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On curvatures of linear coframe bundle with homogeneous lift of a Riemannian metric

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Abstract. *In this paper the homogeneous lift of a Riemannian metric to the linear coframe bundle over a Riemannian manifold is constructed and curvature properties of this metric are investigated.*

Keywords. Riemannian metric · Linear coframe bundle · Homogeneous lift · Levi-Civita connection · Curvature tensor · Scalar curvature

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1 Introduction

It is well - known that the studies connected with Sasaki metric in the fiber bundles are one of the basic foundations of the modern differential geometry [1-7], [10], [13], [14]. However, the Sasaki metric is non homogeneous in the fibres of above mentioned bundles and is not used for study the global properties of these bundles. In [8] R.Miron introduced the homogeneous lift G of a Riemannian metric g to tangent bundle. Similar homogeneous lift of a Riemannian metric g in the cotangent bundle $T^*(M_n)$ was investigated by P.Stavre and L.Popescu [15]. The homogeneous lift of a Riemannian metric g to the tensor bundle of type $(1,1)$ and the curvature properties of the Levi-Civita connection of this metric are studied in [9].

This paper is devoted to the investigation of curvature properties of homogeneous lift of a Riemannian metric in the linear coframe bundle. In 2 we briefly describe the definitions and results that are needed later, after which the homogeneous lift $\tilde{g}$ of a Riemannian metric g to the linear coframe bundle $F^*(M_n)$ is constructed in 3. The Levi-Civita connection of homogeneous lift $\tilde{g}$ is studied in 4. Curvature properties of homogeneous lift of a Riemannian metric are investiqated in 5.

2 Preliminaries

In this section, we summarize all the basic definitions and results, which be used later. In the following all manifolds, maps, tensor fields, connections and metrics under consideration are supposed to be differentiable of class C^∞.

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Let (M_n, g) be an $n-$dimensional Riemannian manifold and

$$F^*(M_n) = \{(x, u^*) \,|x \in M_{\ n}, u^* : \text{ basis (coframe) for } T_x^*(M_n)\}$$

be the linear coframe bundle over M_n(see, [11, 12]). We denote by π the natural projection of $F^*(M_n)$ to M_n defined by $\pi(x, u^*) = x$. If $(U; x^1, x^2, ..., x^n)$ is a system of local coordinates in M_n, then a coframe $u^* = (X^\alpha) = (X^1, X^2, ..., X^n)$ for $T_x^*(M_n)$ can be expressed uniquely in the form $X^\alpha = X_i^\alpha (dx^i)_x$ and hence

$$(\pi^{-1}(U); x^1, x^2, ..., x^n, X_1^1, X_2^1, ..., X_n^n)$$

is a system of local coordinates in $F^*(M_n)$ (see, [11]). We note that indices $i, j, k, ..., \alpha, \beta, \gamma, ...$ have range in $\{1, 2, ..., n\}$, while indices $A, B, C, ...$ have range in

$$\left\{1, ..., n, n+1, ..., n+n^2\right\}.$$

We put $h_\alpha = \alpha \cdot n + h$ $(h_\alpha = n+1, n+2, ..., n+n^2)$. Summation over repeated indices is always implied.

We denote by $\Im_s^r(M_n)$ the set of all differentiable tensor fields of type (r, s) on M_n. Also we consider a symmetric linear connection ∇ on M_n with components Γ_{ij}^k.

Let a vector field $V \in \Im_0^1(M_n)$ and a covector field (1-form) $\omega \in \Im_1^0(M_n)$ are given on M_n. In terms of local components, if $V = V^i \partial_i$ and $\omega = \omega_i dx^i$. Then the complete and horizontal lifts ${}^C V, {}^H V \in \Im_0^1(F^*(M_n))$ of V and the $\beta-$th vertical lifts ${}^{V_\beta}\omega \in \Im_0^1(F^*(M_n))$ $(\beta = 1, 2, ..., n)$ of ω defined by

$$\begin{aligned} {}^C V &= V^i \partial_i - X_m^\alpha \partial_i V^m \partial_{i_\alpha}, \\ {}^H V &= V^i \partial_i + X_m^\alpha \Gamma_{ik}^m V^k \partial_{i_\alpha}, \qquad (2.1) \\ {}^{V_\beta}\omega &= \sum_i \delta_\beta^\alpha \omega_i \partial_{i_\alpha}, \qquad (2.2) \end{aligned}$$

with respect to the natural frame $\{\partial_i, \partial_{i_\alpha}\} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial X_i^\alpha}\right\}$, respectively (see [11] for more details).

The vertical lift of a smooth function f on M_n is a function ${}^V f$ on $F^*(M_n)$ defined by ${}^V f = f \circ \pi$.

