mathbb{R}^n)$. For arbitrary $x \in \mathbb{R}^n$ and $r > 0$, set $B = B(x,r)$, $2B = B(x,2r)$.

We present f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\,\chi_{2B}(y), \quad f_2(y) = f(y)\,\chi_{\complement_{(2B)}}(y),$$

and have

$$\|M_\alpha f\|_{L_{q,w^q}(B)} \le \|M_\alpha f_1\|_{L_{q,w^q}(B)} + \|M_\alpha f_2\|_{L_{q,w^q}(B)}.$$

Since $f_1 \in L_{p,w^p}(\mathbb{R}^n)$, $M_\alpha f_1 \in L_{q,w^q}(\mathbb{R}^n)$ and from the boundedness of M_α from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$ (see [28]) it follows that:

$$\|M_\alpha f_1\|_{L_{q,w^q}(B)} \le \|M_\alpha f_1\|_{L_{q,w^q}} \lesssim \|f_1\|_{L_{p,w^p}} = \|f\|_{L_{p,w^p}(2B)}. \tag{3.3}$$

From (3.3) we obtain

$$\|M_\alpha f_1\|_{L_{q,w^q}(B)} \lesssim (w^q(B(x,r)))^{\frac{1}{q}} \sup_{t\ge r} \|f\|_{L_{p,w^p}(B(x,t))}(w^q(B(x,t)))^{-\frac{1}{q}}. \tag{3.4}$$

Let z be an arbitrary point in $B \equiv B(x,r)$. If $B(z,t) \cap {}^{\complement}B(x,2r) \ne \varnothing$, then $t > r$. Indeed, if $y \in B(z,t) \cap {}^{\complement}B(x,2r)$, then we get $t > |y-z| \ge |x-y| - |x-z| > 2r - r = r$.

On the other hand, $B(z,t) \cap {}^{\complement}B(x,2r) \subset B(x,2t)$. Indeed, if $y \in B(z,t) \cap {}^{\complement}B(x,2r)$, then we get $|x-y| \le |y-z| + |x-z| < t + r < 2t$. Hence, for all $z \in B$

$$\begin{aligned}
M_\alpha f_2(z) &= \sup_{t>0} |B(z,t)|^{-1+\frac{\alpha}{n}} \int_{B(z,t)} |f_2(y)|dy \\
&\le \sup_{t>r} |B(x,2t)|^{-1+\frac{\alpha}{n}} \int_{B(z,t)\cap {}^{\complement}B(x,2r)} |f(y)|dy \\
&\le \sup_{t>r} |B(x,2t)|^{-1+\frac{\alpha}{n}} \int_{B(x,2t)} |f(y)|dy \\
&= \sup_{t>2r} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy.
\end{aligned}$$

By applying Hölder's inequality, for all $z \in B$ we get

$$\begin{aligned}
M_\alpha f_2(z) &\le \sup_{t>2r} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy \\
&\lesssim \sup_{t>r} |B(x,t)|^{-1+\frac{\alpha}{n}} \|f\|_{L_{p,w^p}(B(x,t))}\, \|w^{-1}\|_{L_{p'}(B(x,t))} \\
&\lesssim \sup_{t>r} \|f\|_{L_{p,w^p}(B(x,t))}\, (w^q(B(x,t)))^{-\frac{1}{q}}.
\end{aligned}$$

Thus, the function $M_\alpha f_2(z)$, with fixed x and r, is dominated by the expression not depending on z. Then

$$\|M_\alpha f_2\|_{L_{q,w^q}(B(x,r))} \lesssim (w^q(B(x,r)))^{\frac{1}{q}} \sup_{t>r} \|f\|_{L_{p,w^p}(B(x,t))}(w^q(B(x,t)))^{-\frac{1}{q}}. \tag{3.5}$$

We then obtain (3.1) from (3.4) and (3.5).

Let $p = 1$, and $w \in A_{1,q}(\mathbb{R}^n)$.

