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On the boundedness of the $G-$maximal operator and $G-$Riesz potential in the generalized $G-$Morrey spaces

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Abstract. *In this paper, we study the maximal function ($G-$maximal function) and the Riesz potential ($G-$Riesz potential) generated by the Gegenbauer differential operator*

$$G_\lambda = (x^2-1)^{\frac{1}{2}-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx}, \;\; x\in(1,\infty), \;\; \lambda\in(0,1/2).$$

We introduce the generalized Gegenbauer-Morrey spaces (generalized $G-$Morrey spaces) and find the condition for the boundedness $G-$maximal operator and $G-$Riesz potential from the $G-$Morrey spaces $L_{p,\omega,\lambda}$ to $L_{q,\omega,\lambda}$ and $L_{1,\omega,\lambda}$ to weak $G-$Morrey space $WL_{q,\omega,\lambda}$. We obtain the new results of the strong and weak Spanne-Guliyev and Adams-Guliyev type boundedness of the maximal and Riesz potential operators in generalized $C-$Morrey spaces, respectively.

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1 Definition and auxiliary results

Let $f\in L^{loc}_{1,\lambda}(\mathbb{R}_+)$, $\mathbb{R}_+=(0,\infty)$. The maximal operator M_G, the fractional maximal operator M^α_G and the Gegenbauer-Riesz ($G-$Riesz) potential I^α_G are defined in [16, **?**] as follows:

$$M_G f(chx) = \sup_{r>0}\frac{1}{|H(0,r)|_\lambda}\int_0^r A^\lambda_{cht}|f(chx)|sh^{2\lambda}tdt,$$

$$M^\alpha_G f(chx) = \sup_{r>0}|H(0,r)|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int_0^r A^\lambda_{cht}|f(chx)|sh^{2\lambda}tdt,$$

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and

$$I_G^\alpha f(chx) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(cht)dr \right) A_{cht}^\lambda f(chx) sh^{2\lambda} t dt,$$

where $|H(0,r)|_\lambda = \int_0^r sh^{2\lambda} t dt$ is the Lebesgue measure of the interval $(0,r)$,

$$h_r(cht) = \int_1^\infty e^{-\nu(\nu+2\lambda)r} P_\nu^\lambda(cht)(\nu^2-1)^{\lambda-\frac{1}{2}} d\nu$$

and $P_\nu^\lambda(cht)$ is eigenfunction of the operator G_λ,

$$A_{cht}^\lambda f(chx) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \int_0^\pi f(chxcht - shxsht\cos\varphi)(sin\varphi)^{2\lambda-1} d\varphi$$

is the generalized shift operator generated by the Gegenbauer differential operator G_λ.

The operators $M_G \equiv M_G^0$, M_G^α and I_G^α play an important role in harmonic analysis.

Throughout in the paper, we will denote shx, chx by the hyperbolic functions and by $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C such that $0 < A \leq CB$, moreover C can dependent on some parameters. Symbol $A \approx B$ denote that $A \lesssim B$ and $B \lesssim A$.

Further we need the following assertion.

Lemma 1.1 *[18] For $0 < \lambda < \frac{1}{2}$ the following correlations is true:*

$$|H(0,r)|_\lambda \approx \begin{cases} \left(sh\frac{r}{2}\right)^{2\lambda+1}, & 0 < r < 2, \\ (ch\frac{r}{2})^{4\lambda}, & 2 \leq r < \infty. \end{cases}$$

Lemma 1.2 *[17] For any $\gamma > 0$ the following correlations are true:*

$$|H(x,r)|_{\frac{\gamma}{2}} = \int_{x-r}^{x+r} sh^{2\lambda} t dt \approx \begin{cases} \left(sh\frac{x+r}{2}\right)^{\gamma+1}, & 0 < x+r < 2, \\ (sh\frac{x+r}{2})^{2\gamma}, & 2 \leq x+r < \infty. \end{cases}$$

where

$$H(x,r) = \begin{cases} (0, x+r), & 0 < x < r, \\ (x-r, x+r), & x \geq r. \end{cases}$$

Further we need the following statement.