Let (U, x^i) be a coordinate system in M_n. In $U \in M_n$, we put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \theta^{(i)} = dx^i, i = 1, 2, ..., n.$$

Taking account of (2.1) and (2.2), we easily see that the components of ${}^H X_{(i)}$ and ${}^{V_\beta}\theta^{(i)}$ are given by

$$ {}^H X_{(i)} = \left(A_i^H\right) = \begin{pmatrix} \delta_i^h \\ X_j^\alpha \Gamma_{ih}^j \end{pmatrix}, \qquad (2.3)$$

$$ {}^{V_\beta}\theta^{(i)} = \left(A_{i_\alpha}^H\right) = \begin{pmatrix} 0 \\ \delta_\beta^\alpha \delta_h^i \end{pmatrix} \qquad (2.4)$$

with respect to the natural frame $\{\partial_i, \partial_{i_\alpha}\}$, respectively. The set $\left\{{}^H X_{(i)}, {}^{V_\beta}\theta^{(i)}\right\}$ is called the frame adapted to linear connection ∇ on $\pi^{-1}(U) \subset F^*(M_n)$. On putting

$$D_i = {}^H X_{(i)}, \quad D_{i_\alpha} = {}^{V_\alpha}\theta^{(i)},$$

we write the adapted frame as $\{D_I\} = \{D_i, D_{i_\alpha}\}$. From equations (2.1)-(2.4), we see that ${}^HV$ and ${}^{V_\beta}\omega$ have respectively, components

$$ {}^HV = \begin{pmatrix} V^i \\ 0 \end{pmatrix}, \tag{2.5} $$

$$ {}^{V_\beta}\omega = \begin{pmatrix} 0 \\ \delta^\alpha_\beta \omega_i \end{pmatrix} \tag{2.6} $$

with respect to the adapted frame $\{D_I\}$.

The Lie-bracket of the linear coframe bundle $F^*(M_n)$ over M_n are expressed in the form

$$ [{}^HX, {}^HY] = {}^H[X,Y] + \sum_{\sigma=1}^{n} \left({}^{V_\sigma}X \circ R(X,Y)\right), $$

$$ [{}^HX, {}^{V_\beta}\omega] = {}^{V_\beta}(\nabla_X \omega), $$

$$ [{}^{V_\beta}\omega, {}^{V_\gamma}\theta] = 0 $$

for all $X, Y \in \Im^1_0(M_n)$ and $\omega, \theta \in \Im^0_1(M_n)$, where $\beta, \gamma = 1, 2, ..., n$ and R is the curvature tensor field of ∇ defined by

$$ R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z $$

for all $X, Y, Z \in \Im^1_0(M_n)$.

3 Homogeneous lift of a Riemannian metrics

The diagonal lift ${}^Dg$ of a Riemannian metric g on M_n to the linear coframe bundle $F^*(M_n)$ defined by formula (see, [3])

$$ {}^Dg = g_{ij} dx^i \otimes dx^j + \delta_{\alpha\beta} g^{ij} \delta X^\alpha_i \otimes \delta X^\beta_j $$

and satisfies the following conditions:

$$ {}^Dg({}^HX, {}^HY) = {}^V(g(X,Y)) = g(X,Y) \circ \pi. $$

$$ {}^Dg\left({}^{V_\alpha}\omega, {}^{V_\beta}\theta\right) = \delta^{\alpha\beta V}\left(g^{-1}(\omega,\theta)\right) = \delta^{\alpha\beta}\left(g^{-1}(\omega,\theta)) \circ \pi\right), $$

$$ {}^Dg({}^HX, {}^{V_\beta}\theta) = 0 $$

for all $X, Y \in \Im^1_0(M_n)$ and $\omega, \theta \in \Im^0_1(M_n)$, where $\delta X^\alpha_i = dX^\alpha_i - \Gamma^m_{ki} X^\alpha_m dx^k$ and g^{ij} denote contravariant components of g.

We easily see, that the metric ${}^Dg$ is not $0-$homogeneous on the fibers of the linear coframe bundle $F^*(M_n)$, i.e. for any $\lambda \in R_+$,

$$ {}^Dg(x, u^*) \neq {}^Dg(x, \lambda u^*). $$

Now, we define a new lift $\tilde{g}$ of a Riemannian metric g on M_n, to the coframe bundle $F^*(M_n)$ as follows:

$$ \tilde{g} = g_{ij} dx^i \otimes dx^j + \tfrac{1}{h} \delta_{\alpha\beta} g^{ij} \delta X^\alpha_i \otimes \delta X^\beta_j, $$

where h is a function defined as

$$h=\sum_{\alpha=1}^{n}\left\|X^{\alpha}\right\|^{2}=\sum_{\alpha=1}^{n}g^{ij}X_{i}^{\alpha}X_{j}^{\alpha}=\sum_{\alpha=1}^{n}g^{-1}\left(X^{\alpha},X^{\alpha}\right). \tag{3.1}$$

It is easy to see that $\tilde{g}$ is $0-$homogeneous with respect to X_i^{α}, i.e.

$$\tilde{g}\left(x,\lambda u^{*}\right)=g_{ij}dx^{i}\otimes dx^{j}+\tfrac{\lambda^{2}}{\lambda^{2}\cdot h}\delta_{\alpha\beta}g^{ij}\delta X_{i}^{\alpha}\otimes\delta X_{j}^{\beta}=\tilde{g}\left(x,u^{*}\right),\forall\lambda\in R_{+}.$$

The metric $\tilde{g}$ has components

$$\left(\tilde{g}_{IJ}\right)=\begin{pmatrix} g_{ij} & 0 \\ 0 & \frac{1}{h}\delta_{\alpha\beta}g^{ij}\end{pmatrix} \tag{3.2}$$

with respect to the adapted frame $\{D_I\}$ in $F^{*}\left(M_n\right)$.

