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Global bifurcation from intervals of solutions of nonlinear Sturm-Liouville problems with indefinite weight

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Abstract. *We consider the global bifurcation of solutions of nonlinear Sturm-Liouville problems with indefinite weight function. We prove the existence of four families of global continua of solutions corresponding to the usual nodal properties and bifurcating from intervals of the line of trivial solutions.*

Keywords. nonlinear Sturm-Liouville problem, indefinite weight function, bifurcation interval, global continua of solutions.

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1 Introduction

We consider the nonlinear Sturm-Liouville equation

$$(\ell y) \equiv -(p(x)y')' + q(x)y = \lambda\rho(x)y + f(x, y, y', \lambda),\ x \in (0, 1), \tag{1.1}$$

with boundary conditions

$$\alpha_0 y(0) - \beta_0 y'(0) = 0, \tag{1.2}$$

$$\alpha_1 y(1) + \beta_1 y'(1) = 0, \tag{1.3}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $p \in C^1([0, 1]; (0, +\infty))$, $q \in C([0, 1]; [0, +\infty))$, $r \in C([0, 1]; \mathbb{R})$ such that $\text{meas}\{x \in [0, 1] : \sigma\rho(x) > 0\} > 0$ for each $\sigma \in \{+, -\}$, $\alpha_i, \beta_i \in \mathbb{R},\ i = 0,\ 1$, are constants such that $|\alpha_i| + |\beta_i| > 0$ and $\alpha_i\beta_i \geq 0$ for $i = 0,\ 1$. We also suppose that the nonlinear term $f \in C([0, 1]\times\mathbb{R}^3; \mathbb{R})$ satisfies the following conditions:

$$uf(x, u, s, \lambda) \leq 0; \tag{1.4}$$

there exist constant $M > 0$ and small number $\sigma_0 > 0$ such that

$$\left|\frac{f(x, u, s, \lambda)}{u}\right| \leq K,\ (x, u, s, \lambda) \in [0, 1] \times \mathbb{R}^3, u \neq 0, |u| + |s| \leq \sigma_0. \tag{1.5}$$

It is known that almost all processes occurring in nature are described by nonlinear differential equations with the corresponding initial and boundary conditions. For this reason,

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the study of such problems is important and necessary. It should be noted that the nonlinear Sturm-Liouville eigenvalue problems arise in many applications, for example, the problem (1.1)-(1.3) with indefinite weight arise from population modeling. In this model, weight function changes sign corresponding to the fact that the intrinsic population growth rate is positive at same points and is negative at others, for details, see [8, 10].

The nonlinear eigenvalue problem (1.1)-(1.3) in the case when $\rho > 0$ in a more general boundary conditions was considered in [7, 13, 14]. In these papers prove the existence of unbounded continua of nontrivial solutions in $\mathbb{R} \times C^1$ bifurcating from points and intervals of the line of trivial solutions corresponding to the eigenvalues of the linear problem obtained from (1.1)-(1.3) by setting $f \equiv 0$. Similar results in nonlinear eigenvalue problems for elliptic partial differential equations and for ordinary differential equations of fourth order with definite and indefinite weight functions were obtained in [1-4, 7, 12-15].

Problem (1.1)-(1.3) in the case when nonlinear term f satisfies $o(|u| + |u'|)$ condition near $(u, u') = (0, 0)$ was investigated in a paper [6] where we show that there existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1$ bifurcating from points of the line of trivial solutions corresponding to the all eigenvalues of the linear problem (1.1)-(1.3) with $f \equiv 0$.

The nonlinear problem (1.1)-(1.3) was considered in a recent paper [5], where the global bifurcation of solutions was studied only in classes of positive and negative functions. In this case, the global continua of solutions are bifurcated from the intervals of the line of trivial solutions corresponding to the principal eigenvalues of the linear problem (1.1)-(1.3) with $f \equiv 0$. The reason for studying bifurcation of solutions only in classes of positive and negative functions is the fact that earlier we were not able to find bifurcation intervals corresponding to other eigenvalues of the linear problem (1.1)-(1.3) with $f \equiv 0$.

In the present paper we were able to find bifurcation intervals corresponding to all eigenvalues of the linear problem (1.1)-(1.3) with $f \equiv 0$. Moreover, we show the existence of four families of unbounded continua of solutions of problem (1.1)-(1.3) corresponding to the usual nodal properties and bifurcating from these intervals of the line of trivial solutions.

