

Existence of solutions for a resonant problem under Landesman-Lazer type conditions involving more general elliptic operators in divergence form

Rabil Ayazoglu (Mashiyev) * · Sidika Şule Şener · Tuba
Ağırman Aydın

Received: 26.09.2019 / Revised: 28.11.2019 / Accepted: 27.12.2019

Abstract. *The present paper is concerned with the resonant problem*

$$-\operatorname{div}(a(x, \nabla u)) = \lambda_1 |u|^{p-2} u + f(x, u) - g(x) \quad \text{in } \Omega,$$

where Ω is a bounded domain with smooth boundary in \mathbb{R}^N ($N \geq 2$), $p \in (1, \infty)$ and $\operatorname{div}(a(x, \nabla u))$ is a general elliptic operators in divergence form. By assuming a Landesman-Lazer type condition and using a variational method based on the Minimum Principle, we show the existence of a weak solution in the Sobolev space $W_0^{1,p}(\Omega)$.

Mathematics Subject Classification (2010): 35J20, 35J60, 58E05

Keywords. Resonance, Elliptic operators, p -Laplacian type operator, Landesman-Lazer type conditions, Minimum Principle.

1 Introduction and preliminaries

In this work, we obtain existence and multiplicity results for equations involving more general elliptic operators in divergence form

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = \lambda_1 |u|^{p-2} u + f(x, u) - g(x) & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases} \quad (1.1)$$

where Ω is a bounded domain with smooth boundary in \mathbb{R}^N ($N \geq 2$), $p \in (1, \infty)$ and $\operatorname{div}(a(x, \nabla u))$ is a more general elliptic operators in divergence form, $g \in L^{p'}(\Omega)$, where p' is the conjugate exponent of p with $1/p + 1/p' = 1$ and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a bounded

* Corresponding author

R. Ayazoglu (Mashiyev)
Faculty of Education, Bayburt University, 69000 Bayburt, Turkey
E-mail: rabilmashiyev@gmail.com

S.Ş. Şener
Faculty of Science, Ataturk University, 25000 Erzurum, Turkey
E-mail: senersule@atauni.edu.tr

T.A. Aydın
Faculty of Education, Bayburt University, 69000 Bayburt, Turkey
E-mail: tubayaydin@bayburt.edu.tr

Carathéodory function. Such operators arise, for example, from the expression of the p -Laplacian in curvilinear coordinates. In the case of the p -Laplacian, this is usually achieved by using the uniform convexity of the Sobolev space $E := W_0^{1,p}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} := \|u\|_E = \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

In order to extend this idea to more general equations, we introduce a notion of uniformly convex functional.

Let X is a Banach space.

Definition 1.1 We shall say that the convex functional $A : X \rightarrow \mathbb{R}$ is uniformly convex on the (convex) set $\Omega \subset X$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$A\left(\frac{x+y}{2}\right) \leq \frac{1}{2}A(x) + \frac{1}{2}A(y) - \delta(\varepsilon)$$

for $x, y \in \Omega$ and $\|x - y\|_X > \varepsilon$.

If functional A is uniformly convex on every ball of X , we shall say that functional A is locally uniformly convex.

Example 1 $x \rightarrow x^p$ is strongly p -monotone if $p \geq 2$.

Moreover, although the well-known Poincaré inequality, i.e.,

$$\|u\|_p \leq C \|\nabla u\|_p, \quad (1.2)$$

holds true in $L^p(\Omega)$, where $C > 0$ is a constant and $\|u\|_{L^p(\Omega)} := \|u\|_p$.

Moreover, let λ_1 denote the first eigenvalue for $-\Delta_p$ on Ω with zero Dirichlet boundary condition which has the variational characterization

$$\lambda_1 = \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega) \setminus \{0\} \text{ with } \int_{\Omega} |u|^p dx = 1 \right\}.$$

Recall that λ_1 is simple, positive and there exists a unique positive eigenfunction ϕ_1 whose norm in $W_0^{1,p}(\Omega)$ equals one (see [1]).

Resonance problems of quasilinear elliptic partial differential equations have been studied extensively in the usual Sobolev spaces. Since the celebrated paper by Landesman and Lazer (see [11]), many existence results were obtained under various nonlinearity growth conditions and the Landesman–Lazer conditions (see [2], [3], [6], [8], [9], [12], [14] and references therein).

As we know, the geometry of the problem to (1.1) depends strongly on the values of r in the estimate below

$$|F(x, u)| \leq C(h(x) + |u|^q),$$

where C is a positive constant, and $F(x, s) = \int_0^s f(x, t) dt$ and $h \in L^{p'}(\Omega)$ with $h(x) \geq 0$ for any $x \in \Omega$. We can discuss three distinct cases:

- (i) $q < p$ (sublinear-like),
- (ii) $q > p$ (superlinear-like),
- (iii) $q = p$ (of resonance type).