Lemma 1.3 *The Chebyshev type inequality*

$$\left|\left\{x \in \mathbb{R}_+ : T^\lambda f(chx) > \alpha\right\}\right|_\lambda \leq \frac{1}{\alpha} \int_{\mathbb{R}_+} T^\lambda f(chx) sh^{2\lambda} x dx$$

holds for all $\alpha > 0$.

Proof. Since

$$T^\lambda f(chx) \geq \alpha \chi_{\{T^\lambda f(chx) > \alpha\}}(chx),$$

we have

$$\int_{\mathbb{R}_+} T^\lambda f(chx) sh^{2\lambda} x dx \geq \alpha \int_{\mathbb{R}_+} \chi_{\{T^\lambda f(chx)>\alpha\}}(chx) sh^{2\lambda} x dx$$
$$= \alpha |\{x \in \mathbb{R}_+ : T^\lambda f(chx) > \alpha\}|_\lambda.$$

Theorem 1.1 *[19] (Fefferman-Stein type inequality)*

(i) *For every nonnegative measurable functions f and g on $\mathbb{R}_+$ every $1 \leq p < \infty$ and every $0 < t < \infty$*

$$\int_{\mathbb{R}_+} A^\lambda_{cht}(M_G f(chx))^p g(chx) sh^{2\lambda} x dx \lesssim \int_{R_+} A^\lambda_{cht} f(chx)^p M_G g(chx) sh^{2\lambda} x dx.$$

(ii) *For any measurable function on $\mathbb{R}_+$ $f \geq 0$ and $g \geq 0$*

$$\int_{\{x\in\mathbb{R}_+ : A^\lambda_{cht} M_G f(chx) > \alpha\}} g(chx) sh^{2\lambda} x dx \lesssim \frac{1}{\alpha} \int_{\mathbb{R}_+} A^\lambda_{cht} f(chx) M_G g(chx) sh^{2\lambda} x dx.$$

2 Generalized Gegenbauer-Morrey spaces

In connection with elliptic partial differential equations, C. Morrey proposed a weak condition for the solution to be continuous enough in 1938. Later on, his condition became a family of normed spaces and they are called Morrey spaces [21]. Although the notion is originally from the partial differential equations, the space turned out to be important in many branches of mathematics. Therefore this theory is devote many work [1]-[16], [22]-[26].

In the [16] introduced Gegenbauer-Morrey space as the set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0} \left(r^\gamma \int_0^r A^\lambda_{cht} |f(chx)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}}$$

where $1 \leq p < \infty$, $\lambda \in (0, 1/2)$, $0 \leq \gamma \leq 2\lambda + 1$, and also the weak space $WL_{p,\lambda,\gamma}(\mathbb{R}_+)$ with the finite norm

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{x\in\mathbb{R}_+, t>0} \left(t^{-\gamma} \left| \left\{ y \in (0,t) : |A^\lambda_{chy}|f(chx)| > r \right\} \right|_\lambda \right)^{\frac{1}{p}}$$

$$= \sup_{r>0} r \sup_{x\in\mathbb{R}_+, t>0} \left(t^{-\gamma} \int_{\{y\in(0,t) : A^\lambda_{chy}|f(cht)| > r\}} sh^{2\lambda} y dy \right)^{\frac{1}{p}}.$$

In [16] the following statements were proved.

Theorem 2.1 *[16] Let $0 < \alpha < 2\lambda + 1$, $0 < \gamma < 2\lambda + 1 - \alpha$ and $1 \leq p < \frac{2\lambda+1}{\alpha}$.*

(i) If $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$ is necessary and sufficient for the boundedness of I^α_G from $L_{p,\lambda,\gamma}(\mathbb{R}_+)$ to $L_{q,\lambda,\gamma}(\mathbb{R}_+)$.

(ii) If $p = 1 < \frac{2\lambda+1-\gamma}{\alpha}$, the condition $1 = \frac{1}{q} = \frac{\alpha}{2\lambda+1-\gamma}$ is necessary and sufficient for the boundedness of I^α_G from $L_{1,\lambda,\gamma}(\mathbb{R}_+)$ to $WL_{q,\lambda,\gamma}(\mathbb{R}_+)$.

Everywhere in the sequel the functions $\omega(x, r)$, $\omega_1(x, r)$ and $\omega_2(x, r)$ used in the body of the paper are nonnegative measurable function on $\mathbb{R}_+$.

By analogy with [11], we introduce the following notation.

