Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **39** (4), 1-12 (2019).

On solvability of an inverse boundary value problem for the pseudo hyperbolic equation

Gulnar N. Isgenderova* · **Afag F. Huseynova**

Received: 06.03.2018 / Revised: 15.08.2019 / Accepted: 31.10.2019

Abstract. *In the paper an inverse boundary value problem for pseudo hyperbolic equation of the fourth order . First, the given problem is reduced to an equivalent problem in a certain sense. Then, using the Fourier method the equivalent problem is reduced to solving the system of integral equations. The existence and uniqueness of a solution to the system of integral equation is proved by the contraction mapping principle. This solution is also the unique solution to the equivalent problem. Finally, by equivalence, the theorem of existence and uniqueness of a classical solution to the given problem is proved .*

Keywords. : inverse boundary value problem, the pseudo hyperbolic equation, Fourier method, classical solution

Mathematics Subject Classification (2010): 30E25, 35L20

1 Introduction

There are many cases where the needs of the practice bring about the problems of determining coefficients or the right hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control in industry etc., which makes them an active field of contemporary mathematics. Inverse problems for various types of PDEs have been studied in many papers. Among them we should mention the papers of A.N. Tikhonov [1], M.M. Lavrentyev [2, 3], V.K. Ivanov [4] and their followers. For a comprehensive overview, the reader should see the monograph by A.M. Denisov [5].

2 Problem statement and its reduction to an equivalent problem

Consider for the equation

$$u_{tt}(x,t) - \alpha u_{ttxx}(x,t) + \beta u_{xxxx}(x,t) = a(t)u(x,t) + b(t)u_t(x,t) + f(x,t) \qquad (2.1)$$

* Corresponding author

G.N. Isgenderova
Baku State University, Baku, Azerbaijan,
E-mail: gulnar1890@mail.ru

A.F. Huseynova
Baku State University, Baku, Azerbaijan,
E-mail: huseynova_bsu@mail.ru

in the domain $D_T = \{(x,t) : 0 \le x \le 1, \quad 0 \le t \le T\}$ an inverse boundary problem with the initial conditions

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x)\ \ (0 \le x \le 1), \tag{2.2}$$

the boundary conditions

$$u_x(0,t) = 0,\ u(1,t) = 0,\ \ u_{xxx}(0,1) = 0,\ u_{xx}(1,1) = 0\ \ (0 \le t \le T) \tag{2.3}$$

and with the additional conditions

$$\int_0^1 u(x,t)dx = h_1(t)\ \ (0 \le t \le T), \tag{2.4}$$

$$u(0,t) = h_2(t)\ \ (0 \le t \le T), \tag{2.5}$$

where $\alpha > 0$, $\beta > 0$ are the given numbers, $f(x,t), \varphi(x), \psi(x), h_1(t), h_2(t)$ are the given functions, and $u(x,t), a(t), b(t)$ are the required functions. The condition (2.5) is a non-local integral condition of first kind, i.e. the one not involving values of unknown functions at the domain's boundary points.

Definition 2.1 The classic solution of problem (2.1) - (2.5) is the triple $\{u(x,t), a(t), b(t)\}$ of the functions $u(x,t)$, $a(t)$ and $b(t)$ with the following properties:

1) the function $u(x,t)$ is continuous in D_T together with all its derivatives contained in equation (2.1);

2) the functions $a(t)$ and $b(t)$ are continuous on $[0,T]$;

3) all the conditions of (2.1) - (2.5) are satisfied in the ordinary sense.

The following lemma holds.

Theorem 2.1 Let $f(x,t) \in C(D_T), \varphi(x) \in C[0,1], \psi(x) \in C[0,1], h_i(t) \in C^2[0,T](i = 1,2)$, $h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \ne 0 (0 \le t \le T)$ and the consistency condition

$$\int_0^1 \varphi(x)dx = h_1(0), \int_0^1 \psi(x)dx = h_1'(0), \varphi(0) = h_2(0), \psi(0) = h_2'(0).$$