For the cases (i), (ii) and some other mixed cases there are many papers so we refer the reader to [10], [18] for (i), to [5], [7] for (ii), and to [13], [16], [17] for mixed cases. For the case (iii), which is the main subject of the present paper, the solution of (1.1) depend

in an essential manner on the asymptotic behavior of f . Let assume, for example, that f is asymptotic linear, that is $\frac{f(x,u)}{|u|^{p-2}u}$ has a finite limit as $|u| \rightarrow \infty$. If the term $\lambda_1 + \frac{f(x,u)}{|u|^{p-2}u}$ meets the eigenvalue λ_1 , then problem (1.1) is said to be with resonance at infinity. For the treatment of resonance and the existence of a solution, it is sufficient that $g \in L^{p'}(\Omega)$ satisfy the Landesman-Lazer's condition.

Assume that $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$, $A = A(x, \xi)$, i.e. $a(x, \xi) = \nabla_\xi A(x, \xi)$, and that there are positive real number C_0 and nonnegative measurable function h on Ω such that $h \in L^{p'}(\Omega)$ for a.e. $x \in \Omega$. We can give the following examples for the operators A and a :

(i) Set $A(x, \xi) = \frac{1}{p} |\xi|^p$, $a(x, \xi) = |\xi|^{p-2} \xi$, where $p \geq 2$. Then, we get the p -Laplace operator $\operatorname{div}(|\nabla u|^{p-2} \nabla u)$.

(ii) Set $A(x, \xi) = \frac{1}{p} \left[\left(1 + |\xi|^2\right)^{\frac{p}{2}} - 1 \right]$, $a(x, \xi) = \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} \xi$, where $p \geq 2$.

Then, we obtain the generalized mean curvature operator $\operatorname{div} \left(\left(1 + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u \right)$.

Suppose that a and A satisfy the following hypotheses:

(A₁) The following inequality holds

$$|a(x, \xi)| \leq C_0 \left(h(x) + |\xi|^{p-1} \right) \quad \forall x \in \Omega, \xi \in \mathbb{R}^N, h \in L^{p'}(\Omega)$$

for some constant $C_0 > 0$;

(A₂) A is p -uniformly convex: There exists a constant $k > 0$ such that

$$A\left(x, \frac{\xi + \psi}{2}\right) \leq \frac{1}{2} A(x, \xi) + \frac{1}{2} A(x, \psi) - k |\xi - \psi|^p$$

for all $x \in \Omega$ and $\xi, \psi \in \mathbb{R}^N$;

(A₃) The following inequality holds

$$|\xi|^p \leq a(x, \xi) \cdot \xi \leq pA(x, \xi)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$;

(A₄) $A(x, 0) = 0$.

Moreover, to construct our basic results, we also suppose the following assumptions exist.

(g₁) $g \in L^{p'}(\Omega)$;

(f₁) $|f(x, s)| \leq \gamma(x)$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, where $\gamma \in L^{p'}(\Omega)$;

(f₂) $\limsup_{s \rightarrow +\infty} f(x, s) = f^{+\infty}(x) \in L^\infty(\Omega)$, $\liminf_{s \rightarrow -\infty} f(x, s) = f^{-\infty}(x) \in L^\infty(\Omega)$;

(f₃) $\int_\Omega f^{+\infty}(x) \phi_1(x) dx < \int_\Omega g(x) \phi_1(x) dx < \int_\Omega f^{-\infty}(x) \phi_1(x) dx$.

As we know, under (f₂), problem (1.1) may not have a solution. However, in [11] Landesman and Lazer have showed that the condition (f₃) (so called Landesman-Lazer's condition) is a sufficient condition for the existence of solution of (1.1).

In this paper by introducing a of Landesman-Lazer type condition (see (f₃)) we shall prove an existence result for a p -Laplacian type operator on resonance in bounded domain with the nonlinearities f and g to be functions. We also point out that in that papers, the property $a(x, \xi) \cdot \xi = pA(x, \xi)$, which may not hold under our assumptions by (A₃), play an important role in the arguments.

Define the energy functional $I_{\lambda_1} : E \rightarrow \mathbb{R}$ associated to (1.1) by

$$I_{\lambda_1}(u) = \int_\Omega A(x, \nabla u) dx - \frac{\lambda_1}{p} \int_\Omega |u|^p dx - \int_\Omega F(x, u) dx + \int_\Omega g u dx,$$

where $F(x, u) = \int_0^u f(x, t) dt$.

Letting

$$A(u) = \int_{\Omega} A(x, \nabla u) dx,$$

and

$$J_{\lambda_1}(u) = \frac{\lambda_1}{p} \int_{\Omega} |u|^p dx + \int_{\Omega} F(x, u) dx - \int_{\Omega} g u dx.$$

As we know, standard arguments imply that $J_{\lambda_1} \in C^1(E, \mathbb{R})$ and its derivative given by

$$\langle J'_{\lambda_1}(u), v \rangle = \lambda_1 \int_{\Omega} |u|^{p-2} u v dx + \int_{\Omega} f(x, u) v dx - \int_{\Omega} g v dx$$

for all $u, v \in E$.