Definition 2.1 *Let $1 \le p < \infty$. The generalized Gegenbauer-Morrey (G-Morrey) space $\mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$ associated with the Gegenbauer differential operator G_λ as the set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm*

$$\|f\|_{\mathcal{M}_{p,\omega,\lambda}} = \sup_{\substack{x\in\mathbb{R}_+ \\ 0<arcshr<2}} \omega(x,r)^{-1} r^{-\frac{2\lambda+1}{p}} \|f\|_{L_{p,\lambda}(0,arcshr)} + + \sup_{\substack{x\in\mathbb{R}_+ \\ 2\le arcshr<\infty}} \omega(x,r)^{-1} r^{-\frac{4\lambda}{p}} \|f\|_{L_{p,\lambda}(0,arcshr)},$$

where

$$\|f\|_{L_{p,\lambda}(0,arcshr)} = \left(\int_0^{arcshr} A^\lambda_{cht} |f(chx)|^p sh^{2\lambda} t dt\right)^{1/p}.$$

Definition 2.2 *Let $1 \le p < \infty$. We denote by $WL_{p,\omega,\lambda}(\mathbb{R}_+)$ the weak space $L_{p,\omega,\lambda}(\mathbb{R}_+)$ defined as the if locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$, with the finite norm*

$$\|f\|_{W\mathcal{M}_{p,\omega,\lambda}} = \sup_{\substack{x\in\mathbb{R}_+ \\ 0<arcshr<2}} \omega(x,r)^{-1} r^{-\frac{2\lambda+1}{p}} \|f\|_{WL_{p,\lambda}(0,arcshr)} + + \sup_{\substack{x\in\mathbb{R}_+ \\ 2\le arcshr<\infty}} \omega(x,r)^{-1} r^{-\frac{r\lambda}{p}} \|f\|_{WL_{p,\lambda}(0,arcshr)},$$

where

$$\|f\|_{WL_{p,\lambda}(0,arcshr)} = \sup_{t>0} t \left\{ y \in (0,arcshr) : A^\lambda_{cht} |f(chx)| > t \right\}_\lambda^{1/p} = \sup_{t>0} \left(\int_{\{y\in(0,arcshr): A^\lambda_{chy}|f(chx)|>t\}} sh^{2\lambda} y dy \right)^{1/p}.$$

3 $G-$maximal operator in the spaces $\mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$

The following local Guliyev type estimate for the $G-$maximal function (see [11]) is valid.

Theorem 3.1 *Let $1 \le p < \infty$ and $f \in L^{loc}_{p,\lambda}(\mathbb{R}_+)$.*

(i) Then for $p > 1$

$$\|M_G f\|_{L_{p,\lambda}(0,arcshr)} \lesssim \begin{cases} t^{\frac{2\lambda+1}{p}} \int_{arcsht}^\infty r^{-\frac{2\lambda+1}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshr)} dr, & 0 < arcsht < 2, \\ t^{\frac{4\lambda}{p}} \int_{arcsht}^\infty r^{-\frac{4\lambda}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshr)} dr, & 2 \le arcsht < \infty \end{cases} \tag{3.1}$$

and for $p = 1$

$$\|M_G f\|_{WL_{1,\lambda}(0,arcshr)} \lesssim \begin{cases} t^{2\lambda+1} \int_{arcsht}^\infty r^{-2\lambda-2} \|f\|_{L_{1,\lambda}(0,arcshr)} dr, & 0 < arcsht < 2, \\ t^{4\lambda} \int_{arcsht}^\infty r^{-4\lambda-1} \|f\|_{L_{1,\lambda}(0,arcshr)} dr, & 2 \le arcsht < \infty \end{cases} \tag{3.2}$$

Proof. Let $1 < p < \infty$. We represent f as $f_1 + f_2$,

$$f_1(chy) = f(chy)\chi_{(0,arcsht)}(chy), \quad f_2(chy) = f(chy)\chi_{(arcsht,\infty)}(chy) \tag{3.3}$$

and we have

$$\|M_G f\|_{L_{p,\lambda}(0,arcsht)} \le \|M_G f_1\|_{L_{p,\lambda}(0,arcsht)} + \|M_G f_2\|_{L_{p,\lambda}(arcsht,\infty)} \tag{3.4}$$