Then,

$$\|M_\alpha f\|_{WL_{q,w^q}(B)} \le \|M_\alpha f_1\|_{WL_{q,w^q}(B)} + \|M_\alpha f_2\|_{WL_{q,w^q}(B)}.$$

Since $f_1 \in L_{1,w}(\mathbb{R}^n)$, $M_\alpha f_1 \in WL_{q,w^q}(\mathbb{R}^n)$ and from the boundedness of M_α from $L_{1,w}(\mathbb{R}^n)$ to $WL_{q,w^q}(\mathbb{R}^n)$ (see [28]) it follows that:

$$\|M_\alpha f_1\|_{WL_{q,w^q}(B)} \le \|M_\alpha f_1\|_{WL_{q,w^q}} \lesssim \|f_1\|_{L_{1,w}} = \|f\|_{L_{1,w}(2B)}. \tag{3.6}$$

From (3.6) we obtain

$$\|M_\alpha f_1\|_{WL_{q,w^q}(B)} \lesssim (w^q(B(x,r)))^{\frac{1}{q}} \sup_{t>r} \|f\|_{L_{1,w}(B(x,t))} (w^q(B(x,t)))^{-\frac{1}{q}}. \tag{3.7}$$

On the other hand,

$$\|M_\alpha f_2\|_{WL_{q,w^q}(B(x,r))} \le \|M_\alpha f_2\|_{L_{q,w^q}(B(x,r))}$$
$$\lesssim (w^q(B(x,r)))^{\frac{1}{q}} \sup_{t>r} \|f\|_{L_{1,w}(B(x,t))} (w^q(B(x,t)))^{-\frac{1}{q}}. \tag{3.8}$$

We then obtain (3.2) from (3.7) and (3.8).

For the operator M_α the following Spanne-Guliyev type result on the space $M_{p,\varphi}(w)$ is valid.

Theorem 3.2 *Let* $1 \le p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w \in A_{p,q}(\mathbb{R}^n)$, *and* (φ_1, φ_2) *satisfy the condition*

$$\sup_{t>r} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)(w^p B((x,s)))^{1/p}}{(w^q(B(x,t)))^{1/q}} \le C\varphi_2(x,r), \tag{3.9}$$

where C does not depend on u and r. Then the operator M_α is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{q,\varphi_2}(w^q)$ for $p = 1$. Moreover, for $p > 1$

$$\|M_\alpha f\|_{M_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M_{p,\varphi_1}(w^p)},$$

and for $p = 1$

$$\|M_\alpha f\|_{WM_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M_{1,\varphi_1}(w)}.$$

Proof. For $p > 1$ from Theorem 2.1 and Theorem 3.1 we get

$$\begin{aligned}\|M_\alpha f\|_{M_{q,\varphi_2}(w^q)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} \|f\|_{L_p,w^p} w(B(x,t))^{-\frac{1}{q}} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \|w\|^{-1}_{L_q(B(x,r))} \|f\|_{L_p,w^p} \\ &\lesssim \|f\|_{M_{p,\varphi_1}(w^p)}\end{aligned}$$

and for $p = 1$

$$\begin{aligned}\|M_\alpha f\|_{WM_{q,\varphi_2}(w^q)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} \|f\|_{L_1,w} w(B(x,t))^{-\frac{1}{q}} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \|w\|^{-1}_{L_q(B(x,r))} \|f\|_{L_1,w} \\ &\lesssim \|f\|_{M_{p,\varphi_1}(w^p)}.\end{aligned}$$

Remark 3.1 Note that, in the case $w \equiv 1$, Theorems 3.1 and 3.2 were proved in [14], see also [1, 2, 18–20].

4 Commutators of fractional maximal operators in the spaces $M_{p,\varphi}(\mathbb{R}^n, w)$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 4.1 *Suppose that* $b \in L_1^{\rm loc}(\mathbb{R}^n)$*, and let*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{\rm loc}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Lemma 4.1 *[28] Let* $w \in A_\infty$*. Then the norm* $\|\cdot\|_*$ *is equivalent to the norm*

$$\|b\|_{*,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{w(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r),w}|\, w(y)\, dy,$$

where

$$b_{B(x,r),w} = \frac{1}{w(B(x,r))} \int_{B(x,r)} b(y)\, w(y)\, dy.$$

The following lemma is proved in [13].