We say that $u \in E$ is a *weak solution* of problem (1.1) if

$$\int_{\Omega} a(x, \nabla u) \nabla v dx - \lambda_1 \int_{\Omega} |u|^{p-2} u v dx - \int_{\Omega} f(x, u) v dx + \int_{\Omega} g v dx = 0$$

for all $v \in E$.

2 Auxiliary results

Lemma 2.1

(i) A verifies the growth condition

$$|A(x, \xi)| \leq C_0(h(x) |\xi| + |\xi|^p),$$

for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(ii) A is p -homogeneous

$$A(x, z\xi) \leq A(x, \xi) z^p,$$

for all $z \geq 1, \xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Proof.

(i) For any $\xi \in \mathbb{R}^N$, we have

$$A(x, \xi) = \int_0^1 \frac{d}{dt} A(x, t\xi) dt = \int_0^1 a(x, t\xi) \cdot \xi dt$$

By hypothesis (A_1) , we have

$$\begin{aligned} |A(x, \xi)| &\leq \int_0^1 |a(x, t\xi)| \cdot |\xi| dt \leq C_0 \int_0^1 (h(x) + |\xi|^{p-1} |t|^{p-1}) |\xi| dt \\ &\leq C_0 \int_0^1 (h(x) |\xi| + |\xi|^p |t|^{p-1}) dt \\ &\leq C_0(h(x) |\xi| + |\xi|^p). \end{aligned}$$

(ii) To see that, let us define $g(t) = A(t\xi)$. Then, by (A_3)

$$g'(t) = a(x, t\xi) \cdot \xi = \frac{1}{t} a(x, t\xi) \cdot t\xi \leq \frac{p}{t} A(x, t\xi) = \frac{p}{t} g(t),$$

then

$$\frac{g'(t)}{g(t)} \leq \frac{p}{t},$$

and integrating both side over $(1, z)$, we have

$$\log g(z) - \log g(1) \leq p \log z.$$

Then,

$$\frac{g(z)}{g(1)} \leq z^p,$$

so we conclude that

$$A(x, z\xi) \leq A(x, \xi) z^p.$$

The proof is complete.

Lemma 2.2

(i) The functional Λ is well-defined on E .

(ii) The functional Λ is of class $C^1(E, \mathbb{R})$ and

$$\langle \Lambda'(u), \varphi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi dx,$$

for all $u, \varphi \in E$.

(iii) The functional Λ is weakly lower semi-continuos on E .

(iv) For all $u, v \in E$

$$\Lambda\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(v) - k \|u-v\|_E^p.$$

(v) For all $u, v \in E$

$$\Lambda(u) - \Lambda(v) \geq \langle \Lambda'(v), u-v \rangle.$$

Proof. (i) By (i) in Lemma 2.1 and (1.2), we have

$$\begin{aligned} \Lambda(u) &= \int_{\Omega} A(x, \nabla u) dx \leq C_0 \int_{\Omega} h(x) |\nabla u| dx + C_0 \int_{\Omega} |\nabla u|^p dx \\ &\leq C_0 C \|h\|_{p'} \|u\|_E + C_0 C \|u\|_E^p < \infty. \end{aligned}$$

Hence, Λ is well defined on E .

(ii) Let $u, \varphi \in E$, $x \in \Omega$, and $0 < |r| < 1$. Then, by the mean value theorem, there exists $v \in [0, 1]$ such that

$$\begin{aligned} &\left| \frac{A(x, \nabla u(x) + r \nabla \varphi(x)) - A(x, \nabla u(x))}{r} \right| \\ &= \left| \int_0^1 a(x, \nabla u(x) + vr \nabla \varphi(x)) \nabla \varphi(x) dv \right| \\ &\leq C_0 \int_0^1 (h(x) + |\nabla u(x) + vr \nabla \varphi(x)|^{p-1}) |\nabla \varphi(x)| dv \\ &\leq C_0 (h(x) + (|\nabla u(x)| + |\nabla \varphi(x)|)^{p-1}) |\nabla \varphi(x)| \\ &\leq C_0 h(x) |\nabla \varphi(x)| + C_0 |\nabla \varphi(x)| (|\nabla u(x)| + |\nabla \varphi(x)|)^{p-1} \\ &\leq C_0 h(x) |\nabla \varphi(x)| + C_0 2^{p-1} |\nabla \varphi(x)| (|\nabla u(x)|^{p-1} + |\nabla \varphi(x)|^{p-1}) \\ &\leq C_0 h(x) |\nabla \varphi(x)| + C_0 2^{p-1} |\nabla \varphi(x)| |\nabla u(x)|^{p-1} + C_0 2^{p-1} |\nabla \varphi(x)|^p. \end{aligned}$$

