On Wiman-Valiron type estimations for parabolic equations

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Abstract. In the paper we established Wiman-Valiron type estimations for parabolic equations of the form

\[ u'(t) + A(t)u(t) = 0 \]

in Hilbert space, where \( A(t) \) is a uniformly positive-definite self-adjoint operator with a discrete spectrum. Imposing asymptotic character conditions on the distribution function \( N(\lambda) \) of the eigen-values of the operator \( A(t) \), we derive estimations for the norm of the solution of the equation that in particular characterize the behavior of the solution as \( t \to 0 \) depending on the behavior of the Fourier coefficients of initial data.

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Mathematics Subject Classification (2010):

1 Introduction

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function,

\[ M(r) = \max_{|z|=r} |f(z)|; \mu(r) = \max_{n} |a_n| r^n \]

be a maximum of the modulus and maximum term of the function \( f(z) \) in the circle of radius \( r \). As known, the inequality \( \mu(r) \leq M(r) \) always holds. But it is very important to estimate \( M(r) \) through \( \mu(r) \) from above Wiman [8] and Valiron [7] fist established the classical result: The following inequality is valid

\[ M(r) \leq \mu(r)(\log \mu(r))^{\frac{1}{2} + \varepsilon}, \varepsilon > 0. \]  

(1.1)

And this inequality can violate only on some set \( E \subset (0, \infty) \) of finite logarithmic measure, \( \int \frac{dr}{r} < \infty \).

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In 1963, the American mathematician Rosenbloom [3] established more accurate and overall result: for some class of increasing functions $\varphi(y), y > 0$, satisfying the condition of the form
\[ \int_0^\infty \left( \int_0^y \varphi(t) \, dt \right) \frac{1}{y} \, dy < \infty, \tag{1.2} \]
the following estimation
\[ \frac{M(r)}{\sqrt{\varphi(\log M(r))}} \leq c \mu(r), c = \text{const} > 0 \tag{1.3} \]
is valid.

Hence, in particular for the function $\varphi(y) = y^k$, the Wiman-Valiron result (1.1) is obtained. In the monograph of Suleymanov [4], published in M.V. Lomonosov MSU publishing house in 2012, theory of Wiman-Valiron-Rosenbloom type estimations was constructed for solving evolution equations of type
\[ u'(t) \pm A(t)u(t) = 0 \tag{1.4} \]
in Hilbert space, where $A(t)$ in particular, is a self-adjoint positive operator with a discrete spectrum, where the role of the functions $M(r)$ and $\mu(r)$ are played by the functions
\[ M(t) = \|u(t)\|, \mu(t) = \max_k |(u(t), \varphi_k(t))|, \]
where $\{\varphi_k(t)\}_{k \geq 1}$ is a complete orthonormed system of eigen-functions of the operator $A(t)$. These estimations, in particular, characterize the behavior of the solution of equation (1.4) as $t \to 0$ or $t \to \infty$ depending on the behavior of Fourier coefficients of initial data.

2 Formulation and proof of the main result.

In the present paper we establish the Rosenbloom type estimations (1.3) for parabolic equations of the form
\[ u'(t) + A(t)u(t) = 0 \tag{2.1} \]
in Hilbert space and study the behavior of the solution as $t \to 0$. Equation (2.1), in particular, contains the heat equation ($A(t) = -\Delta$) in the domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary.

Let $u(t)$ be the solution of equation (2.1). Denote
\[ \mu(t) = \max_k |(u(t), \varphi_k(t))|; g(t) = \frac{1}{2} \log (u(t), u(t)). \]

Let $N(\lambda)$ be the number of eigen-values $\lambda_k$ of the operator $A(t)$ such that $\lambda_k \leq \lambda, \lambda > 0$. The following lemma was proved in Suleymanovs above mentioned monograph ([4], p.88):

The key lemma. The following nonlinear differential equation
\[ e^{2g(t)} \leq \mu^2(t) \cdot \Delta N \left( g'(t), g''(t) \right), \]
where
\[ \Delta N(a, b) = N \left( a + c \sqrt{b + k(t)a} \right) - N \left( a - c \sqrt{b + k(t)a} \right), \]
\[ k(t) > 0, k(t) \in L_1(0, \infty), c = \text{const} > 0 \]
is valid. Note that for proving the lemma, in Suleymanov’s monograph a special method based on theory of probability was offered.