From (3.2) it easily follows that if g is a Riemannian metric in M_n, then $\tilde{g}$ is a Riemannian metric in $F^{*}(M_n)$. The metric $\tilde{g}$ is similar to that of the Riemannian metric studied by Miron in the slit tangent bundle $T(M_n)\setminus\{0\}$ [8] (for the slit cotangent bundle $T^{*}(M_n)\setminus\{0\}$ and slit $(1,1)-$ tensor bundle $T_1^1(M_n)\setminus\{0\}$ see [15] and [9], respectively).

Remark 3.1 Since $u^{*}=(X^1,X^2,...,X^n)\neq 0$ is a basis of the cotangent space $T_x^{*}(M_n)$, the condition $h\neq 0$ is fulfilled at each point $x\in M_n$ and the bundle of linear coframes $F^{*}(M_n)$ is naturally a slit bundle. This means that the metric $\tilde{g}$ is defined in the linear coframe bundle $F^{*}(M_n)$.

It is easily to verify that the inverse matrix $\left(\tilde{g}^{IJ}\right)$ of matrix $\left(\tilde{g}_{IJ}\right)$ is as follows:

$$\left(\tilde{g}^{IJ}\right)=\begin{pmatrix} g_{ij} & 0 \\ 0 & h\delta^{\alpha\beta}g_{ij}\end{pmatrix} \tag{3.3}$$

with respect to the adapted frame $\{D_I\}$ in $F^{*}\left(M_n\right)$.

Also, we can represent the metric $\tilde{g}$ by the following global formulas:

$$\begin{gathered}\tilde{g}\left({}^{H}X,{}^{H}Y\right)=g\left(X,Y\right)\circ\pi. \\ \tilde{g}\left({}^{V_{\alpha}}\omega,{}^{V_{\beta}}\theta\right)=\tfrac{1}{h}\delta_{\alpha\beta}\left(g^{-1}\left(\omega,\theta\right)\circ\pi\right), \\ \tilde{g}\left({}^{H}X,{}^{V_{\beta}}\theta\right)=0\end{gathered} \tag{3.4}$$

for all vector fields $X,Y\in\Im_0^1\left(M_n\right)$ and covector fields ($1-$ forms) $\omega,\theta\in\Im_1^0\left(M_n\right)$.

Let us consider local $1-$forms $\tilde{\eta}^I$ in $\pi^{-1}\left(U\right)$ defined by

$$\tilde{\eta}^{I}=\bar{A}^{I}\quad_{J}dx^{J},$$

where

$$A^{-1}=\left(\bar{A}^{I}\quad_{J}\right)=\begin{pmatrix}\bar{A}^{i}{}_{j} & \bar{A}^{i}{}_{j_{\beta}} \\ \bar{A}^{i_{\alpha}}{}_{j} & \bar{A}^{i_{\alpha}}{}_{j_{\beta}}\end{pmatrix}=\begin{pmatrix}\delta_{j}^{i} & 0 \\ -X_{m}^{\alpha}\Gamma_{ij}^{m} & \delta_{\beta}^{\alpha}\delta_{i}^{j}\end{pmatrix}. \tag{3.5}$$

The matrix (3.5) is the inverse of the matrix

$$A=\left(A_{K}{}^{J}\right)=\begin{pmatrix}A_{k}{}^{j} & A_{k_{\gamma}}{}^{j} \\ A_{k}{}^{j_{\beta}} & A_{k_{\gamma}}{}^{j_{\beta}}\end{pmatrix}=\begin{pmatrix}\delta_{k}^{j} & 0 \\ X_{m}^{\beta}\Gamma_{jk}^{m} & \delta_{\gamma}^{\beta}\delta_{j}^{k}\end{pmatrix} \tag{3.6}$$

of the transformation $D_K = A_K{}^J \partial_J$ (see (2.3) and (2.4)). It is easy to establish that the set $\left\{\tilde{\eta}^I\right\}$ is the coframe dual to the adapted frame $\{D_K\}$, i.e.

$$\tilde{\eta}^I\left(D_K\right) = \bar{A}^I{}_J A_K{}^J = \delta^I_K.$$

Since the adapted frame is non-holonomic, we put

$$[D_I, D_J] = \Omega_{IJ}{}^K D_K$$

from which we have

$$\Omega_{IJ}{}^K = \left(D_I A_J^L - D_J A_I^L\right) \bar{A}^K_L.$$

According to (3.5) and (3.6), the components of non-holonomic object $\Omega_{IJ}{}^K$ are given by

$$\begin{cases} \Omega_{ij_\beta}{}^{k_\gamma} = -\Omega_{j_\beta i}{}^{k_\gamma} = -\delta^\gamma_\beta \Gamma^j_{ik}, \\ \Omega_{ij}{}^{k_\gamma} = X^\gamma_m R_{ijk}{}^m \end{cases} \tag{3.7}$$

all the others being zero.

4 Levi-Civita connection of $\tilde{g}$

We note that the Levi-Civita connection ∇ of a Riemannian metric g is given by well-known Koszul formula

$$\begin{aligned} 2g\left(\nabla_X Y, Z\right) = Xg\left(Y, Z\right) + Yg\left(Z, X\right) - Zg\left(X, Y\right) + g\left([X, Y], Z\right) \\ -g\left([Y, Z], X\right) + g\left([Z, X], Y\right) \end{aligned} \tag{4.1}$$

for all vector fields $X, Y, Z \in \Im^1_0\left(M_n\right)$.