Further by Theorem 1.1 we obtain

$$\|M_G f_1\|_{L_{p,\lambda}(0,arcsht)} \le \|M_G f_1\|_{L_{p,\lambda}(\mathbb{R}_+)} \lesssim \|f_1\|_{L_{p,\lambda}(\mathbb{R}_+)} \lesssim \|f\|_{L_{p,\lambda}(0,arcsht)} \tag{3.5}$$

from (3.4) we have

$$\begin{aligned}\|M_G f_1\|_{L_{p,\lambda}(0,arcsht)} &\lesssim t^{\frac{2\lambda+1}{p}} \int_t^\infty r^{-\frac{2\lambda+1}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr \\ &\lesssim t^{\frac{2\lambda+1}{p}} \int_{arcsht}^\infty r^{-\frac{2\lambda+1}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr \end{aligned} \tag{3.6}$$

since norm $\|f\|_{L_{p,\lambda}(0,arcshr)}$ is noncreasing by r.

Now we estimate $M_G f_2$. For any $u \in (0,r)$ we get

$$M_G f_2(chu) = \sup_{r>0} \frac{1}{|H(u,r)|_\lambda} \int\limits_{H(u,r)} A^\lambda_{chy}|f_2(chu)| sh^{2\lambda} y dy.$$

Using Lemma 1.2 by $\gamma = 2\lambda$ we have

$$|H(u,r)|_\lambda \approx \begin{cases} \left(sh\frac{u+r}{2}\right)^{2\lambda+1}, & 0 < u+r < 2, \\ \left(sh\frac{u+r}{2}\right)^{4\lambda}, & 2 \le u+r < \infty. \end{cases}$$

$$M_G f_2(chu) \lesssim \begin{cases} \sup\limits_{0<r<2} \left(sh\frac{r}{2}\right)^{-2\lambda-1} \int_{H(u,r)\cap(arcsht,\infty)} A^\lambda_{chy}|f(chu)| sh^{2\lambda} y dy \\ \sup\limits_{2\le r<\infty} \left(sh\frac{r}{2}\right)^{-4\lambda} \int_{H(u,r)\cap(arcsht,\infty)} A^\lambda_{chy}|f(chu)| sh^{2\lambda} y dy \end{cases}$$

Then

$$\begin{aligned} M_G f_2(chu) &\lesssim \begin{cases} \sup\limits_{0<t<r<2} t^{-2\lambda-1} \int_{arcsht}^{u+r} A^\lambda_{chy}|f(chu)| sh^{2\lambda} y dy \\ \sup\limits_{2\le t<r<\infty} t^{-4\lambda} \int_{arcsht}^{u+r} A^\lambda_{chy}|f(chu)| sh^{2\lambda} y dy \end{cases} \\ &\lesssim \begin{cases} \int_{arcsht}^\infty (shy)^{-2\lambda-1} A^\lambda_{chy}|f(chu)| sh^{2\lambda} y dy, & 0 < t < 2, \\ \int_{arcsht}^\infty (shy)^{-4\lambda} A^\lambda_{chy}|f(chu)| sh^{2\lambda} y dy, & 2 \le t < \infty. \end{cases} \end{aligned} \tag{3.7}$$

We choose $\beta > 2\lambda + 1$ and estimate first integral in (3.7)

$$\begin{aligned} &\int_{arcsht}^\infty (shy)^{-2\lambda-1} A^\lambda_{chy}|f(chu)| sh^{2\lambda} y dy \\ &= \beta \int_{arcsht}^\infty (shy)^{\beta-2\lambda-1} A^\lambda_{chy}|f(chu)| \left(\int_{shy}^\infty s^{\beta-1} ds\right) sh^{2\lambda} y dy \\ &= \beta \int_{shy}^\infty s^{\beta-1} \left(\int_{\{u\in\mathbb{R}_+ : arcsht \le y \le arcshs\}} (shy)^{\beta-2\lambda-1} A^\lambda_{chy}|f(chu)| sh^{2\lambda} y dy\right) ds \\ &\lesssim \int_{arcshy}^\infty s^{\beta-1} \|f\|_{L_{p,\lambda}(0,arcshs)} \|shy\|_{L_{p',\lambda}(0,arcshs)}\, ds, \end{aligned} \tag{3.8}$$

here $p+p'=pp'$.