In the following theorem we impose a condition on the asymptotic behavior of the distribution function $N(\lambda)$ of eigen values of the operator $A(t)$, establish Wiman-Valiron type estimations for solving equation (4)

**Theorem 2.1** Let the function satisfy the following conditions

1. $N(\lambda) \leq c\lambda^{s+1}, c > 0, s \geq -1, \lambda \geq 1$.
2. For $\lambda > \delta > 0, \lambda \to \infty$.

$$N(\lambda, \delta) \equiv N(\lambda + \delta) - N(\lambda - \delta) = c\delta (1 + \lambda^{v}) \lambda^{s}, 0 < v < 1.$$  

Let the function $\varphi(y) > 0, y > 0$, do not decrease and the following integral a be finite:

$$\int_{-\infty}^{\infty} \left( \int_{-\infty}^{y} \varphi(t)dt \right)^{-\frac{1}{2(s+1)}} dy < \infty \quad (\varphi)$$

Then, maybe out of some set of finite logarithmic measure, for solving equation (2.1) the following Wiman-Valiron-Rosenbloom type estimation is valid:

$$\frac{\|u(t)\|}{\sqrt{\varphi(t^{-\beta} \log \|u(t)\|)}} \leq \mu(t)t^{-\gamma}, \quad (B - B)$$

where $\gamma >, 0 < \beta < 1, t \to 0$ ( $\gamma$ is calculated specifically).

**Proof.** We immediately note that the functions for which the asymptotic formulas of the form

$$N(\lambda) = c\lambda^{s/m} + O\left(\lambda^{\frac{n-1}{m}}\right), \lambda \to \infty \quad (2.2)$$

are true satisfy conditions 1 and 2 on the function $N(\lambda)$.

Such formulas were established for self-adjoint positive elliptic operators of order in bounded domain $\Omega \subset R^{n}$ in the papers of Weyl, Courant, Hörmander, Agmon, Seeley, Shubin, Ivriy, Birman, Solomyak, Vasilyev, Clark, Kostyuchenko, Levitan, Mikhailets, Safarov and others.

For example, in the paper of Safarov and Netrusov ([2], 2005) the unimprovable asymptotic formula

$$N(\lambda) = \lambda^{n} + O\left(\lambda^{n-\alpha}\right), 0 < \alpha < 1. \quad (2.3)$$

was established for the Laplace-Dirichlet operator $(-\Delta_{D})$ in domain $\Omega \subset R^{n}$ with a smooth boundary for the function $N(\lambda)$.

For formula (2.2) condition 2 of the theorem is fulfilled for while for $s = \frac{n}{m} - 1, v = \frac{1}{m}$, formula (2.3) it is fulfilled for $s = n - 1 - \alpha, \nu = \alpha$.

The functions

$$\varphi(y) = y^{2s+1+\varepsilon}, \text{or \hspace{0.5cm}} \varphi(y) = y^{2s+1}(\log y)^{2(s+1)+\varepsilon}$$

satisfy the condition $(\varphi)$. Having calculated the derivatives $g', g''$, we find

$$g' < 0, g'' > 0, g'' - kg' > 0$$

Introduce a new variable:

$$\xi(t) = \int_{0}^{t} \exp\left(\int_{0}^{y} k(\tau) d\tau\right) dy.$$
Hence we determine the inverse function \( t = t(\xi) \). For simplicity we consider that \( k(t) = k = \text{const} > 0 \). Assume:
\[
\tilde{g}(\xi) = g(t(\xi)) = g(t).
\]

We find:
\[
\tilde{g}(\xi') = g'(t)e^{-kt},
\]
\[
\sqrt{g'' + k g'} = e^{kt}\sqrt{\tilde{g}''(\xi)}.
\]