2 Preliminary and some properties of eigenvalues and eigenfunctions of the linear Sturm-Liouville problems with indefinite weight

We consider the following linear eigenvalue problem which obtained from (1.1)-(1.3) by setting $f \equiv 0$

$$\begin{cases} -(p(x)y'(x))' + q(x)y(x) = \lambda\rho(x)y(x), \; x \in (0, \; 1), \\ y \in B.C., \end{cases} \tag{2.1}$$

where by $B.C.$ we denote the set of boundary conditions (1.2), (1.3). It is a classical result (see [11; Ch. 10]) that the eigenvalues of problem (2.1) are all real, simple and form two unbounded sequences

$$0 > \lambda_1^- > \lambda_2^- > ... > \lambda_k^- > \dots$$

and

$$0 < \lambda_1^+ < \lambda_2^+ < ... < \lambda_k^+ < \dots \; .$$

Moreover, for each $k \in \mathbb{N}$ and each $\sigma \in \{+, -\}$ the eigenfunction $y_k^\sigma(x)$ corresponding to eigenvalue λ_k^σ, has exactly $k-1$ simple noda zeros in $(0, 1)$.

Alongside the boundary-value problem (2.1) we shall consider the spectral problem

$$\begin{cases} (\ell y)(x) - \lambda\rho(x)y(x) = \mu y(x), \; x \in (0, 1), \\ y \in B.C., \end{cases} \tag{2.2}$$

for each fixed $\lambda \in \mathbb{R}$. It is known (see [11]) that for each $\lambda \in \mathbb{R}$ the eigenvalues of problem (2.2) are real, simple and form an infinitely increasing sequence

$$\mu_1(\lambda) < \mu_2(\lambda) < \ \ldots \ < \mu_k(\lambda) < \ \ldots \ ;$$

for each $k \in \mathbb{N}$ the eigenfunction $y_k(x, \lambda)$ corresponding to the eigenvalue $\mu_k(\lambda)$ has $k-1$ simple nodal zeros in $(0, 1)$.

For each $k \in \mathbb{N}$ the k-th eigenvalue $\mu_k(\lambda),\ k \in \mathbb{N}$, of problem (2.2) can be characterized as follows (see [9]):

$$\mu_k(\lambda) = \max_{V^{(k-1)}} \min_{y \in B.C.} \left\{ R_\lambda[y] : \int_0^1 y(x)\, \varphi(x) dx = 0,\ \varphi \in V^{(k-1)} \right\}, \tag{2.3}$$

where

$$R_\lambda[y] = \frac{\int_0^1 \left(y'^2 + q(x)y^2 - \lambda\rho(x)y^2\right)dx + N[y]}{\int_0^1 y^2 dx},$$

$$N[y] = \begin{cases} \frac{\alpha_0}{\beta_0} y^2(0) + \frac{\alpha_1}{\beta_1} y^2(1), & if\ \beta_0 \neq 0,\ \ \beta_1 \neq 0, \\ \frac{\alpha_0}{\beta_0} y^2(0), & if\ \beta_0 = 0,\ \ \beta_1 \neq 0, \\ \frac{\alpha_1}{\beta_1} y^2(1), & if\ \beta_0 \neq 0,\ \ \beta_1 = 0, \\ 0, & if\ \beta_0 = \beta_1 = 0 \end{cases} \tag{2.4}$$

and $V^{(k-1)}$ denotes any set of $(k-1)$ linearly independent functions with $\varphi_j(x) \in B.C.$, $1 \leq j \leq k-1$. It follows from this max-min characterization that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2.1) which corresponds to an eigenfunction having $k-1$ simple nodal zeros in $(0, 1)$, if and only if $\mu_k(\lambda) = 0$. Note that for each $k \in \mathbb{N}$ the eigenvalue $\mu_k(\lambda)$ of problem (2.2) and the corresponding eigenfunction $y_k(x, \lambda)$ are continuous functions of the parameter $\lambda \in \mathbb{R}$.