By using (2.5), (2.6), (3.1), (3.3), (3.7) and (4.1), we have

Theorem 4.1 *Let M_n be a Riemannian manifold with metric g and $\tilde{\nabla}$ be the Levi-Civita connection of the linear coframe bundle $F^*(M_n)$ equipped with the metric $\tilde{g}$. Then $\tilde{\nabla}$ satisfies*

$$\begin{gathered} i) \quad \tilde{\nabla}_{{}^H X}{}^H Y = {}^H\left(\nabla_X Y\right) + \tfrac{1}{2}\sum_{\sigma=1}^n {}^{V_\sigma}\left(X^\sigma \circ R\left(X, Y\right)\right), \\ ii) \quad \tilde{\nabla}_{{}^H X}{}^{V_\alpha}\omega = {}^{V_\alpha}\left(\nabla_X \omega\right) + \tfrac{1}{2h}\sum_{\sigma=1}^n \delta_{\alpha\sigma}{}^H\left(X^\sigma \circ R\left(\ \ , X\right)\tilde{\omega}\right), \\ iii) \quad \tilde{\nabla}_{{}^{V_\alpha}\omega}{}^H Y = \tfrac{1}{2h}\sum_{\sigma=1}^n \delta_{\alpha\sigma}{}^H\left(X^\sigma \circ R\left(\ \ , Y\right)\tilde{\omega}\right), \\ iv) \quad \tilde{\nabla}_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta = -\tilde{g}\left({}^{V_\alpha}\omega, \sum_{\sigma=1}^n {}^{V_\sigma}X^\sigma\right){}^{V_\beta}\theta - \tilde{g}\left({}^{V_\beta}\theta, \sum_{\sigma=1}^n {}^{V_\sigma}X^\sigma\right){}^{V_\alpha}\omega \\ +\tilde{g}\left({}^{V_\alpha}\omega, {}^{V_\beta}\theta\right)\sum_{\sigma=1}^n {}^{V_\sigma}X^\sigma, \end{gathered} \tag{4.2}$$

for all $X, Y \in \Im^1_0\left(M_n\right)$, $\omega, \theta \in \Im^0_1\left(M_n\right)$, where $\tilde{\omega} = g^{-1} \circ \omega \in \Im^1_0\left(M_n\right)$, $\tilde{X}^\alpha = g^{-1} \circ X^\alpha \in \Im^1_0\left(M_n\right)$.

We note that the analogue of Theorem 4.1 for the case of a diagonal lift ${}^{D}g$ of a Riemannian metric g to the linear coframe bundle $F^{*}(M_n)$, was proved in [3].
Let us put

$$\tilde{\nabla}_{D_I} D_J = \tilde{\Gamma}_{IJ}^{K} D_K$$

with respect to the adapted frame $\{D_K\}$ of linear coframe bundle $F^{*}(M_n)$, where $\tilde{\Gamma}_{IJ}^{K}$ denote the components of the Levi-Civita connection $\tilde{\nabla}$. Then by using the Theorem 4.1, we immediately get the following.

Theorem 4.2 *Let (M_n, g) be a Riemannian manifold and $\tilde{\nabla}$ be the Levi-Civita connection of the linear coframe bundle $F^{*}(M_n)$ equipped with the homogeneous lift $\tilde{g}$ of a Riemannian metric g on M_n. The particular values of $\tilde{\Gamma}_{IJ}^{K}$ for different indices, on taking account of (4.2) are then found to be*

$$\tilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k}, \quad \tilde{\Gamma}_{i_\alpha j_\beta}^{k} = \tilde{\Gamma}_{i_\alpha j}^{k_\gamma} = 0,$$

$$\tilde{\Gamma}_{ij}^{k_\gamma} = \tfrac{1}{2} \sum_{\sigma=1}^{n} \delta_\sigma^\gamma X_m^\sigma R_{ijk}{}^{m}, \qquad \tilde{\Gamma}_{ij_\beta}^{k_\gamma} = -\delta_\gamma^\beta \Gamma_{ik}^{j},$$

$$\tilde{\Gamma}_{i_\alpha j}^{k} = \tfrac{1}{2h} \sum_{\sigma=1}^{n} \delta_{\alpha\sigma} X_m^\sigma R_{\cdot j \cdot}^{k\ im}, \quad \tilde{\Gamma}_{ij_\beta}^{k} = \tfrac{1}{2h} \sum_{\sigma=1}^{n} \delta_{\beta\sigma} X_m^\sigma R_{\cdot i \cdot}^{k\ jm}, \tag{4.3}$$

$$\tilde{\Gamma}_{i_\alpha j_\beta}^{k_\gamma} = -\tfrac{1}{h} \left(g^{im} \left(\sum_{\sigma=1}^{n} \delta_\sigma^\alpha X_m^\sigma \right) \delta_\gamma^\beta \delta_k^j + g^{jm} \left(\sum_{\sigma=1}^{n} \delta_\sigma^\beta X_m^\sigma \right) \delta_\gamma^\alpha \delta_k^i \right.$$
$$\left. - g^{ij} \delta_{\alpha\beta} \left(\sum_{\sigma=1}^{n} \delta_{\sigma\gamma} X_k^\sigma \right) \right),$$

with respect to the adapted frame $\{D_K\}$, where $R_{\cdot i \cdot}^{k\ jm} = g^{kl} g^{js} R_{lis}{}^{m}$.