But

$$\|sh(\cdot)\|_{L_{p',\lambda}(0,arcshs)} = \left(\int_0^{arcshs} (shy)^{(\beta-2\lambda-1)p'} sh^{2\lambda} ydy\right)^{1/p'}$$
$$\lesssim s^{\beta-2\lambda-1+\frac{2\lambda+1}{p'}} = s^{\beta-2\lambda-1+(2\lambda+1)\left(1-\frac{2}{p}\right)} = s^{\beta-\frac{2\lambda+1}{p}}.$$

Then from (3.8) we obtain

$$\int_{arcsht}^{\infty} (sh\frac{y}{2})^{-2\lambda-1} A_{chy}^{\lambda} |f(chu)| sh^{2\lambda} ydy \lesssim \int\limits_{arcsht}^{\infty} s^{-\frac{2\lambda+1}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshs)} ds.$$

Analogous estimate second integral in (3.7) and we have

$$M_G f_2(chu) \lesssim \begin{cases} \int_{arcsht}^{\infty} s^{-\frac{2\lambda+1}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshs)} ds, & 0 < arcsht < 2, \\ \int_{arcsht}^{\infty} s^{-\frac{4\lambda}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshs)} ds, & 2 \le arcsht < \infty. \end{cases} \tag{3.9}$$

Applying of Theorem 1.1 by $g(chx) \equiv 1$ and (3.9) we get

$$\|M_G f_2\|_{L_{p,\lambda}(0,arcshs)}$$
$$\lesssim \begin{cases} \int_{arcsht}^{\infty} s^{-\frac{2\lambda+1}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshs)} ds \cdot \|1\|_{L_{p,\lambda}(0,arcsht)}, & 0 < arcsht < 2, \\ \int_{arcsht}^{\infty} s^{-\frac{4\lambda}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshs)} ds \cdot \|1\|_{L_{p,\lambda}(0,arcsht)}, & 2 \le arcsht < \infty. \end{cases} \tag{3.10}$$

But by Lemma 1.1

$$\|1\|_{L_{p,\lambda}(0,arcsht)} \lesssim \begin{cases} sh(arcsht)^{\frac{2\lambda+1}{p}} = t^{\frac{2\lambda+1}{p}}, & 0 < arcsht < 2, \\ sh(arcsht)^{\frac{4\lambda}{p}} = t^{\frac{4\lambda}{p}}, & 2 \le arcsht < \infty. \end{cases} \tag{3.11}$$

Taking into account (3.11) in (3.10) and also (3.6) in (3.4) we obtain (3.1),

Let $p=1$. It is obvious that for any interval

$$\|M_G f\|_{WL_{1,\lambda}(0,arcsht)} \le \|M_G f_1\|_{WL_{1,\lambda}(0,arcsht)} + \|M_G f_2\|_{WL_{1,\lambda}(0,arcsht)}.$$

By boundedness of the operator M_G from $L_{1,\lambda}(\mathbb{R}_+)$ to $WL_{1,\lambda}(\mathbb{R}_+)$ (see [18] Theorem 2.2) we have

$$\|M_G f_1\|_{WL_{1,\lambda}(0,arcsht)} \lesssim \|f\|_{L_{1,\lambda}(0,arcsht)}.$$

Note that inequality (3.10) also true in the case $p=1$. Then by (3.10) and (3.11), we get inequality (3.2).

Therefore we get the following Spanne-Guliyev type theorem for the $G-$maximal operator in generalalzied $G-$Morrey spaces (see [11]).

Theorem 3.2 *Let $1 \le p < \infty$ and the function $\omega_1(x,r)$ and $\omega_2(x,r)$ satisfy the condition*

$$\int_{arcsht}^{\infty} \omega_1(x,r) \frac{dr}{r} \lesssim \omega_2(x,r). \tag{3.12}$$

Then for $p>1$ the maximal operator M_G is bounded from $M_{p,\omega_1,\lambda}(\mathbb{R}_+)$ to $M_{p,\omega_2,\lambda}(\mathbb{R}_+)$ and for $p=1$ M_G is bounded from $M_{1,\omega_1,\lambda}(\mathbb{R}_+)$ to $WM_{1,\omega_2,\lambda}(\mathbb{R}_+)$.