By help of the (1.2), we can see $h(x)|\nabla\varphi(x)|$, $|\nabla\varphi(x)||\nabla u(x)|^{p-1}$, and $|\nabla\varphi(x)|^p$ are integrable on Ω , so the right-hand side is integrable on Ω . Applying the Lebesgue Dominated convergence theorem, we have

$$\begin{aligned}\langle A'(u), \varphi \rangle &= \lim_{r \rightarrow 0} \int_{\Omega} \frac{A(x, \nabla u + r \nabla \varphi) - A(x, \nabla u)}{r} dx \\ &= \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi dx\end{aligned}$$

Next, let show the continuity of A' on E . Suppose $u_n \rightarrow u$ in E and let define $\theta(x, u) = a(x, \nabla u)$. Using the hypothesis (A_1) , we conclude that $\theta(x, u_n) \rightarrow \theta(x, u)$ in $(L^{p'}(\Omega))^N$ a.e. $x \in \Omega$. Then, we have

$$|\langle A'(u_n) - A'(u), \varphi \rangle| \leq \|\theta(x, u_n) - \theta(x, u)\|_{p'} \|\nabla \varphi\|_p,$$

and so

$$\|A'(u_n) - A'(u)\| \leq \|\theta(x, u_n) - \theta(x, u)\|_{p'} \rightarrow 0,$$

as $n \rightarrow \infty$.

(iii) By corollary III.8 in Brezis [4], it is enough to show that A is lower semi-continuous. Since A is convex (by condition (A_2)), we deduce that for any $v \in E$, the following inequality holds

$$\int_{\Omega} A(x, \nabla v) dx \geq \int_{\Omega} A(x, \nabla u) dx + \int_{\Omega} a(x, \nabla u) \cdot (\nabla v - \nabla u) dx.$$

Using condition (A_1) , we have

$$\begin{aligned}& \int_{\Omega} A(x, \nabla v) dx \\ & \geq \int_{\Omega} A(x, \nabla u) dx - \int_{\Omega} |a(x, \nabla u)| |\nabla v - \nabla u| dx \\ & \geq \int_{\Omega} A(x, \nabla u) dx - C_0 \int_{\Omega} h(x) |\nabla(v-u)| dx - C_0 \int_{\Omega} |\nabla(v-u)| |\nabla u|^{p-1} dx \\ & \geq \int_{\Omega} A(x, \nabla u) dx - C_1 \|h\|_{p'} \|\nabla(v-u)\|_p - C_2 \|\nabla(v-u)\|_p \left\| |\nabla u|^{p-1} \right\|_{p'} \\ & \geq \int_{\Omega} A(x, \nabla u) dx - C_3 \|v-u\|_E - C_4 \|v-u\|_E \|u\|_E^{p-1} \\ & \geq \int_{\Omega} A(x, \nabla u) dx - C_3 \|v-u\|_E - C_5 \|v-u\|_E \\ & \geq \int_{\Omega} A(x, \nabla u) dx - \varepsilon\end{aligned}$$

for all $v \in E$ with $\|v-u\|_E < \delta = \frac{\varepsilon}{C_3 + C_5}$. So, we deduce that A is weakly lower semi-continuous.

(iv) Using condition (A_2) , we have

$$\begin{aligned} \Lambda\left(\frac{u+v}{2}\right) &= \int_{\Omega} A\left(x, \frac{\nabla u + \nabla v}{2}\right) dx \\ &\leq \frac{1}{2} \int_{\Omega} A(x, \nabla u) dx + \frac{1}{2} \int_{\Omega} A(x, \nabla v) dx - k \int_{\Omega} |\nabla u - \nabla v|^p dx \\ &\leq \frac{1}{2} \Lambda(u) + \frac{1}{2} \Lambda(v) - k \|u - v\|_E^p. \end{aligned}$$

(v) Since Λ is convex (by condition (A_2)), we can find $t \in (0, 1)$ such that

$$\begin{aligned} &\frac{\Lambda(v + t(u - v)) - \Lambda(v)}{t} \\ &= \frac{\Lambda((1-t)v + tu) - \Lambda(v)}{t} \\ &\leq \frac{(1-t)\Lambda(v) + t\Lambda(u) - \Lambda(v)}{t} = \Lambda(u) - \Lambda(v). \end{aligned}$$

Letting $t \rightarrow 0$, we have

$$\lim_{t \rightarrow 0} \frac{\Lambda(v + t(u - v)) - \Lambda(v)}{t} = \langle \Lambda'(v), u - v \rangle.$$

Thus, we obtain

$$\langle \Lambda'(v), u - v \rangle \leq \Lambda(u) - \Lambda(v).$$

The proof of Lemma 2.2 is complete.