5 Curvature properties of Levi-Civita connection $\tilde{\nabla}$

Let $\tilde{R}$ be a curvature tensor field of the Levi-Civita connection $\tilde{\nabla}$. Then we have

$$\tilde{R}(D_I, D_J) D_K = \tilde{\nabla}_I \tilde{\nabla}_J D_K - \tilde{\nabla}_J \tilde{\nabla}_I D_K - \Omega_{IJ}{}^{L} \tilde{\nabla}_L D_K,$$

where $\tilde{\nabla}_I = \tilde{\nabla}_{D_I}$. The curvature tensor field $\tilde{R}$ has components

$$\tilde{R}_{IJK}{}^{L} = D_I \tilde{\Gamma}_{JK}^{L} - D_J \tilde{\Gamma}_{IK}^{L} + \tilde{\Gamma}_{IP}^{L} \tilde{\Gamma}_{JK}^{P} - \tilde{\Gamma}_{JP}^{L} \tilde{\Gamma}_{IK}^{P} - \Omega_{IJ}{}^{P} \tilde{\Gamma}_{PK}^{L}$$

with respect to the adapted frame $\{D_I\}$. Taking account (3.7) and (4.3), we obtain:

$$\tilde{R}_{ijk}{}^{l} = R_{ijk}{}^{l} - \tfrac{1}{2h} \sum_{\sigma=1}^{n} \left(X_m^\sigma X_r^\sigma \right) R_{ijp}{}^{m} R_{\cdot\ k\ \cdot}^{l\ \cdot\ pr}$$
$$+ \tfrac{1}{4h} \sum_{\sigma=1}^{n} X_m^\sigma X_r^\sigma \left(R_{jkp}{}^{m} R_{\cdot\ i\ \cdot}^{l\ \ pr} - R_{ikp}{}^{m} R_{\cdot\ j\ \cdot}^{l\ \ pr} \right),$$

$$\tilde{R}_{i_\alpha jk}{}^{l} = -\tfrac{1}{2h} X_m^\alpha \nabla_j R_{\cdot\ k\ \cdot}^{l\ \ im},$$

$$\tilde{R}_{ij_\beta k}{}^{l} = \tfrac{1}{2h} X_m^\beta \nabla_i R_{\cdot\ k\ \cdot}^{l\ \ jm},$$

$$\tilde{R}_{ijk_\gamma}{}^{l} = \frac{1}{2h} X_m^\gamma \left(\nabla_i R^{l}_{\cdot\, j\, \cdot}{}^{km} - \nabla_j R^{l}_{\cdot\, i\, \cdot}{}^{km} \right),$$

$$\tilde{R}_{i_\alpha j_\beta k}{}^{l} = \frac{1}{h^2} \left(X^{\beta j} X_m^\alpha R^{l}_{\cdot\, k\, \cdot}{}^{im} - X^{\alpha i} X_m^\beta R^{l}_{\cdot\, k\, \cdot}{}^{jm} \right)$$

$$+ \frac{1}{2h} \delta_{\alpha\beta} \left(R^{l}_{\cdot\, k\, \cdot}{}^{ji} - R^{l}_{\cdot\, k\, \cdot}{}^{ij} \right) + \frac{1}{4h^2} X_m^\alpha X_r^\beta \left(R^{l}_{\cdot\, p\, \cdot}{}^{im} \; R^{p}_{\cdot\, k\, \cdot}{}^{jr} \right.$$

$$\left. - R^{l}_{\cdot\, p\, \cdot}{}^{jr} \; R^{\,p}_{\cdot\, k\, \cdot}{}^{im} \right),$$

$$\tilde{R}_{i_\alpha j k_\gamma}{}^{l} = \frac{1}{2h} \delta_\alpha^\gamma R^{l}_{\cdot\, j\, \cdot}{}^{ki} + \frac{1}{4h^2} X_m^\alpha X_r^\gamma R^{l}_{\cdot\, p\, \cdot}{}^{im} \; R^{p}_{\cdot\, j\, \cdot}{}^{kr}$$

$$- \frac{1}{2h^2} \sum_{\tau=1}^{n} X_r^\tau R^{l}_{\cdot\, j\, \cdot}{}^{pr} \left(g^{im} \; X_m^\alpha \delta_\tau^\gamma \delta_p^k + g^{km} X_m^\gamma \delta_\tau^\alpha \delta_p^i - g^{ik} \delta_{\alpha\gamma} \; X_p^\tau \right),$$

$$\tilde{R}_{i j_\beta k_\gamma}{}^{l} = - \frac{1}{2h} \delta_\beta^\gamma R^{l}_{\cdot\, i\, \cdot}{}^{kj} - \frac{1}{4h^2} X_m^\beta X_r^\gamma R^{l}_{\cdot\, p\, \cdot}{}^{jm} \; R^{p}_{\cdot\, i\, \cdot}{}^{kr}$$

$$+ \frac{1}{2h^2} \sum_{\tau=1}^{n} X_r^\tau R^{l}_{\cdot\, i\, \cdot}{}^{pr} \left(g^{jm} \; X_m^\beta \delta_\tau^\gamma \delta_p^k + g^{km} X_m^\gamma \delta_\tau^\beta \delta_p^j - g^{jk} \delta_{\beta\gamma} \; X_p^\tau \right),$$