Proof. Let $1 < p < \infty$ and $f \in \mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$. By Theorem 3.1 we have which completes the proof for $1 < p < \infty$.

Let $p = 1$ and $f \in \mathcal{M}_{1,\omega_1,\lambda}(\mathbb{R}_+)$. By Theorem 3.1 we have

$$\begin{aligned}\|M_G f\|_{W\mathcal{M}_{1,\omega_2,\lambda}} &= \sup_{\substack{x\in\mathbb{R}_+\\ 0<arcsht<2}} \omega_2(x,t)^{-1} t^{-2\lambda-1} \|M_G f\|_{WL_{1,\lambda}(0,arcsht)} \\ &+ \sup_{\substack{x\in\mathbb{R}_+\\ 2\le arcsht<\infty}} \omega_2(x,t)^{-1} t^{-4\lambda} \|M_G f\|_{WL_{1,\lambda}(0,arcsht)} \\ &\lesssim \sup_{\substack{x\in\mathbb{R}_+\\ 0<arcsht<2}} \omega_2(x,t)^{-1} \int_{arcsht}^{\infty} r^{-2\lambda-2} \|f\|_{L_{1,\lambda}(0,arcsht)}\, dr \\ &+ \sup_{\substack{x\in\mathbb{R}_+\\ 2\le arcsht<\infty}} \omega_2(x,t)^{-1} \int_{arcsht}^{\infty} r^{-4\lambda-1} \|f\|_{L_{1,\lambda}(0,arcsht)}\, dr.\end{aligned}$$

Hence

$$\begin{aligned}\|M_G f\|_{W\mathcal{M}_{1,\omega_2,\lambda}} &\lesssim \|f\|_{\mathcal{M}_{1,\omega_1,\lambda}} \Bigg(\sup_{\substack{x\in\mathbb{R}_+\\ 0<arcsht<2}} \omega_2(x,t)^{-1} \int_{arcsht}^{\infty} \omega_1(x,r) \frac{dr}{r} \\ &+ \sup_{\substack{x\in\mathbb{R}_+\\ 2\le arcsht<\infty}} \omega_2(x,t)^{-1} \int_{arcsht}^{\infty} \omega_1(x,r) \frac{dr}{r} \Bigg) \\ &\lesssim \|f\|_{\mathcal{M}_{1,\omega_1,\lambda}}\end{aligned}$$

by (4.12), which completes the proof for $p = 1$.

4 $G-$Riesz potential in the spaces $\mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$

In this section, we shall give the Spanne-Guliyev and Adams-Guliyev type boundedness of the $G-$Riesz potential operator on the generalized $G-$Morrey spaces $\mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$, including weak versions.

41 Spanne-Guliyev type result

The following local Guliyev type estimate for the $G-$Riesz potential (see [11]) is valid.

Theorem 4.1 *Let $1 \le p < \infty$, $0 < \alpha < \frac{2\lambda+1}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$ and $f \in L^{loc}_{p,\lambda}(\mathbb{R}_+)$. Then for $p > 1$ and $t > 0$*

$$\|I_G^\alpha f\|_{L_{q,\lambda}(0,arcsht)} \lesssim \begin{cases} t^{\frac{2\lambda+1}{q}} \int_{arcsht}^{\infty} r^{-\frac{2\lambda+1}{q}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr, & 0 < arcsht < 2, \\ t^{\frac{4\lambda}{q}} \int_{arcsht}^{\infty} r^{-\frac{4\lambda}{q}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr, & 2 \le arcsht < \infty. \end{cases} \tag{4.1}$$

and for $p = 1$

$$\|I_G^\alpha f\|_{WL(0,arcsht)} \lesssim \begin{cases} t^{\frac{2\lambda+1}{q}} \int_{arcsht}^\infty r^{-\frac{2\lambda+1}{q}-1} \|f\|_{L_{1,\lambda}(0,arcshr)}\, dr, & 0 < arcsht < 2, \\ t^{\frac{4\lambda}{q}} \int_{arcsht}^\infty r^{-\frac{4\lambda}{q}-1} \|f\|_{L_{1,\lambda}(0,arcshr)}\, dr, & 2 \le arcsht < \infty. \end{cases} \tag{4.2}$$