Then the problem of finding a classical solution to the problem (2.1)-(2.5) is equivalent to the problem of finding functions $u(x,t)$ and $a(t)$) with the properties 1) and 2) of the definition of the classical solution from the relations (2.1)-(2.3) and satisfying

$$a(t)h_1(t) + b(t)h_1'(t) + \int_0^1 f(x,t)dx = h_1''(t) - \alpha u_{ttx}(1,t) + \beta u_{xxx}(1,t)\ (0 \le t \le T), \tag{2.6}$$

$$a(t)h_2(t) + b(t)h_2'(t) + f(0,t) = h_2''(t) - \alpha u_{ttxx}(0,t) + \beta u_{xxxx}(0,t)\ \ (0 \le t \le T). \tag{2.7}$$

Proof. Let $u(x,t), a(t), b(t)$ be a classical solution to the problem (2.1)-(2.5). Further, assuming $h_i(t) \in C^2[0,T](i = 1,2)$ and differentiating (2.4) and (2.5), we have

$$\int_0^1 u_t(x,t)dx = h_1'(t)\ \ (0 \le t \le T), \ \int_0^1 u_{tt}(x,t)dx = h_1''(t)\ \ (0 \le t \le T), \tag{2.8}$$

$$u_t(0,t) = h_2'(t) \ \ (0 \le t \le T), \ \ u_{tt}(0,t) = h_2''(t) \ \ (0 \le t \le T). \tag{2.9}$$

Integrating equation (2.1) with respect to x from 0 to 1 and taking into account (2.4), we have

$$\int_0^1 u_{tt}(x,t)dx - \alpha u_{ttx}(1,t) + \beta u_{xxx}(1,t)$$

$$= a(t)\int_0^1 u(x,t)dx + b(t)\int_0^1 u_t(x,t)dx + \int_0^1 f(x,t)dx \ \ (0 \le t \le T). \tag{2.10}$$

From (2.10), by (2.4) and (2.8), we conclude that the relation (2.6) is fulfilled.

Now, setting $x = 0$ in (2.1) we obtain

$$u_{tt}(0,t) - \alpha u_{ttxx}(0,t) + \beta u_{xxxx}(0,t)$$

$$= a(t)u(0,t) + b(t)u_t(0,t) + f(0,t) \ \ (0 \le t \le T). \tag{2.11}$$

Hence, taking into account (2.5) and (2.9), we conclude that (2.7) is fulfilled.

Now suppose that $u(x,t), a(t), b(t)$ is a solution to the problem (2.1)-(2.3), (2.6), (2.7). Then from (2.6) and (2.10) we find that

$$\frac{d^2}{dt^2}\left(\int_0^1 u(x,t)dx - h_1(t)\right)$$

$$= b(t)\frac{d}{dt}\left(\int_0^1 u(x,t)dx - h_1(t)\right) + a(t)\left(\int_0^1 u(x,t)dx - h_1(t)\right) \ \ (0 \le t \le T). \tag{2.12}$$

By (2.2) and $\int_0^1 \varphi(x)dx = h_1(0)$, $\int_0^1 \psi(x)dx = h_1'(0)$, it is obvious that

$$\int_0^1 u(x,0)dx - h_1(0) = \int_0^1 \varphi(x)dx - h_1(0) = 0,$$

$$\int_0^1 u_t(x,0)dx - h_1'(0) = \int_0^1 \psi(x)dx - h_1'(0) = 0. \tag{2.13}$$

Since the problem (2.13), (2.14) has only a trivial solution, $\int_0^1 u(x,t)dx - h_1(t) = 0$ $(0 \le t \le T)$, i.e. the condition (2.4) is fulfilled.

From (2.7) and (2.11) we obtain

$$\frac{d^2}{dt^2}\left(u(0,t) - h_2(t)\right) = b(t)\frac{d}{dt}\left(u(0,t) - h_2(t)\right)$$

$$+a(t)\frac{d}{dt}\left(u(0,t)-h_2(t)\right) \quad (0\le t\le T). \tag{2.13}$$

By (2.2) and $\varphi(0)=h_2(0), \psi(0)=h_2'(0)$ we have

$$\begin{array}{l} u(0,0)-h_2(0)=\varphi(0)-h_2(0)=0, \\ u_t(0,0)-h_2'(0)=\psi(0)-h_2'(0)=0. \end{array} \tag{2.14}$$

From (2.13) and (2.14) we conclude that the condition (2.5) is fulfilled. The theorem is proved.