Lemma 2.3 (*Palais-Smale condition $(PS)_c$, see [15]*). Let X be a real Banach space. A functional $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition, $(PS)_c$ condition for short, if any sequence $\{u_n\}$ in X such that

$$|I(u_n)| \leq c \text{ and } I'(u_n) \rightarrow 0, \quad (2.1)$$

has a convergent subsequence, where $c \in \mathbb{R}$.

Lemma 2.4 (*The Minimum Principle, see [15]*). Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. Assume that

- (i) I is bounded from below, $c = \inf I$,
- (ii) I satisfies (PS) condition.

Then, there exists $u_0 \in X$ such that $I(u_0) = c$.

3 The main results and proofs

The main theorem which we deal with in the present paper is

Theorem 3.1 Suppose the conditions (A_1) - (A_4) and (f_1) - (f_3) hold. Then, the problem (1.1) has at least one nontrivial weak solution in E .

Lemma 3.1 I_{λ_1} is well-defined on E and of class $C^1(E, \mathbb{R})$, and its derivative given by

$$\langle I'_{\lambda_1}(u), v \rangle = \int_{\Omega} a(x, \nabla u) \nabla v dx - \lambda_1 \int_{\Omega} |u|^{p-2} uv dx - \int_{\Omega} f(x, u) v dx + \int_{\Omega} g v dx$$

for all $u, v \in E$.

Proof. This comes from (i) and (ii) in Lemma 2.2 and properties of J_{λ_1} .

Lemma 3.2 I_{λ_1} satisfies $(PS)_c$ condition on E provided that the condition (f_3) hold.

Proof. First, we prove that $\{u_n\}$ is bounded in E . We assume by contradiction that $\|u_n\|_E \rightarrow \infty$ as $n \rightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|_E}$ for all n . Thus $\{v_n\}$ is bounded in E . Since E is reflexive, we can assume that there exists a subsequence which we still denote by $\{v_n\}$ converging weakly to a certain v in E . Since we have the embedding $E \hookrightarrow L^p(\Omega)$ (compact), then $\{v_n\}$ converges strongly to a certain v in $L^p(\Omega)$.

$$I_{\lambda_1}(u_n) = \int_{\Omega} A(x, \nabla u_n) dx - \frac{\lambda_1}{p} \int_{\Omega} |u_n|^p dx - \int_{\Omega} F(x, u_n) dx + \int_{\Omega} g u_n dx. \quad (3.1)$$

Taking into account (A_3) together with (2.1), and dividing (3.1) by $\|u_n\|_E^p$, we have

$$\begin{aligned} & \frac{1}{p} \int_{\Omega} \frac{|\nabla u_n|^p}{\|u_n\|_E^p} dx - \frac{\lambda_1}{p} \int_{\Omega} \frac{|u_n|^p}{\|u_n\|_E^p} dx - \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|_E^p} dx + \int_{\Omega} \frac{g u_n}{\|u_n\|_E^p} dx \\ & \leq \frac{c}{\|u_n\|_E^p}. \end{aligned} \quad (3.2)$$

Now we take the limit of both sides, we have

$$\limsup_{n \rightarrow +\infty} \left(\frac{1}{p} \int_{\Omega} |\nabla v_n|^p dx - \frac{\lambda_1}{p} \int_{\Omega} |v_n|^p dx - \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|_E^p} dx + \int_{\Omega} \frac{g u_n}{\|u_n\|_E^p} dx \right) \leq 0. \quad (3.3)$$

Moreover, using (f_1) and L'Hôpital's rule, we have

$$\int_{\Omega} \frac{F(x, u_n)}{\|u_n\|_E^p} dx = \frac{1}{\|u_n\|_E^{p-1}} \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|_E} dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Similarly, considering that $g \in L^{p'}(\Omega)$, we also get

$$\int_{\Omega} \frac{g u_n}{\|u_n\|_E^p} dx = \frac{1}{\|u_n\|_E^{p-1}} \int_{\Omega} g v_n dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, it follows

$$\limsup_{n \rightarrow +\infty} \left(\int_{\Omega} \frac{F(x, u_n)}{\|u_n\|_E^p} dx + \int_{\Omega} \frac{g u_n}{\|u_n\|_E^p} dx \right) = 0. \quad (3.4)$$

Then, considering (3.3) together with (3.4), we get

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} |\nabla v_n|^p dx \leq \lambda_1 \int_{\Omega} |v_n|^p dx.$$

Since $\{v_n\}$ converges strongly to v in $L^p(\Omega)$, we have

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} |v_n|^p dx = \int_{\Omega} |v|^p dx.$$

Thus, we can write

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} |\nabla v_n|^p dx \leq \lambda_1 \int_{\Omega} |v|^p dx.$$

Using the weak lower semi-continuity of norm, we have

$$\begin{aligned} \lambda_1 \int_{\Omega} |v|^p dx &\leq \int_{\Omega} |\nabla v|^p dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} |\nabla v_n|^p dx \\ &\leq \limsup_{n \rightarrow +\infty} \int_{\Omega} |\nabla v_n|^p dx \leq \lambda_1 \int_{\Omega} |v|^p dx. \end{aligned}$$

Hence, we get $\{v_n\}$ converges strongly to a certain v in E , and

$$\int_{\Omega} |\nabla v|^p dx = \lambda_1 \int_{\Omega} |v|^p dx.$$

This implies, by the definition of ϕ_1 , that $v = \pm\phi_1$.