We determine \( \lambda(t) \) and \( \delta(t) \) as follows:
\[
\lambda(t) = \left[|g'| - \frac{1}{2}\sqrt{\tilde{g}'' + k|g'|}\right] = e^{kt}\left[|\tilde{g}'| - \frac{1}{2}\sqrt{\tilde{g}''}\right]
\]
\[
\delta(t) = \sqrt{g'' + k|g'|} = e^{kt}\sqrt{\tilde{g}''}.
\]

Then we get:
\[
\lambda(t) = e^{kt}\left[|\tilde{g}'| - \frac{3}{2}\sqrt{\tilde{g}''} + \sqrt{\tilde{g}''}\right] = e^{kt}\left[|\tilde{g}'| - \frac{3}{2}\sqrt{\tilde{g}''}\right] + \delta(t)
\]

We consider the two cases:

1. \(|\tilde{g}'| - \frac{3}{2}\sqrt{\tilde{g}''} + \sqrt{\tilde{g}''} > 0\); 2. \(|\tilde{g}'| - \frac{3}{2}\sqrt{\tilde{g}''} \leq 0\).

Let condition 1) be fulfilled. Then it is clear that \( \lambda(t) > \delta(t) \) and \( \lambda < |\tilde{g}'| \) (since \( \lambda \leq 1 \) then \( |\tilde{g}'| > 1 \)). Then from (1) we get:
\[
\Delta N (\lambda, \delta) \leq c\sqrt{\tilde{g}''} (1 + |\tilde{g}'|^v) |\tilde{g}'|^s \leq 2c\sqrt{\tilde{g}''} |\tilde{g}'|^{s+v}.
\]

**Lemma 2.1** Under the theorem conditions, the set
\[
E_1 = \left\{ \xi > 0 : \tilde{g}''(\xi) |\tilde{g}'(\xi)|^{2s} > \xi^{\frac{s}{2}} \sqrt{\varphi(\xi^\beta \tilde{g}(\xi))} \right\}
\]
has a finite measure. The proof of this lemma is similar to the proof of the similar lemma from Suleymanovs monograph ([4], p.74) with regard to the parameter \( \beta \) and \( s \) was changed by \( s + v \). Thus, (**) by out \( E_1 \) of the following inequality holds:
\[
\Delta N (\tilde{g}', \tilde{g}''') \leq c\xi^{-\frac{s}{2}} \sqrt{\varphi(\xi^\beta \tilde{g}(\xi))}.
\]

Passing in the right hand side, to the variable \( t \) we get that out of the set \( E \) of finite logarithmic measure, the following inequality is fulfilled:
\[
\Delta N (g', g''') \leq c\xi^{-\frac{s}{2}} \sqrt{\varphi(t^\beta g(t))}.
\]

Then by virtue of the key lemma we get
\[
e^{2\tilde{g}(t)} \leq \mu^2(t)\xi^{-\frac{s}{2}} \sqrt{\varphi(t^{-\beta} g(t))}
\]
or the same,
\[
\|u(t)\| \leq \mu(t)\xi^{-\frac{s}{2}} \sqrt{\varphi(t^{-\beta} \log \|u(t)\|)}.
\]

The theorem is proved.
Corollary. Assume $\varphi(y) = y^{2s+1+\varepsilon}$. We get:

$$
\|u(t)\| \leq \mu(t) t^{-\gamma} (\log \|u(t)\|)^k,
$$

where

$$
\gamma = \alpha + \beta \left(\frac{2k + 1 + \varepsilon}{4}\right), k = \frac{2s + 1 + \varepsilon}{4}
$$

The last estimation as $t \to 0$ is equivalent to the estimation

$$
\|u(t)\| \leq \mu(t) t^{-\gamma} (\log t^{-\gamma} \mu(t))^k.
$$

Remark. Assuming here $\bar{\mu}(t) = \bar{\gamma} \mu(t), k = \frac{1}{2} + \varepsilon$, we get the exact form of the Wiman-Valiron estimation in theory of entire functions:

$$
\|u(t)\| \leq \bar{\mu}(t) (\log \bar{\mu}(t))^{\frac{1}{2} + \varepsilon}, t \to +0.
$$

Case 20) of the theorem is studied in the same way.

References