$$\tilde{R}_{ijk}{}^{l_\mu} = \frac{1}{2} X_m^\mu \left(\nabla_i R_{jkl}{}^{m} - \nabla_j R_{\cdot ikl}{}^{m} \right),$$

$$\tilde{R}_{i_\alpha jk}{}^{l_\mu} = \frac{1}{2} \delta_\alpha^\mu R_{jkl}{}^{i} - \frac{1}{2h} \sum_{\tau=1}^{n} X_r^\tau R_{jkp}{}^{r} \left(g^{im} \; X_m^\alpha \delta_{\mu\tau} \delta_l^p \right. \tag{5.1}$$

$$\left. + g^{pm} X_m^\tau \delta_{\alpha\mu} \delta_l^i - g^{ip} \delta_{\alpha\tau} X\,{}_l^\mu \right) - \frac{1}{4h} X_m^\mu X_r^\alpha R_{jpl}{}^{m} R^{p}_{\cdot\, k\, \cdot}{}^{ir},$$

$$\tilde{R}_{i j_\beta k}{}^{l_\mu} = - \frac{1}{2} \delta_\beta^\mu R_{ikl}{}^{j} + \frac{1}{2h} \sum_{\tau=1}^{n} X_r^\tau R_{ikp}{}^{r} \left(g^{jm} \; X_m^\beta \delta_{\mu\tau} \delta_l^p \right.$$

$$\left. + g^{pm} X_m^\tau \delta_{\alpha\mu} \delta_l^j - g^{jp} \delta_{\beta\tau} X\,{}_l^\mu \right) + \frac{1}{4h} X_m^\mu X_r^\alpha R_{jpl}{}^{m} R^{p}_{\cdot\, k\, \cdot}{}^{ir},$$

$$\tilde{R}_{ijk_\gamma}{}^{l_\mu} = \delta_\mu^\gamma R_{jil}{}^{k} + \frac{1}{4h} X_m^\mu \left(R_{ipl}{}^{m} X_r^\gamma R^{p}_{\cdot\, j\, \cdot}{}^{kr} - R_{jpl}{}^{m} X_r^\gamma R^{\,p}_{\cdot\, i\, \cdot}{}^{kr} \right)$$

$$- \frac{1}{h} \sum_{\tau=1}^{n} X_m^\tau R_{ijp}{}^{m} \left(g^{pr} \; X_r^\tau \delta_\mu^\gamma \delta_l^k + g^{kr} X_r^\gamma \delta_\mu^\tau \delta_l^p - g^{pk} \delta_{\tau\gamma} \; X_l^\mu \right),$$

$$\tilde{R}_{i_\alpha j_\beta k_\gamma}{}^{l_\mu} = - \frac{2X^{\alpha i}}{h^3} \left(X^{\gamma k} \delta_\mu^\beta \delta_l^j - g^{jk} \delta_{\beta\gamma} X_l^\mu \right)$$

$$+ \frac{2X^{\beta j}}{h^3} \left(X^{\gamma k} \delta_\mu^\alpha \delta_l^i - g^{ik} \delta_{\alpha\gamma} X_l^\mu \right) + \frac{1}{h^2} \left[X^{\gamma k} \left(X^{\beta j} \delta_\mu^\alpha \delta_l^i - X^{\alpha i} \delta_\mu^\beta \delta_l^j \right) \right.$$

$$\left. - \sum_{\tau=1}^{n} X^{\tau p} X_p^\tau \left(g^{jk} \delta_\mu^\alpha \delta_{\beta\gamma} \delta_l^i - g^{ik} \delta_\mu^\beta \delta_{\alpha\gamma} \delta_l^j \right) + X_l^\mu \left(X^{\alpha i} \delta_{\beta\gamma} g^{jk} - X^{\beta j} \delta_{\alpha\gamma} g^{\,ik} \right) \right],$$

$$\tilde{R}_{i_\alpha j_\beta k_\gamma}{}^{l} = \tilde{R}_{i_\alpha j_\beta k}{}^{l_\mu} = \tilde{R}_{i_\alpha j k_\gamma}{}^{l_\mu} = \tilde{R}_{i j_\beta k_\gamma}{}^{l_\mu} = 0.$$

Therefore, the following theorem holds.

Theorem 5.1 *Let (M_n, g) be a Riemannian manifold and $F^*(M_n)$be its linear coframe bundle with metric $\tilde{g}$. Then the coefficients of the Riemannian curvature tensor of metric $\tilde{g}$ with respect to the adapted frame $\{D_I\}$ are given by (5.1) , where $X^{\alpha i} = g^{im} X_m^\alpha$.*