Moreover, using (2.1), we can obtain the following two inequalities

$$-cp \leq pI_{\lambda_1}(u_n) \leq cp \quad (3.5)$$

and

$$-\varepsilon_n \|u_n\|_E \leq \langle I'_{\lambda_1}(u_n), u_n \rangle_{E^*} \leq \varepsilon_n \|u_n\|_E, \quad (3.6)$$

where $\varepsilon_n \rightarrow 0$ and E^* is dual space of E .

By considering the following two cases, we shall conclude that $\{u_n\}$ is bounded in E .

We then consider the following two cases.

Case 1. Suppose that $v_n \rightarrow -\phi_1$. Since $u_n \rightarrow -\infty$ as $n \rightarrow +\infty$, by (f_2) , we have

$$f(x, u_n) \rightarrow f_{-\infty}(x) \text{ a.e. } x \in \Omega,$$

and

$$\frac{F(x, u_n)}{u_n} \rightarrow f_{-\infty}(x) \text{ a.e. } x \in \Omega. \quad (3.7)$$

Hence, letting n tend to infinity, and considering the Lebesgue Dominated Theorem, we get

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \int_{\Omega} \left(f(x, u_n) \frac{u_n}{\|u_n\|_E} - \frac{pF(x, u_n)}{u_n} \frac{u_n}{\|u_n\|_E} \right) dx \\ &= \lim_{n \rightarrow +\infty} \int_{\Omega} \left(f(x, u_n) v_n - \frac{pF(x, u_n)}{u_n} v_n \right) dx = (p-1) \int_{\Omega} f_{-\infty}(x) \phi_1 dx. \end{aligned} \quad (3.8)$$

On the other hand, by summing up (3.5) and (3.6),

$$\begin{aligned} &-cp - \varepsilon_n \|u_n\|_E \\ &\leq \int_{\Omega} (a(x, \nabla u_n) \nabla u_n - pA(x, \nabla u_n)) dx + \lambda_1 p \int_{\Omega} \frac{1}{p} |u_n|^p dx \\ &\quad - \lambda_1 \int_{\Omega} |u_n|^p dx + \int_{\Omega} (pF(x, u_n) dx - f(x, u_n) u_n) dx + (1-p) \int_{\Omega} g u_n dx \\ &\leq \int_{\Omega} (pF(x, u_n) dx - f(x, u_n) u_n) dx + (1-p) \int_{\Omega} g u_n dx. \end{aligned}$$

Dividing by $\|u_n\|_E$, we have

$$\frac{-cp}{\|u_n\|_E} - \varepsilon_n \leq \int_{\Omega} \left(p \frac{F(x, u_n)}{u_n} v_n - f(x, u_n) v_n \right) dx + (1-p) \int_{\Omega} g v_n dx. \quad (3.9)$$

Since $g \in L^{p'}(\Omega)$ and $\|v_n - (-\phi_1)\|_E \rightarrow 0$, we obtain

$$\lim_{n \rightarrow +\infty} \int_{\Omega} g v_n dx = - \int_{\Omega} g \phi_1 dx. \quad (3.10)$$

In (3.9), taking limit to both sides and using (3.8) and (3.10), we get

$$(1-p) \int_{\Omega} f_{-\infty}(x) \phi_1 dx - (1-p) \int_{\Omega} g \phi_1 dx \geq 0$$

that is, since $p > 1$,

$$\int_{\Omega} g \phi_1 dx \geq \int_{\Omega} f_{-\infty}(x) \phi_1 dx,$$

which contradicts (f_3) .

Case 2. Suppose that $v_n \rightarrow \phi_1$. Since $u_n \rightarrow +\infty$ as $n \rightarrow +\infty$, by (f_2) , we have

$$f(x, u_n) \rightarrow f^{+\infty}(x) \text{ a.e. } x \in \Omega,$$

and

$$\frac{F(x, u_n)}{u_n} \rightarrow f^{+\infty}(x) \text{ a.e. } x \in \Omega. \quad (3.11)$$

Hence, letting n tend to infinity, and considering the Lebesgue Dominated Theorem, we get

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{\Omega} \left(f(x, u_n) \frac{u_n}{\|u_n\|_E} - \frac{pF(x, u_n)}{u_n} \frac{u_n}{\|u_n\|_E} \right) dx \\ &= \lim_{n \rightarrow +\infty} \int_{\Omega} \left(f(x, u_n) v_n - \frac{pF(x, u_n)}{u_n} v_n \right) dx \\ &= (1-p) \int_{\Omega} f^{+\infty}(x) \phi_1 dx. \end{aligned} \quad (3.12)$$