Lemma 4.2 *1 Let* $w \in A_\infty$ *and* $b \in BMO(\mathbb{R}^n)$*. Let also* $1 \le p < \infty$*,* $x \in \mathbb{R}^n$*,* $k > 0$ *and* $r_1, r_2 > 0$*. Then,*

$$\Big(\frac{1}{w(B(x,r_1))} \int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{kp} w(y) dy\Big)^{\frac{1}{p}} \le C\Big(1 + \Big|\ln\frac{r_1}{r_2}\Big|\Big)^k \|b\|_*^k,$$

where $C > 0$ *is independent of* f*,* w*,* x*,* r_1 *and* r_2*.*

2 Let $w \in A_p$ *and* $b \in BMO(\mathbb{R}^n)$*. Let also* $1 < p < \infty$*,* $x \in \mathbb{R}^n$*,* $k > 0$ *and* $r_1, r_2 > 0$*. Then,*

$$\Big(\frac{1}{w^{1-p'}(B(x,r_1))} \int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{kp'}\, w(y)^{1-p'} dy\Big)^{\frac{1}{p'}} \le C\Big(1 + \Big|\ln\frac{r_1}{r_2}\Big|\Big)^k \|b\|_*^k,$$

where $C > 0$ *is independent of* b*,* w*,* x*,* r_1 *and* r_2*.*

Remark 4.1 (1) Let $b \in BMO(\mathbb{R}^n)$. Then

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy\right)^{\frac{1}{p}} \tag{4.1}$$

for $1 < p < \infty$.

(2) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\big|b_{B(x,r)} - b_{B(x,\tau)}\big| \le C\|b\|_* \log\frac{\tau}{r} \ \text{ for } \ 0 < 2r < \tau, \tag{4.2}$$

where C is independent of f, x, r and τ.

The commutator generated by $b \in L_1^{loc}(\mathbb{R}^n)$ and the operator M_α is defined by

$$M_{b,\alpha}(f)(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{Q}} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy. \tag{4.3}$$

For the kth-order fractional maximal commutator operator $M_{b,\alpha,k}$ (see [14])

$$M_{b,\alpha,k}(f)(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{Q}} \int_{B(x,r)} |b(x) - b(y)|^k |f(y)|dy$$

the following Guliyev weighted local estimates are valid (see [13]).

Theorem 4.1 *Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $b \in BMO(\mathbb{R}^n)$, and $w \in A_{p,q}(\mathbb{R}^n)$. Then the inequality*

$$\|M_{b,\alpha,k} f\|_{L_{q,w^q}(B(x,r))} \lesssim \|b\|_*^k \, w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} \ln^k \left(e + \frac{t}{r}\right) w^q(B(x,t))^{-\frac{1}{q}} \|f\|_{L_{p,w^p}(B(x,t))}$$

holds for any ball $B(x,r)$ and for all $f \in L_{p,w^p}^{loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. We write f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\,\chi_{2B}(y), \quad f_2(y) = f(y)\,\chi_{\complement_{(2B)}}(y).$$

Hence,

$$\|M_{b,\alpha,k} f\|_{L_{q,w^q}(B)} \leq \|M_{b,\alpha,k} f_1\|_{L_{q,w^q}(B)} + \|M_{b,\alpha,k} f_2\|_{L_{q,w^q}(B)}$$

From the boundedness of $M_{b,\alpha,k}$ from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$ (see [3]) it follows that:

$$\|M_{b,\alpha,k} f_1\|_{L_{q,w^q}(B)} \leq \|M_{b,\alpha,k} f_1\|_{L_{q,w^q}} \lesssim \|b\|_*^k \, \|f_1\|_{L_{p,w^p}} = \|b\|_*^k \, \|f\|_{L_{p,w^p}(2B)}. \tag{4.4}$$

From (4.4) we obtain

$$\|M_{b,\alpha,k} f_1\|_{WL_{q,w^q}(B)} \lesssim \|b\|_*^k (w^q(B(x,r)))^{\frac{1}{q}} \sup_{t \geq r} \|f\|_{L_{p,w^p}(B(x,t))} (w^q(B(x,t)))^{-\frac{1}{q}}. \tag{4.5}$$

Let z be an arbitrary point in $B \equiv B(x,r)$. If $B(z,t) \cap {}^{\complement}B(x,2r) \neq \varnothing$, then $t > r$. Indeed, if $y \in B(z,t) \cap {}^{\complement}B(x,2r)$, then we get $t > |y-z| \geq |x-y| - |x-z| > 2r - r = r$.