3 Solvability of inverse boundary-value problem

The first component $u(x,t)$ of the solution $u(x,t), a(t), b(t)$ to problem (2.1)-(2.3), (2.6),(2.7) will be sought in the form:

$$u(x,t)=\sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x \ \left(\lambda_k=\frac{\pi}{2}(2k-1)\right), \tag{3.1}$$

where

$$u_k(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx \quad (k=1,2,...).$$

Then, applying the formal scheme of the Fourier method, from (2.1), (2.2), we get:

$$(1+\alpha\lambda_k^2)u_k''(t)+\beta\lambda_k^4 u_k(t)=F_k(t,u,a,b) \quad (k=1,2,...;\ 0\le t\le T), \tag{3.2}$$

$$u_k(0)=\varphi_k,\ u_k'(0)=\psi_k \quad (k=1,2,...)\,, \tag{3.3}$$

where

$$F_k(t,u,a,b)=f_k(t)+a(t)u_k(t)+b(t)u_k'(t),\ \ f_k(t)=2\int_0^1 f(x,t)\cos\lambda_k x dx\,,$$

$$\varphi_k=2\int_0^1 \varphi(x)\cos\lambda_k x dx\,,\ \ \psi_k=2\int_0^1 \psi(x)\cos\lambda_k x dx \quad (k=1,2,...)\,.$$

Solving the problem (3.2), (3.3) we find:

$$u_k(t)=\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t$$

$$+\frac{1}{\beta_k(1+\alpha\lambda_k^2)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau \quad (k=1,2,...), \tag{3.4}$$

where

$$\beta_k = \frac{\beta\lambda_k^2}{\sqrt{1+\alpha\lambda_k^2}} \ (k=1,2,...).$$

After substituting the expressions $u_k(t)$ $(k=1,2,...)$ from (3.4) into (3.1), for the component $u(x,t)$ of the solution $u(x,t), a(t), b(t)$ to the problem (2.1)-(2.3), (2.6), (2.7) we ge

$$u(x,t) = \sum_{k=1}^{\infty} \left\{ \varphi_k \cos\beta_k t + \frac{1}{\beta_k}\psi_k \sin\beta_k t \right.$$

$$\left. + \frac{1}{\beta_k(1+\alpha\lambda_k^2)} \int_0^t F(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau \right\} \cos\lambda_k x. \tag{3.5}$$

Now, from (2.6), (2.7) and (3.1) we have

$$a(t) = \left(h_1''(t) - \int_0^1 f(x,t)dx \right) h_2'(t) - \left(h_2''(t) - f(0,t) \right) h_1'(t)$$

$$+ \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k}(-1)^k h_2'(t) - h_1'(t) \right) \left(\alpha\lambda_k^2 u_k''(t) + \beta\lambda_k^4 u_k(t) \right), \tag{3.6}$$

$$b(t) = h_1(t)\left(h_2''(t) - f(0,t) \right) - h_2(t) \left(h_1''(t) - \int_0^1 f(x,t)dx \right)$$

$$+ \sum_{k=1}^{\infty} \left(h_1(t) - \frac{1}{\lambda_k}(-1)^k h_2(t) \right) \left(\alpha\lambda_k^2 u_k''(t) + \beta\lambda_k^4 u_k(t) \right). \tag{3.7}$$

By (3.2) and (3.4) we have

$$\alpha\lambda_k^4 u_k''(t) + \beta\lambda_k^4 u_k(t) = -u_k''(t) + F_k(t,u,a,b)$$

$$= \frac{\beta\lambda_k^4}{1+\alpha\lambda_k^2}u_k(t) + \frac{\alpha\lambda_k^2}{1+\alpha\lambda_k^2}F_k(t,u,a,b) = \frac{\alpha\lambda_k^2}{1+\alpha\lambda_k^2}F_k(t,u,a,b) + \frac{\beta\lambda_k^4}{1+\alpha\lambda_k^2}$$

$$\times \left[\varphi_k \cos\beta_k t + \frac{1}{\beta_k}\psi_k \sin\beta_k t + \frac{1}{\beta_k(1+\alpha\lambda_k^2)} \int_0^T F(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau \right]. \tag{3.8}$$