Lemma 2.1 (see [5, Lemma 2.1]) *Let* $y_k(x, \lambda),\ k \in \mathbb{N}$, *be an eigenfunction of* (2.2) *corresponding to the* k*-th eigenvalue* $\mu_k(\lambda)$. *Then* $\mu_k(\lambda) \in C^\infty(\mathbb{R})$ *and*

$$\frac{d\mu_k(\lambda)}{d\lambda} = -\frac{\int_0^1 \rho(x) y_k^2(x, \lambda)\, dx}{\int_0^1 y_k^2(x, \lambda)\, dx},\ \lambda \in \mathbb{R},\ k \in \mathbb{N}. \tag{2.5}$$

Alongside the spectral problem (2.2) we consider the following linear eigenvalue problems

$$\begin{cases} (\ell y)(x) + \psi(x)y(x) = \lambda\rho(x)y(x),\ x \in (0, 1), \\ y \in B.C., \end{cases} \tag{2.6}$$

$$\begin{cases} (\ell y)(x) + \psi(x)y(x) - \lambda\rho(x)y(x) = \mu y(x),\ x \in (0, 1), \\ y \in B.C., \end{cases} \tag{2.7}$$

where $\psi \in C\left([0, 1]; [0, +\infty\right)$.

Let

$$R_{\lambda,\psi}[y] = \frac{\int_0^1 \left(y'^2 + q(x)y^2 - \lambda\rho(x)y^2 + \psi(x)y^2\right) dx + N[y]}{\int_0^1 y^2 dx}. \tag{2.8}$$

Since $\psi \in C([0,1];[0,+\infty))$ and $N[y] \geq 0$ (see (2.4)), it follows from (2.3) that

$$0 \leq \mu_{k,\psi}(\lambda) - \mu_k(\lambda) \leq M_\psi, \tag{2.9}$$

where $\mu_{k,\psi}(\lambda)$ is the k-th eigenvalue of the spectral problem (2.7) and $M_\psi = \sup\limits_{x\in[0,1]} \psi(x)$.

It is obvious that for each $k \in \mathbb{N}$ the eigenvalues $\lambda^+_{k,\psi}$ and $\lambda^-_{k,\psi}$ of problem (2.6) are positive and negative zeros of function $\mu_{k,\psi}(\lambda)$ respectively.

Multiplying both sides of equation in (2.1) by $\overline{y(x)}$, integrating the resulting equality in the range from 0 to 1, using the formula for integration by parts, and by taking into account the boundary conditions (1.2) and (1.3), we obtain

$$\int_0^1 \left(y'^2 + q(x)y^2\right) dx + N[y] = \lambda \int_0^1 \rho(x) y^2 dx.$$

Consequently, for each $k \in \mathbb{N}$ we have

$$\int_0^1 \rho(x)(y_k^+(x))^2 dx > 0 \ \text{ and } \ \int_0^1 \rho(x)(y_k^-(x))^2 dx < 0. \tag{2.10}$$

Then it follows from (2.5) that the function $\mu_k(\lambda)$ $(\mu_{k,\psi}(\lambda))$ decreases in the interval $(0,+\infty)$ and increases in the interval $(-\infty,0)$.

By λ^+_{k,M_ψ} and λ^-_{k,M_ψ}, $k \in \mathbb{N}$, we denote the kth positive and negative eigenvalues of the following spectral problem

$$\begin{cases} (\ell y)(x) + M_\psi y(x) = \lambda \rho(x) y(x), \ x \in (0,1), \\ y \in B.C. \end{cases} \tag{2.11}$$

Thus, by the above reasoning, we are convinced of the validity of the following lemma.

Lemma 2.2 *The following relations hold:*

$$\lambda_k^+ \leq \lambda^+_{k,\psi} \leq \lambda^+_{k,M_\psi}, \ \ \lambda^-_{k,M_\psi} \leq \lambda^-_{k,\psi} \leq \lambda^-_k. \tag{2.12}$$

Remark 2.1 Since the class of continuous functions $C[0,1]$ is dense in $L_1[0,1]$ relations in (2.12) hold for $\psi \in L_1[0,1]$.

3 Global bifurcation from zeros of solutions of the nonlinear eigenvalue problem (1.1)-(1.3)

Let $E = C^1[0,1] \cap B.C.$ be a Banach space with the usual norm $||u||_1 = ||u||_\infty + ||u'||_\infty$, where $||u||_\infty = \max\limits_{x\in[0,1]} |u(x)|$.

By $S_k^{\sigma,\nu}$, $\sigma,\ \nu \in \{+,-\}$, we denote the set of functions $u \in E$ satisfying the following conditions:

(i) the function u has exactly $k-1$ simple zeros in $(0,1)$, all zeros of u in $[0,1]$ being nodal;

(ii) $\sigma \int\limits_0^1 \rho(x) u^2(x) dx > 0$;

(iii) the function νu is positive in a deleted neighborhood of 0.