By summing up (3.5) and (3.6) again, we have

$$\begin{aligned} & cp + \varepsilon_n \|u_n\|_E \\ & \geq \int_{\Omega} (pA(x, \nabla u_n) - a(x, \nabla u_n) \nabla u_n) dx - \lambda_1 p \int_{\Omega} \frac{1}{p} |u_n|^p dx \\ & \quad + \lambda_1 \int_{\Omega} |u_n|^p dx + \int_{\Omega} (f(x, u_n) u_n - pF(x, u_n)) dx + (p-1) \int_{\Omega} g u_n dx \\ & \geq \int_{\Omega} (f(x, u_n) u_n - pF(x, u_n)) dx + (p-1) \int_{\Omega} g u_n dx. \end{aligned}$$

Dividing by $\|u_n\|_E$, we get

$$\int_{\Omega} \left(f(x, u_n) v_n - p \frac{F(x, u_n)}{u_n} v_n \right) dx + (p-1) \int_{\Omega} g v_n dx \leq \frac{cp}{\|u_n\|_E} + \varepsilon_n \quad (3.13)$$

Since $g \in L^{p'}(\Omega)$ and $\|v_n - \phi_1\|_E \rightarrow 0$, we have

$$\lim_{n \rightarrow +\infty} \int_{\Omega} g v_n dx = \int_{\Omega} g \phi_1 dx \quad (3.14)$$

In (3.13), taking limit to both sides and using (3.12) and (3.14), we conclude

$$(1-p) \int_{\Omega} f^{+\infty}(x) \phi_1 dx + (p-1) \int_{\Omega} g \phi_1 dx \leq 0,$$

that is, since $p > 1$,

$$\int_{\Omega} g \phi_1 dx \leq \int_{\Omega} f^{+\infty}(x) \phi_1 dx,$$

which contradicts (f_3) .

From the two cases above, we conclude $\{u_n\}$ is bounded in E . Hence, $u_n \rightharpoonup u$ in E . Now, we shall show $u_n \rightarrow u$ in E . By the embedding $E \hookrightarrow L^p(\Omega)$ and (2.1), it follows

$$\langle I'_{\lambda_1}(u_n), u_n - u \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

On the other hand, we have

$$\begin{aligned} & \langle I'_{\lambda_1}(u_n), u_n - u \rangle \\ &= \int_{\Omega} a(x, \nabla u_n) (\nabla u_n - \nabla u) dx - \lambda_1 \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx \\ & \quad - \int_{\Omega} f(x, u_n) (u_n - u) dx + \int_{\Omega} g(u_n - u) dx, \end{aligned}$$

thus

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) (\nabla u_n - \nabla u) dx = \langle I'_{\lambda_1}(u_n), u_n - u \rangle + \lambda_1 \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx \\ & \quad + \int_{\Omega} f(x, u_n) (u_n - u) dx - \int_{\Omega} g(u_n - u) dx. \end{aligned}$$

Using the fact that $\{u_n\}$ converges strongly to a certain u in $L^p(\Omega)$ then $\|u_n\|_p \leq C, (C > 0)$ we have

$$\begin{aligned} \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx &\leq \left| \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx \right| \leq \left\| |u_n|^{p-1} \right\|_{p'} \|u_n - u\|_p \\ &\leq \|u_n\|_p^{p-1} \|u_n - u\|_p \leq C^{p-1} \|u_n - u\|_p. \end{aligned}$$

Moreover, by the embedding $E \hookrightarrow L^p(\Omega)$, we have $\|u_n - u\|_p \rightarrow 0$ as $n \rightarrow +\infty$. Hence, we deduce

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx = 0.$$

Using similar arguments and considering the hypotheses on f (see (f_1)) and g (see (g_1)), we also deduce that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n) (u_n - u) dx = 0,$$

and

$$\lim_{n \rightarrow +\infty} \int_{\Omega} g(u_n - u) dx = 0.$$

All the above pieces of information imply

$$\int_{\Omega} a(x, \nabla u_n) (\nabla u_n - \nabla u) dx = 0,$$

that is,

$$\lim_{n \rightarrow +\infty} \langle A'(u_n), u_n - u \rangle = 0.$$

By using (v) in Lemma 2.2, we have

$$0 = \lim_{n \rightarrow \infty} \langle \Lambda'(u_n), u - u_n \rangle \leq \lim_{n \rightarrow \infty} (\Lambda(u) - \Lambda(u_n)) = \Lambda(u) - \lim_{n \rightarrow \infty} \Lambda(u_n)$$

or

$$\lim_{n \rightarrow \infty} \Lambda(u_n) \leq \Lambda(u).$$

This fact and relation (iii) in Lemma 2.2 imply

$$\lim_{n \rightarrow \infty} \Lambda(u_n) = \Lambda(u).$$

We assume by contradiction that $\{u_n\}$ does not converge strongly to u in E . Then, there exists $\varepsilon > 0$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that $\|u_{n_m} - u\|_E \geq \varepsilon$. On the other hand, by (iv) in Lemma 2.2, we have

$$\frac{1}{2}\Lambda(u_{n_m}) + \frac{1}{2}\Lambda(u) - \Lambda\left(\frac{u_{n_m} + u}{2}\right) \geq k\|u_{n_m} - u\|_E \geq k\varepsilon.$$