On the other hand, $B(z,t) \cap {}^{\complement}B(x,2r) \subset B(x,2t)$. Indeed, if $y \in B(z,t) \cap {}^{\complement}B(x,2r)$, then we get $|x-y| \leq |y-z| + |x-z| < t + r < 2t$. Hence, for all $z \in B$

$$\begin{aligned} M_{b,\alpha,k} f_2(z) &= \sup_{t>0} |B(z,t)|^{-1+\frac{\alpha}{n}} \int_{B(z,t)} |b(y) - b(z)|^k |f_2(y)|\, dy \\ &= \sup_{t>0} |B(z,t)|^{-1+\frac{\alpha}{n}} \int_{B(z,t) \cap {}^{\complement}B(x,2r)} |b(y) - b(z)|^k |f(y)|\, dy \\ &\lesssim \sup_{t>r} |B(x,2t)|^{-1+\frac{\alpha}{n}} \int_{B(x,2t)} |b(y) - b(z)|^k |f(y)|\, dy \\ &= \sup_{t>2r} |B(x,2t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(y) - b(z)|^k |f(y)|\, dy. \end{aligned}$$

Therefore, for all $z \in B$ we have

$$M_{b,\alpha,k} f_2(z) \lesssim \sup_{t>2r} |B(x,2t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(y)-b(z)|^k |f(y)|\, dy.$$

Thus, the function $M_\alpha f_2(z)$, with fixed u and r, is dominated by the expression not depending on z. Then

$$\begin{aligned}
&\|M_{b,\alpha,k} f_2\|_{L_{q,w^q}(B)} \\
&\lesssim \Big(\int_B \Big(\sup_{t>2r} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(y)-b(z)|^k |f(y)| dy \Big)^q w^q(z)\, dz \Big)^{\frac{1}{n}} \\
&\lesssim \Big(\int_B \Big(\sup_{t>2r} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(y)-b_{B(x,r),w}|^k |f(y)| dy \Big)^q w^q(z)\, dz \Big)^{\frac{1}{n}} \\
&+ \Big(\int_B \Big(\sup_{t>2r} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(z)-b_{B(x,r),w}|^k |f(y)| dy \Big)^q w^q(z)\, dz \Big)^{\frac{1}{n}} \\
&= J_1 + J_2.
\end{aligned}$$

Let us estimate J_1. Applying Hölder's inequality and by Lemma 4.2 we get

$$\begin{aligned}
J_1 &= w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(y)-b_{B(x,r),w}|^k |f(y)| dy \\
&\approx w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} t^{\alpha-Q} \int_{B(x,t)} |b(y)-b_{B(x,r),w}|^k |f(y)| dy \\
&\le (w^q(B(x,r)))^{\frac{1}{q}} \sup_{t>2r} t^{\alpha-Q} \Big(\int_{B(x,t)} |b(y)-b_{B(x,r),w}|^{kp'} w(y)^{-p'} dy \Big)^{\frac{1}{p'}} \|f\|_{L_{p,w^p}(B(x,t))} \\
&\lesssim \|b\|_*^k \, w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} t^{\alpha-Q} \Big(1+\ln\frac{t}{r}\Big)^k \|w^{-1}\|_{L_{p'}(B(x,t))} \|f\|_{L_{p,w^p}(B(x,t))} \\
&\lesssim \|b\|_*^k \, w^q(B(x,r))^{\frac{1}{n}} \sup_{t>2r} t^{\alpha-Q} \Big(1+\ln\frac{t}{r}\Big)^k (w^q(B(x,t)))^{-\frac{1}{n}} \, t^{\frac{n}{p}+\frac{n}{p'}} \|f\|_{L_{p,w^p}(B(x,t))} \\
&= \|b\|_*^k \, w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} \ln^k \Big(e+\frac{t}{r}\Big) w^q(B(x,t))^{-\frac{1}{q}} \|f\|_{L_{p,w^p}(B(x,t))}.
\end{aligned}$$

In order to estimate I_2 we get

$$\begin{aligned}
J_2 &= \Big(\int_B \Big(\sup_{t>2r} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(z)-b_{B(x,r),w}|^k |f(y)| dy \Big)^q w^q(z)\, dz \Big)^{\frac{1}{q}} \\
&\approx \Big(\int_B |b(z)-b_{B(x,r),w}|^{kq} w^q(z) dz \Big)^{\frac{1}{q}} \sup_{t>2r} t^{\alpha-Q} \int_{B(x,t)} |f(y)| dy.
\end{aligned}$$