Let $S_{k,\sigma} = S_k^{\sigma,+} \cup S_k^{\sigma,-}$, $k \in \mathbb{N}$, $\sigma \in \{+,-\}$. It follows from the definition of the sets $S_k^{\sigma,\nu}$ and $S_{k,\sigma}$, $k \in \mathbb{N}$, σ, $\nu \in \{+,-\}$, that these sets are open sets in E and $S_k^{\sigma,\nu} \cap S_m^{\varrho,\theta} = \emptyset$, $S_{k,\sigma} \cap S_{m,\varrho} = \emptyset$, where $(k,\sigma,\nu) \neq (m,\varrho,\theta)$, $(k,\sigma) \neq (m,\varrho)$, respectively. Moreover, if $u \in \partial S_{k,\sigma}$, then either $\int\limits_0^1 \rho(x) u^2(x)\,dx = 0$ or the function u has a double zero in $[0,1]$.

We say that $(\lambda, 0) \in R \times \{0\}$ is a bifurcation point of problem (1.1)-(1.3) with respect to the set $\mathbb{R} \times S_k^{\sigma,\nu}$ $(\mathbb{R} \times S_{k,\sigma})$ if there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \in \mathbb{R} \times S_k^{\sigma,\nu}$ $(\mathbb{R} \times S_{k,\sigma})$ of solutions of this problem such that $\lambda_n \to \lambda$ and $u_n \to 0$ as $n \to \infty$ (see [2]).

We denote by $\mathcal{C}$ the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (1.1)-(1.3). It follows from Theorem 2.1 that $y_k^\sigma \in S_{k,\sigma}$ for each $k \in \mathbb{N}$ and each $\sigma \in \{+,-\}$, where y_k^σ is an eigenfunction corresponding to the eigenvalue λ_k^σ of problem (2.1). Hence y_k^σ, $\sigma \in \{+,-\}$, is made unique by requiring that $y_k^\sigma \in S_k^{+,\sigma}$ and $||y_k^\sigma||_1 = 1$.

Lemma 3.1 (see [5, Lemma 3.1]) *If* $(\lambda, y) \in \mathbb{R} \times E$ *is a solution of* (1.1)-(1.3) *such that* $y \in \partial S_k^{\sigma,\nu}$, $k \in \mathbb{N}$, σ, $\nu \in \{+,-\}$, *then* $y \equiv 0$.

Alongside the problem (1.1)-(1.3) we introduce the approximate problem

$$\begin{cases} \ell y(x) = \lambda \rho(x) y(x) + f(x, |y|^\varepsilon y, y', \lambda), \ x \in (0,\ 1), \\ y \in B.C., \end{cases} \tag{3.1}$$

where $\varepsilon \in (0,1]$.

By (1.3) it follows that

$$f(x, |u|^\varepsilon u, v, \lambda) = o\,(|u| + |s|) \text{ as } |u| + |s| \to 0,$$

uniformly in $x \in [0,1]$ and $\lambda \in \mathbb{R}$. Hence by [1, Theorem 2] for each $k \in \mathbb{N}$, each $\sigma \in \{+,-\}$ and each $\nu \in \{+,-\}$ there exists an unbounded continua $D_k^{\sigma,\nu}$ of solutions of problem (3.1) such that

$$(\lambda_k^\sigma, 0) \in D_k^{\sigma,\nu} \subset (\mathbb{R} \times S_k^{\sigma,\nu}) \cup (\lambda_k^\sigma, 0). \tag{3.2}$$

Remark 3.1 If $(\lambda, y) \in D_k^{\nu,+}$, then it follows from (1.4) that $\lambda > 0$, and if $(\lambda, y) \in D_k^{\nu,-}$, then, $\lambda < 0$.

Lemma 3.2 *For each* $k \in \mathbb{N}$, *each* $\nu \in \{+,-\}$, *each* $\sigma \in \{+,-\}$ *and for every* $0 < r < \tau_0$ *problem* (1.1)-(1.3) *has a solution* $(\lambda_{k,r}^{\sigma,\nu}, y_{k,r}^{\sigma,\nu})$ *such that*

$$\lambda_{k,r}^{\sigma,\nu} \in I_k^\sigma,\ y_{k,r}^{\sigma,\nu} \in S_k^{\sigma,\nu} \text{ and } ||y_{k,r}^{\sigma,\nu}||_1 = r,$$

where

$$I_k^+ = [\lambda_k^+, \lambda_{k,M}^+],\ \ I_k^- = [\lambda_{k,M}^-, \lambda_k^-].$$

The proof of this lemma is similar to that of [5, Lemma 3.2] by the use of Lemmas 2.2, 3.1 and Remarks 2.1, 3.1.