Let $\tilde{R}_{IJ} = \tilde{R}_{KIJ}{}^{K}$ are the components of the Ricci tensor field with respect to the adapted frame $\{D_I\}$, then the scalar curvature $\tilde{r} = \tilde{g}^{IJ}\tilde{R}_{IJ}$ based on equalities (3.4) and (5.1), is given by

$$\tilde{r} = \tilde{g}^{ij}\tilde{R}_{ij} + \tilde{g}^{i_\alpha j_\beta}\tilde{R}_{i_\alpha j_\beta} = g^{ij}\left(\tilde{R}_{kij}{}^{k} + \tilde{R}_{k_\gamma ij}{}^{k_\gamma}\right)$$

$$+hg_{ij}\delta^{\alpha\beta}\tilde{R}_{ki_\alpha j_\beta}{}^{k} + hg_{ij}\delta^{\alpha\beta}\tilde{R}_{k_\gamma i_\alpha j_\beta}{}^{k_\gamma} =$$

$$r - \tfrac{1}{2h}\left(\sum_{\sigma=1}^{n} X_m^\sigma X_r^\sigma\right) g^{ij}g^{kq}g^{ps}R_{kip}{}^{m}R_{qjs}{}^{r} + \tfrac{1}{4h}\left(\sum_{\sigma=1}^{n} X_m^\sigma X_r^\sigma\right) g^{ij}g^{kq}g^{ps}R_{qks}{}^{r}R_{ijp}{}^{m}$$

$$-\tfrac{1}{4h}\left(\sum_{\sigma=1}^{n} X_m^\sigma X_r^\sigma\right) g^{ij}g^{kq}g^{ps}R_{qis}{}^{r}R_{ijp}{}^{m} + \tfrac{1}{2}g^{ij}\delta_\gamma^\gamma R_{ijk}{}^{k}$$

$$-\tfrac{1}{2h}\left(\sum_{\tau=1}^{n} X_m^\tau X_r^\tau\right) g^{ij}g^{km}R_{ijk}{}^{r} - \tfrac{1}{2h}\left(\sum_{\tau=1}^{n} X_m^\tau X_r^\tau\right) g^{ij}g^{pm}R_{ijp}^{r}\delta_{\gamma\gamma}\delta_k^k$$

$$+\tfrac{1}{2h}\left(\sum_{\tau=1}^{n} X_k^\tau X_r^\tau\right) g^{ij}g^{kp}R_{ijp}{}^{r} - \tfrac{1}{4h}\left(\sum_{\gamma=1}^{n} X_m^\gamma X_r^\gamma\right) g^{ij}g^{pq}g^{ks}R_{qjs}{}^{r}R_{ipk}{}^{m}$$

$$-\tfrac{1}{2}g_{ij}g^{kq}\delta_\beta^\beta R_{qks}{}^{i} - \tfrac{1}{4h}\left(\sum_{\beta=1}^{n} X_m^\beta X_r^\beta\right) g_{ij}g^{kq}g^{is}g^{pt}g^{ja}R_{qps}{}^{m}R_{tka}{}^{r}$$

$$+\tfrac{1}{2h}\left(\sum_{\tau=1}^{n} X_m^\tau X_r^\tau\right) g_{ij}g^{kq}g^{js}g^{im}R_{qks}{}^{r} + \tfrac{1}{2h}\left(\sum_{\tau=1}^{n} X_m^\tau X_r^\tau\right) g_{ij}g^{kq}g^{is}g^{jm}R_{qks}{}^{r}$$

$$-\tfrac{1}{2h}\delta_\alpha^\alpha\delta_i^i\left(\sum_{\tau=1}^{n} X_r^\tau X_p^\tau\right) g^{kq}g^{ps}R_{qks}{}^{r} - \tfrac{2}{h^2}\left(\sum_{\tau=1}^{n} X^{\tau k}X^{\tau j}\right) g_{kj}$$

$$+\tfrac{2}{h^2}\delta_i^i\left(\sum_{\tau=1}^{n} X^{\tau i}X^{\tau j}\right) g_{ij}\delta_\gamma^\gamma\delta_k^k - \tfrac{2}{h^2}\left(\sum_{\tau=1}^{n} X^{\tau k}X_k^\tau\right) + \tfrac{1}{h}\delta\left(\sum_{\tau=1}^{n} X^{\tau i}X^{\tau j}\right) g_{ij}\delta_\gamma^\gamma\delta_k^k$$

$$-\tfrac{1}{h}\left(\sum_{\tau=1}^{n} X^{\tau j}X^{\tau k}\right) g_{kj} - \tfrac{1}{h}\delta_i^i\left(\sum_{\tau=1}^{n} X^{\tau p}X_p^\tau\right) \delta_\gamma^\gamma\delta_k^k\delta_\alpha^\alpha$$

$$+\tfrac{1}{h}\delta_i^i\left(\sum_{\tau=1}^{n} X^{\tau p}X_p^\tau\right)\delta_\alpha^\alpha + \tfrac{1}{h}\delta_i^i\left(\sum_{\tau=1}^{n} X^{\tau k}X_k^\tau\right)\delta_\alpha^\alpha - \tfrac{1}{h}\left(\sum_{\tau=1}^{n} X^{\tau k}X_k^\tau\right)$$

$$= r + \tfrac{1}{2h}\left(1-n^2\right)\left(\sum_{\tau=1}^{n} X_r^\tau X_p^\tau\right) g^{kq}g^{ps}R_{qks}{}^{r}$$

$$+\tfrac{2}{h^2}\left(n^2-1\right)\left(\sum_{\tau=1}^{n} X^{\tau i}X^{\tau j}\right) g_{ij} + \tfrac{2}{h^2}\left(n^2-1\right)\left(\sum_{\tau=1}^{n} X^{\tau k}X_k^\tau\right)$$