We substitute expression (3.8) into (3.6), (3.7) and have

$$a(t) = [h(t)]^{-1} \left\{ \left(h_1''(t) - \int_0^1 f(x,t)dx \right) h_2'(t) - \left(h_2''(t) - f(0,t) \right) h_2'(t) \right.$$

$$+\sum_{k=1}^{\infty}\left[\frac{\beta\lambda_k^4}{1+\alpha\lambda_k^2}\left[\varphi_k os\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t\right.\right.$$

$$\left.+\frac{1}{\beta_k(1+\alpha\lambda_k^2)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau\right]$$

$$\left.+\frac{\alpha\lambda_k^2}{1+\alpha\lambda_k^2}F_k(t;u,a,b)\right]\left(\frac{(-1)^k}{\lambda_k}h_2'(t)-h_1'(t)\right)\Bigg\}, \tag{3.9}$$

$$b(t)=[h(t)]^{-1})\left\{\left(h_2''(t)-f(0,t)\right)h_1(t)-\left(h_1''(t)-\int_0^1 f(x,t)dx\right)h_2(t)\right.$$

$$+\sum_{k=1}^{\infty}\left[\frac{\beta\lambda_k^4}{1+\alpha\lambda_k^2}\left[\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t\right.\right.$$

$$\left.+\frac{1}{\beta_k(1+\alpha\lambda_k^2)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau\right]$$

$$\left.+\frac{\alpha\lambda_k^2}{1+\alpha\lambda_k^2}F_k(t;u,a,b)\right]\left(h_1(t)-\frac{(-1)^k}{\lambda_k}h_2(t)\right)\Bigg\}. \tag{3.10}$$

Thus, the problem (2.1)-(2.3), (2.6), (2.7) is reduced to solving the system (3.5), (3.9),(3.10) with respect to the unknown functions $u(x,t)$, $a(t)$ and $b(t)$.

Using the definition of the solution of the problem (2.1)-(2.3), (2.8),(2.9), we prove the following lemma.

Lemma 3.1 If $\{u(x,t),a(t),b(t)\}$ is any solution to the problem (2.1)-(2.3), (2.6), (2.7), then the functions

$$u_k(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx\quad(k=1,2,...)$$

satisfy the system (3.4)) in $[0,T]$.

Remark 3.1 It follows from lemma 1 that to prove the uniqueness of the solution to the problem (2.1)-(2.3), (2.6), (2.7), it suffices to prove the uniqueness of the solution to the system (3.5), (3.9),(3.10).

Differentiating (3.4) twice, we get

$$u_k'(t)=-\beta_k\varphi_k\sin\beta_k t+\psi_k\cos\beta_k t+\frac{1}{1+\alpha\lambda_k^2}\int_0^t F_k(\tau,u,a,b)\cos\lambda_k(t-\tau)d\tau. \tag{3.10}$$

In order to investigate the problem (2.1)-(2.3), (2.6), (2.7), consider the following spaces.

Denote by $B_{2,T}^{5,4}$ [6] the set of all functions of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\cos\lambda_k x\ (\lambda_k=\frac{\pi}{2}(2k-1))$$

defined on D_T such that the functions $u_k(t)(k=1,2,...)$ are continuous on $[0,T]$ and

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k'(t)\right\|\right)^2\right)^{\frac{1}{2}}<\infty.$$

The norm on this set is given by

$$\|u(x,t)\|_{B_{2,T}^5}=\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k'(t)\right\|\right)^2\right)^{\frac{1}{2}}.$$

Denote by $E_T^{5,4}$ the space $B_{2,T}^{5,4}\times C[0,T]$ of the vector-functions $z(x,t)=u(x,t),a(t),b(t)$ with the norm

$$\|z\|_{E_T^{5,4}}=\|u\|_{B_{2,T}^{5,4}}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

It is known that $B_{2,T}^5$ and E_T^5 are Banach spaces.

Now, in the space E_T^5 consider the operator

$$\Phi(u,a,b)=\{\Phi_1(u,a,b),\Phi_2(u,a,b),\Phi_3(u,a,b)\},$$

where

$$\Phi_1(u,p,q)=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_k(t)\cos\lambda_k x,\ \Phi_2(u,a,b)=\tilde{a}(t),\ \Phi_3(u,a,b)=\tilde{b}(t)$$

and $\tilde{u}_k(t),\ \tilde{a}(t),\ \tilde{b}(t)$ equal to the right hand sides of (3.4), (3.9), (3.10), respectively.