Letting $m \rightarrow \infty$ in the above inequality, we obtain

$$\limsup_{n \rightarrow \infty} \Lambda\left(\frac{u_{n_m} + u}{2}\right) \leq \Lambda(u) - k\varepsilon^p.$$

Moreover, we have $\left\{\frac{u_{n_m} + u}{2}\right\}$ converges weakly to u in E . Using (iii) in Lemma 2.2, we obtain

$$\Lambda(u) \leq \liminf_{n \rightarrow \infty} \Lambda\left(\frac{u_{n_m} + u}{2}\right),$$

and that is a contradiction. It follows that $\{u_n\}$ converges strongly to u in E . The proof of Lemma 3.2 is complete.

Lemma 3.3 *The functional I_{λ_1} is coercive on E provided (f_3) holds.*

Proof. We firstly note that, in the proof of the Lemma 3.2, we showed that if $I_{\lambda_1}(u_n)$ is a sequence bounded from above with $\|u_n\|_E \rightarrow \infty$ then (up to a subsequence), $v_n = \frac{u_n}{\|u_n\|_E} \rightarrow \pm\phi_1$ in E . Using this fact, we shall prove that I_{λ_1} is coercive on E provided (f_3) holds. Indeed, if I_{λ_1} is not coercive, it is possible to choose a sequence $\{u_n\} \subset E$ such that $\|u_n\|_E \rightarrow \infty$, $I_{\lambda_1}(u_n) \leq M$, where M is a constant, and $v_n = \frac{u_n}{\|u_n\|_E} \rightarrow \pm\phi_1$ in E .

By the assumption (A_3) , we have

$$\begin{aligned} & I_{\lambda_1}(u_n) \\ & \geq \frac{1}{p} \int_{\Omega} |\nabla u_n|^p dx - \frac{\lambda_1}{p} \int_{\Omega} |u_n|^p dx - \int_{\Omega} F(x, u_n) dx + \int_{\Omega} g u_n dx \\ & \geq - \int_{\Omega} F(x, u_n) dx + \int_{\Omega} g u_n dx. \end{aligned} \quad (3.15)$$

Now, we shall investigate two cases:

Case 1. Assume that $v_n \rightarrow \phi_1$. Dividing (3.15) by $\|u_n\|_E$ and using (3.11), we have

$$\begin{aligned} & - \int_{\Omega} f^{+\infty}(x) \phi_1 dx + \int_{\Omega} g \phi_1 dx \\ & = \lim_{n \rightarrow +\infty} \left(- \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|_E} dx + \int_{\Omega} g \frac{u_n}{\|u_n\|_E} dx \right) \\ & \leq \limsup_{n \rightarrow +\infty} \frac{I_{\lambda_1}(u_n)}{\|u_n\|_E} \leq \limsup_{n \rightarrow +\infty} \frac{M}{\|u_n\|_E} = 0, \end{aligned}$$

that is,

$$\int_{\Omega} g\phi_1 dx \leq \int_{\Omega} f^{+\infty}(x) \phi_1 dx,$$

which contradicts (f_3) .

Case 2. Assume that $v_n \rightarrow -\phi_1$. Dividing (3.15) by $\|u_n\|_E$ and using (3.7), we have

$$\begin{aligned} & \int_{\Omega} f_{-\infty}(x) \phi_1 dx - \int_{\Omega} g\phi_1 dx \\ &= \lim_{n \rightarrow +\infty} \left(- \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|_E} dx + \int_{\Omega} g \frac{u_n}{\|u_n\|_E} dx \right) \\ &\leq \limsup_{n \rightarrow +\infty} \frac{I_{\lambda_1}(u_n)}{\|u_n\|_E} \leq \limsup_{n \rightarrow +\infty} \frac{M}{\|u_n\|_E} = 0, \end{aligned}$$

that is,

$$\int_{\Omega} f_{-\infty}(x) \phi_1 dx \leq \int_{\Omega} g\phi_1 dx,$$

which contradicts (f_3) . Hence, the functional I_{λ_1} is coercive on E .

Proof of Theorem 3.1 is completed. The coerciveness and $(PS)_c$ condition are enough to prove that I_{λ_1} attains its infimum in E (see Lemma 2.4). Hence, the problem (1.1) has at least a weak solution in E . The proof is complete.

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