$$+\tfrac{1}{h}\left(n^2-1\right)\left(\sum_{\tau=1}^{n} X^{\tau i}X^{\tau j}\right) g_{ij} - \tfrac{1}{h}\left(n^4-n^2\right)\left(\sum_{\tau=1}^{n} X^{\tau p}X_p^\tau\right)$$

$$+\tfrac{1}{h}\left(n^2-1\right)\left(\sum_{\tau=1}^{n} X^{\tau k} X_k^{\tau}\right)=r+\tfrac{1}{2h}\left(1-n^2\right)\left(\sum_{\tau=1}^{n} X_r^{\tau} X_p^{\tau}\right) g^{kq} g^{ps} R_{qks}{}^{r}$$

$$+\tfrac{4}{h^2}\left(n^2-1\right)\left(\sum_{\tau=1}^{n} X^{\tau k} X_k^{\tau}\right)-\tfrac{1}{h}\left(n^4-3n^2+1\right)\left(\sum_{\tau=1}^{n} X^{\tau p} X_p^{\tau}\right)$$

$$=r+\left(1-n^2\right)\left(\tfrac{1}{2h}\left(\sum_{\tau=1}^{n} X_r^{\tau} X_p^{\tau}\right) g^{kq} g^{ps} R_{qks}{}^{r}-\tfrac{4}{h^2}\left(\sum_{\tau=1}^{n} X^{\ \tau k} X_k^{\tau}\right)\right)$$

$$-\tfrac{1}{h}\left(n^4-3n^2+1\right)\left(\sum_{\tau=1}^{n} X^{\tau p} X_p^{\tau}\right).$$

Thus we have

Theorem 5.2 *Let (M_n, g) be a Riemannian manifold and $F^*(M_n)$ be its coframe bundle equipped with metric $\tilde{g}$. Then the scalar curvature of $\tilde{g}$ satisfies the following equation:*

$$\tilde{r}=r+\left(1-n^2\right)\left(\tfrac{1}{2h}\cdot A-\tfrac{4}{h^2}\cdot B\right)-\tfrac{1}{h}\left(n^4-3n^2+1\right)\cdot B, \tag{5.2}$$

where r is the scalar curvature of g and

$$A=\left(\sum_{\tau=1}^{n} X_r^{\tau} X_p^{\tau}\right) g^{kq} g^{ps} R_{qks}{}^{r}, B=\sum_{\tau=1}^{n} X^{\tau k} X_k^{\tau}.$$

Now we suppose that $(M_n, g), n>2$ be a Riemannian manifold of constant curvature κ, i.e.

$$R_{kmj}{}^{s}=\kappa\left(\delta_k^s g_{mj}-\delta_m^s g_{kj}\right) \tag{5.3}$$

and

$$r=n(n-1)\kappa.$$

Then by using (5.2) and (5.3), we deduce the following equation:

$$\tilde{r}=r+\left(1-n^2\right)\left(\tfrac{1}{2h}\cdot A-\tfrac{4}{h^2}\cdot B\right)-\tfrac{1}{h}\left(n^4-3n^2+1\right)\cdot B$$

$$=r+\left(1-n^2\right)\left(\tfrac{1}{2h}\left(\sum_{\tau=1}^{n} X_r^{\tau} X_p^{\tau}\right) g^{kq} g^{ps} R_{qks}^{s}-\tfrac{4}{h^2}\left(\sum_{\tau=1}^{n} X^{\tau k} X_k^{\tau}\right)\right)$$

$$-\tfrac{1}{h}\left(n^4-3n^2+1\right)\left(\sum_{\tau=1}^{n} X^{\tau k} X_k^{\tau}\right)=r$$

$$+\left(1-n^2\right)\left(\tfrac{1}{2h}\left(\sum_{\tau=1}^{n} X_r^{\tau} X_p^{\tau}\right) g^{kq} g^{ps} \kappa\left(\delta_q^s g_{ks}-\delta_k^s g_{qs}\right)-\tfrac{4}{h^2}\left(\sum_{\tau=1}^{n} X^{\tau k} X_k^{\tau}\right)\right)$$

$$-\tfrac{1}{h}\left(n^4-3n^2+1\right)\left(\sum_{\tau=1}^{n} X^{\tau k} X_k^{\tau}\right)=n(n-1)\kappa+B\cdot\left(\tfrac{4}{h^2}\cdot\left(n^2-1\right)\right.$$

$$\left.-\tfrac{1}{h}\left(n^4-3n^2+1\right)\right).$$

Thus we have

Theorem 5.3 *Let (M_n, g), $n > 2$ be a Riemannian manifold of constant curvature κ. Suppose that M_n has the scalar curvature $r = n(n-1)\kappa$. Then $(F^*(M_n), \tilde{g})$ has the scalar curvature*

$$\tilde{r} = n(n-1)\kappa + B \cdot \left(\frac{4}{h^2} \cdot \left(n^2 - 1\right) - \frac{1}{h}\left(n^4 - 3n^2 + 1\right)\right) \tag{5.4}$$

where

$$B = \sum_{\tau=1}^{n} X^{\tau k} X_k^{\tau}.$$

Corollary 5.1 *If the base manifold (M_n, g) has constant curvature, then its linear coframe bundle $F^*(M_n)$ equipped with metric $\tilde{g}$ is not (curvature) homogeneous.*

Proof. The equation (5.4) implies that the scalar curvature $\tilde{r}$ is never constant if κ is a constant. Hence the assertion follows.

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