It is easy to see that

$$\alpha\lambda_k^2\le 1+\alpha\lambda_k^2\le(1+\alpha)\lambda_k^2,\ \frac{\beta}{\sqrt{1+\alpha}}\lambda_k<\beta_k<\frac{\beta}{\sqrt{\alpha}}\lambda_k.$$

Taking into account these relations, by means of simple transformations we find

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\le 2\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left|\varphi_k\right|\right)^2\right)^{\frac{1}{2}}+\frac{2\sqrt{1+\alpha}}{\beta}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left|\psi_k\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\frac{2}{\beta\sqrt{\alpha}}\left(\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+T\left\|a(t)\right\|_{C[0,T]}\right.$$

$$\left.\times\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\right)^2d\tau\right)^{\frac{1}{2}}\right.$$

$$+\left\|b(t)\right\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]}\right)^2d\tau\right)^{\frac{1}{2}},\qquad(3.11)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq\frac{2\beta}{\sqrt{\alpha}}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left|\varphi_k\right|\right)^2\right)^{\frac{1}{2}}+\frac{2}{\alpha}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}$$

$$+\frac{2}{\alpha}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\frac{2}{\alpha}T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\frac{2}{\alpha}T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},\tag{3.12}$$

$$\left\|a(t)\right\|_{C[0,T]}=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(h_1''(t)-\int_0^1 f(x,t)dx\right)h_2'(t)\right\|_{C[0,T]}\right.$$

$$\left.+\left\|\left(h_2''(t)-f(0,t)\right)h_1'(t)\right\|_{C[0,T]}+\left\|\left|h_2'(t)\right|+\left|h_1'(t)\right|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\right.$$

$$\times\left[\frac{\beta}{\alpha}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left|\varphi_k\right|\right)^2\right)^{\frac{1}{2}}+\frac{\beta}{2}\frac{\sqrt{1+\alpha}}{\beta}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left|\psi_k\right|\right)^2\right)^{\frac{1}{2}}+\frac{1}{\alpha\sqrt{\alpha}}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\right.$$

$$+\frac{T}{\alpha\sqrt{\alpha}}\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\frac{T}{\alpha\sqrt{\alpha}}\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|f_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\left.+\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\right\},\tag{3.13}$$

$$\left\|b(t)\right\|_{C[0,T]}=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(h_1''(t)-\int_0^1 f(x,t)dx\right)h_2(t)\right\|_{C[0,T]}\right.$$

$$\left.+\left\|\left(h_2''(t)-f(0,t)\right)h_1(t)\right\|_{C[0,T]}+\left\|\left|h_2(t)\right|+\left|h_1(t)\right|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\right.$$

$$\times\left[\frac{\beta}{\alpha}\Big(\sum_{k=1}^{\infty}\left(\lambda_k^5\left|\varphi_k\right|\right)^2\Big)^{\frac{1}{2}}+\frac{\beta}{2}\frac{\sqrt{1+\alpha}}{\beta}\Big(\sum_{k=1}^{\infty}\left(\lambda_k^4\left|\psi_k\right|\right)^2\Big)^{\frac{1}{2}}+\frac{1}{\alpha\sqrt{\alpha}}\sqrt{T}\Big(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_k(\tau)\right|\right)^2d\tau\Big)^{\frac{1}{2}}\right.$$

$$+\frac{T}{\alpha\sqrt{\alpha}}\left\|a(t)\right\|_{C[0,T]}\Big(\sum_{k=1}^{\infty}\Big(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\Big)^2\Big)^{\frac{1}{2}}+\frac{T}{\alpha\sqrt{\alpha}}\left\|b(t)\right\|_{C[0,T]}\Big(\sum_{k=1}^{\infty}\Big(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]}\Big)^2\Big)^{\frac{1}{2}}$$

$$+\left(\sum_{k=1}^{\infty}\Big(\lambda_k\left\|f_k(t)\right\|_{C[0,T]}\Big)^2\right)^{\frac{1}{2}}+\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\Big(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\Big)^2\right)^{\frac{1}{2}}$$

$$\left.+\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\Big(\lambda_k^4\left\|u_k'(t)\right\|_{C[0,T]}\Big)^2\right)^{\frac{1}{2}}\right]\Bigg\}. \tag{3.14}$$

Suppose that the data of the problem (2.1)-(2.3), (2.6), (2.7) satisfy the following condition

$1^0.\varphi(x)\in C^4[0,1],\varphi^{(5)}(x)\in L_2(0,1),\varphi'(0)=\varphi(1)=\varphi'''(0)=\varphi''(1)=\varphi^{(4)}(0)=0;$