Corollary 3.1 *For each* $\sigma \in \{+,-\}$ *and each* $\nu \in \{+,-\}$ *the set of bifurcation points of nonlinear eigenvalue problem* (1.1)-(1.3) *with respect to the set* $S_k^{\sigma,\nu}$, $k \in \mathbb{N}$, *is nonempty. Moreover, if* $(\lambda, 0)$ *is a bifurcation point of problem* (1.1)-(1.3) *with respect to* $S_k^{\sigma,\nu}$, *then* $\lambda \in I_k^\sigma$.

For each $k \in \mathbb{N}$, each $\sigma \in \{+\,,\,-\}$ and each $\nu \in \{+\,,\,-\}$ we define the set $\hat{D}_k^{\sigma,\nu} \subset \mathcal{C}$ to be the union of all the connected components $D_{k,\lambda}^{\sigma,\nu}$ of the set of solutions of problem (1.1)-(1.3) bifurcating from points $(\lambda, 0) \in I_k^\sigma \times 0$ with respect to the set $S_k^{\sigma,\nu}$. By virtue of Corollary 3.1 for each $\sigma \in \{+\,,\,-\}$ and each $\nu \in \{+\,,\,-\}$ the sets $\tilde{D}_k^{\sigma,\nu}$, $k \in \mathbb{N}$, are nonempty. The set $\hat{D}_k^{\sigma,\nu}$, $k \in \mathbb{N}$, may not be connected in $\mathbb{R} \times E$. But it can be seen that the set $D_k^{\sigma,\nu} = \tilde{D}_k^{\sigma,\nu} \cup (I_k^\sigma \times \{0\})$, $k \in \mathbb{N}$, is connected in $\mathbb{R} \times E$.

Theorem 3.1 *For each $k \in \mathbb{N}$, each $\sigma \in \{+\,,\,-\}$ and each $\nu \in \{+\,,\,-\}$ the connected set $D_k^{\sigma,\,\nu} \subset \mathcal{C}$ is unbounded in $\mathbb{R} \times E$ and lies in $\mathbb{R} \times S_k^{\sigma,\nu}$.*

The proof of this theorem can be proved in accordance with the scheme of the proof of Theorem 1 in [7], by the use of Lemmas 3.1, 3.2, Remarks 2.1, 3.1 and (3.2).

Theorem 3.2 *Suppose the condition* (1.4) *holds for any $(x, u, s, \lambda) \in [0, 1] \times \mathbb{R}^3$ such that $u \neq 0$. Then for each $k \in \mathbb{N}$, each $\sigma \in \{+\,,\,-\}$ and each $\nu \in \{+\,,\,-\}$ the connected component $D_k^{\sigma,\,\nu} \subset \mathcal{C}$ is unbounded in $\mathbb{R} \times E$ and lies in $I_k^\sigma \times S_k^{\sigma,\nu}$.*

Proof. If $(\lambda, y) \in \mathbb{R} \times S_k^{\sigma,\nu}$ is a solution of problem (1.1)-(1.3), then (λ, y) is also a solution of the following linear eigenvalue problem (2.6), where

$$\psi(x) = \begin{cases} -\frac{f(x,y(x),y'(x),\lambda)}{y(x)} & \text{if } \; y(x) \neq 0, \\ 0 & \text{if } \; y(x) = 0. \end{cases} \tag{3.3}$$

By virtue of conditions (1.4) and (1.5) it follows from relation (3.2) that

$$\psi(x) \geq 0 \;\text{ and }\; |\psi(x)| \leq M \text{ for } x \in [0, 1].$$

If $y \in S_k^{+,\nu}$, then from Theorem 1 implies that λ is a k-th positive eigenvalue of problem (2.6), and if $y \in S_k^{-,\nu}$, then λ is a k-th negative eigenvalue of this problem. Thus if $y \in S_k^{\sigma,,\nu}$, then it follows from Lemma 2.2 that $\lambda \in I_k^\sigma$. Then the statement of this theorem follows from Theorem 3.1. The proof of Theorem 3.2 is complete.

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