According to the first part of Lemma 4.2, we get

$$\begin{aligned}
J_2 &\lesssim \|b\|_*^k \Big(1+\ln\frac{r}{r}\Big)^k w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} t^{\alpha-Q} \int_{B(x,t)} |f(y)| dy \\
&\le \|b\|_*^k \, w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} t^{\alpha-Q} \|f\|_{L_{p,w^p}(B(x,t))} \|w^{-1}\|_{L_{p'}(B(x,t))} \\
&= \|b\|_*^k \, w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} \ln^k \Big(e+\frac{t}{r}\Big) w^q(B(x,t))^{-\frac{1}{q}} \|f\|_{L_{p,w^p}(B(x,t))}.
\end{aligned}$$

Summing up J_1 and J_2 , for all $p \in (1, \infty)$ we get

$$\|M_{b,\alpha,k} f_2\|_{L_{q,w^q}(B)} \lesssim \|b\|_*^k \, w^q(B(x,r))^{\frac{1}{q}} \times \sup_{t>2r} \ln^k \Big(e + \frac{t}{r}\Big) w^q(B(x,t))^{-\frac{1}{q}} \|f\|_{L_{p,w^p}(B(x,t))}. \tag{4.6}$$

Finally, from (4.5) and (4.6) we get

$$\begin{aligned}\|M_{b,\alpha,k} f\|_{L_{q,w^q}(B)} &\lesssim \|b\|_*^k (w^q(B(x,r)))^{\frac{1}{q}} \sup_{t\geq r} \|f\|_{L_{p,w^p}(B(x,t))} (w^q(B(x,t)))^{-\frac{1}{q}} \\ &+ \|b\|_*^k \, w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} \ln^k \Big(e + \frac{t}{r}\Big) w^q(B(x,t))^{-\frac{1}{q}} \|f\|_{L_{p,w^p}(B(x,t))} \\ &\lesssim \|b\|_*^k \, w^q(B(x,r))^{\frac{1}{q}} \sup_{t>2r} \ln^k \Big(e + \frac{t}{r}\Big) w^q(B(x,t))^{-\frac{1}{q}} \|f\|_{L_{p,w^p}(B(x,t))}.\end{aligned}$$

For the operator $M_{b,\alpha,k}$ the following Spanne-Guliyev type result on the space $M_{p,\varphi}(w)$ is valid.

Theorem 4.2 *Let* $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w \in A_{p,q}(\mathbb{R}^n)$, $b \in BMO(\mathbb{R}^n)$ *and* (φ_1, φ_2) *satisfy the condition*

$$\sup_{t>r} \ln^k \Big(e + \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)(w^p B((x,s)))^{1/p}}{(w^q(B(x,t)))^{1/q}} \leq C \varphi_2(x,r), \tag{4.7}$$

where C *does not depend on* u *and* r*. Then the operator* $M_{b,\alpha,k}$ *is bounded from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$*. Moreover,*

$$\|M_{b,\alpha,k} f\|_{M_{q,\varphi_2}(w^q)} \lesssim \|b\|_*^k \|f\|_{M_{p,\varphi_1}(w^p)}.$$

Proof. Using the Theorem 2.1 and the Theorem 4.1 we have

$$\begin{aligned}\|M_{b,\alpha,k} f\|_{M_{q,\varphi_2}(w^q)} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w^q(B(x,r))^{\frac{1}{q}} \|M_{b,\alpha,k} f\|_{L_{q,w^q} B(x,r)} \\ &\lesssim \|b\|_*^k \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>2r} \ln^k \Big(e + \frac{t}{r}\Big) w^q(B(x,t))^{-\frac{1}{q}} \|f\|_{L_{p,w^p}(B(x,t))} \\ &\lesssim \|b\|_*^k \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} (w^p(B(x,r)))^{-\frac{1}{p}} \|f\|_{L_{p,w^p}(B(x,r))} \\ &= \|b\|_*^k \|f\|_{M_{p,\varphi_1}(w^p)}.\end{aligned}$$

Corollary 4.1 *Let* $1 < p < q < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w \in A_{p,q}(\mathbb{R}^n)$, $b \in BMO(\mathbb{R}^n)$, $b^- \in L^\infty(\mathbb{R}^n)$ *and* (φ_1, φ_2) *satisfy the condition* (4.7)*. Then the operator* $[b, M_\alpha]^k$ *is bounded from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$*.*

Remark 4.2 Note that, in the case $w \equiv 1$, Theorems 4.1 and 4.2 were proved in [14], see also [19,21,23].

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