$2^0.\psi(x)\in C^3[0,1],\psi^{(4)}(x)\in L_2(0,1),\psi'(0)=\psi(1)=\psi'''(0)=\psi''(1)=0;$

$3^0.f(x,t),f_x(x,t)\in C(D_T),f_{xx}(x,t)\in L_2(D_T),f_x(0,t)=f(1,t)=0\,(0\le t\le T);$

$4^0.h_i(t)\in C^2[0,T]\,(i=1,2),h(t)\equiv h_1(t)h_2'(t)-h_2(t)h_1'(t)\neq 0(0\le t\le T).$

Then, from (3.11)-(3.14), we ge

$$\left\|\tilde{u}(x,t)\right\|_{B_{2,T}^{5,4}}\le A_1(T)+B_1(T)(\left\|a(t)\right\|_{C[0,T]}+\left\|b(t)\right\|_{C[0,T]})\left\|u(x,t)\right\|_{B_{2,T}^{5,4}}, \tag{3.15}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\le A_2(T)+B_2(T)(\left\|a(t)\right\|_{C[0,T]}+\left\|b(t)\right\|_{C[0,T]})\left\|u(x,t)\right\|_{B_{2,T}^{5,4}}, \tag{3.16}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\le A_3(T)+B_3(T)(\left\|a(t)\right\|_{C[0,T]}+\left\|b(t)\right\|_{C[0,T]})\left\|u(x,t)\right\|_{B_{2,T}^{5,4}}, \tag{3.17}$$

where

$$A_1(T)=2(1+\frac{\beta}{\sqrt{\alpha}})\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+(\frac{2\sqrt{1+\alpha}}{\beta}+1)\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}$$

$$+\frac{2}{\sqrt{\alpha}}\sqrt{T}(\frac{1}{\beta}+\frac{1}{\sqrt{\alpha}})\left\|f_{xx}(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T)=\frac{2}{\sqrt{\alpha}}T\Big(\frac{1}{\beta}+\frac{1}{\sqrt{\alpha}}\Big),$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\|(h_1''(t)-\int_0^1 f(x,t)dx)h_2'(t)\right\|_{C[0,T]}+\left\|(h_2''(t)-f(0,t))h_1'(t)\right\|_{C[0,T]}$$

$$+\left\|\left|(h_2'(t)\right|+\left|(h_1'(t)\right|\right\|_{C[0,T]}\times\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$

$$\times\Big[\frac{\beta}{\alpha}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\frac{\sqrt{1+\alpha}}{2}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}$$

$$+\frac{1}{\alpha\sqrt{\alpha}}T\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}+\left\|\left\|f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\Big],$$

$$B_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\times\left\|\left|(h_2'(t)\right|+\left|(h_1'(t)\right|\right\|_{C[0,T]}\times\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times\left(\frac{T}{\alpha\sqrt{\alpha}}+1\right),$$

$$A_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\|(h_1''(t)-\int_0^1 f(x,t)dx)h_2(t)\right\|_{C[0,T]}+\left\|(h_2''(t)-f(0,t))h_1(t)\right\|_{C[0,T]}$$

$$+\left\|\left|(h_1(t)\right|+\left|(h_2(t)\right|\right\|_{C[0,T]}\times\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times\Big[\frac{\beta}{\alpha}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\frac{\sqrt{1+\alpha}}{2}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}$$

$$+\frac{1}{\alpha\sqrt{\alpha}}T\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}+\left\|\left\|f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\Big],$$

$$B_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\times\left\|\left|(h_1(t)\right|+\left|(h_2(t)\right|\right\|_{C[0,T]}\times\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times\left(\frac{T}{\alpha\sqrt{\alpha}}+1\right).$$

It follows from the inequalities (3.15)- (3.17) that

$$\left\|\tilde{u}(x,t)\right\|_{B_{2,T}^{5,4}}+\left\|\tilde{a}(t)\right\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}$$

$$\leq A(T)+B(T)(\left\|a(t)\right\|_{C[0,T]}+\left\|b(t)\right\|_{C[0,T]})\left\|u(x,t)\right\|_{B_{2,T}^{5,4}},\qquad(3.18)$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T)\;\;B(T)=B_1(T)+B_2(T)+B_3(T).$$

Theorem 3.1 Let the conditions 1^0-4^0 be fulfilled and

$$(A(T)+2)^2B(T)<1.\qquad(3.19)$$

Then the problem (2.1)-(2.3), (2.6), (2.7) has a unique solution in the ball $K=K_R(\left\|z\right\|_{E_T^{5,4}}\leq R=A(T)+2)$ of the space $E_T^{5,4}$.

Proof. In the space $E_T^{5,4}$ consider the equation

$$z=\Phi z,\qquad(3.20)$$

where $z = \{u, a, b\}$ and the components $\Phi_i(u, a, b_)$ $(i = 1, 2, 3)$) of the operator $\Phi(u, a, b_)$ are given by the right hand sides of the equations (3.5),(3.9), (3.10). Consider the operator $\Phi(u, a, b_)$ in the ball $K = K_R$ from $E_T^{5,4}$. Similarly to (3.18), we see that for any $z, z_1, z_2 \in K_R$ the following estimates hold:

$$\|\Phi z\|_{E_T^{5,4}} \leq A(T) + B(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]}) \|u(x,t)\|_{B_{2,T}^{5,4}}, \tag{3.21}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^{5,4}} \leq B(T)R \Big(\|a_1(t) - b_2(t)\|_{C[0,T]}$$

$$+ \|b_1(t) - b_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^{5,4}} \Big). \tag{3.22}$$

Then, it follows from (3.19) together with the estimates (3.21) and (3.22) that the operator Φ acts in the ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$ the operator Φ has a unique fixed point $\{u, a, b\}$, that is a unique solution to the equation (3.20), i.e. a unique solution to the system (3.5),(3.9), (3.10).

The function $u(x,t)$, as an element of the space $B_{2,T}^{5,4}$, is continuous and has continuous derivatives $u_x(x,t), u_{xx}(x,t), u_{xxx}(x,t), u_{xxx}x(x,t), u_t(x,t), u_{txxx}(x,t)$ in D_T.

Next, from (3.2) it follows that $u_k''(t)$ $(k = 1, 2, ...)$ is continuous in $[0, T]$ and consequently we have:

$$\left(\sum_{k=1}^{\infty} (\lambda_k^3 \left\| u_k''(t) \right\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \frac{\sqrt{2}}{\alpha} \left\{ \beta \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right.$$

$$\left. + \left\| \|f_x(x,t) + a(t)u_x(x,t) + b(t)u_{tx}(x,t)\|_{C[0,T]} \right\|_{L_2(0,1)} \right\}.$$

From the last relation it is obvious that $u_{tt}(x,t), u_{ttx}(x,t), u_{ttxx}(x,t)$ is continuous in D_T .

It is easy to verify that the equation (2.1) and conditions (2.2), (2.3), (2.6), (2.7) are satisfied in the ordinary sense. Consequently, $\{u(x,t), a(t), b(t)\}$ is a solution to the problem (2.1)-(2.3), (2.6), (2.7), and by Lemma 1 it is unique in the ball $K = K_R$.

By Theorem 1 the unique solvability of the initial problem (2.1)-(2.5) follows from the theorem.

Theorem 3.2 Let all the conditions of Theorem 2 be fulfilled and

$$\int_0^1 \varphi(x)dx = h_1(0), \int_0^1 \psi(x)dx = h_1'(0), \varphi(0) = h_2(0), \psi(0) = h_2'(0).$$

Then the problem (2.1)-(2.5) has a unique classical solution in the ball $K = K_R(\|z\|_{E_T^{5,4}} \leq R = A(T) + 2)$ of the space $E_T^{5,4}$.

References

1. Tikhonov A.N.: On stability of inverse problems, Dokl. AN SSSR, **39** (5), 195-198 (1943). (Russian).
2. Lavrentev M.M.: On an inverse problem for a wave equation, Dokl. AN SSSR, **157** (3), 520-521 (1964) (Russian).
3. Lavrentev M.M., Romanov V.G., Shishatsky S.T.: III-posed problems of mathematical physics and analysis, *M., Nauka,* (1980) (Russian).
4. Ivanov V.K., Vasin V.V., Tanana V.P.: Theory of linear ill-posed problems and its applications, *M. Nauka,* (1978) (Russian).
5. Denisov A.M.: Introduction to theory of inverse problems, *M. MGU,* (1994) (Russian).
6. Khudaverdiev K.I., Veliev A.A.: The study of one-dimensional mixed problem for one class of quasi-hyperbolic third-order equations with a nonlinear operator right side. *Baku, Chashyogly Publ.* 168 p. (2010) (Russian).