Proof. As in the proof of Theorem 3.1, we represent f in form (3.3) and have

$$I_G^\alpha f(chx) = I_G^\alpha f_1(chx) + I_G^\alpha f_2(chx). \tag{4.3}$$

Let $1 < p < \infty, 0 < \alpha < \frac{2\lambda+1}{p}, \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$. By boundedness of the operator I_G^α from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$ (see [12]) we obtain

$$\|I_G^\alpha f_1\|_{L_{q,\lambda}(0,arcsht)} \le \|I_G^\alpha f_1\|_{L_{q,\lambda}(\mathbb{R}_+)} \lesssim \|f_1\|_{L_{p,\lambda}(\mathbb{R}_+)} \lesssim \|f\|_{L_{p,\lambda}(0,arcsht)} \tag{4.4}$$

Taking into account that

$$\|f\|_{L_{p,\lambda}(0,arcsht)} \lesssim \begin{cases} t^{\frac{2\lambda+1}{q}} \int_{arcsht}^\infty r^{-\frac{2\lambda+1}{q}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr, & 0 < arcsht < 2, \\ t^{\frac{4\lambda}{q}} \int_{arcsht}^\infty r^{-\frac{4\lambda}{q}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr, & 2 \le arcsht < \infty. \end{cases}$$

we get

$$\|I_G^\alpha f_1\|_{L_{q,\lambda}(0,arcsht)} \lesssim \begin{cases} t^{\frac{2\lambda+1}{q}} \int_{arcsht}^\infty r^{-\frac{2\lambda+1}{q}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr, & 0 < arcsht < 2, \\ t^{\frac{4\lambda}{q}} \int_{arcsht}^\infty r^{-\frac{4\lambda}{q}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr, & 2 \le arcsht < \infty. \end{cases} \tag{4.5}$$

Further (see [18], Corollary 3.1)

$$\begin{aligned} \|I_G^\alpha f_2\|_{L_{q,\lambda}(0,arcsht)} &\lesssim \left\| \int_{arcsht}^\infty (shy)^{\alpha-2\lambda-1} A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \right\|_{L_{q,\lambda}(0,arcsht)} \\ &\lesssim \int_{arcsht}^\infty (shy)^{\alpha-2\lambda-1} A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \left\| \chi_{(0,arcsht)}(\cdot) \right\|_{L_{q,\lambda}(\mathbb{R}_+)} \end{aligned} \tag{4.6}$$

we choose $\beta > 2\lambda + 1$ and obtain

$$\begin{aligned} &\int_{arcsht}^\infty (shy)^{\alpha-2\lambda-1} A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \\ &= \beta \int_{arcsht}^\infty (shy)^{\beta+\alpha-2\lambda-1} A_{chy}^\lambda |f(chx)| \left(\int_{arcsht}^\infty s^{-\beta-1} ds \right) sh^{2\lambda} y dy \\ &= \beta \int_{arcsht}^\infty \left(\int_{\{x \in R_+ : arcsht \le y \le arcshs\}} (shy)^{\beta+\alpha-2\lambda-1} A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \right) s^{-\beta-1} ds \\ &\lesssim \int_{arcsht}^\infty s^{-\beta-1} \left\| A_{chy}^\lambda f \right\|_{L_{p,\lambda}(0,arcshs)} \cdot \left\| (shy)^{\beta+\alpha-2\lambda-1} \right\|_{L_{p',\lambda}(0,arcshs)} ds \\ &\lesssim \int_{arcsht}^\infty s^{-\beta-1} s^{\beta+\alpha-\frac{2\lambda+1}{p}} \|f\|_{L_{p,\lambda}(0,arcshs)} ds \\ &\lesssim \int_{arcsht}^\infty s^{\alpha-\frac{2\lambda+1}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshs)} ds. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) we obtain

$$\|I_G^\alpha f_2\|_{L_{q,\lambda}(0,arcsht)} \tag{4.8}$$

$$\lesssim \int_{arcsht}^{\infty} s^{\alpha-\frac{2\lambda+1}{p}-1}\|f\|_{L_{p,\lambda}(0,arcshs)}\,ds\cdot\left\|\chi_{(0,arcsht)}(\cdot)\right\|_{L_{q,\lambda}(\mathbb{R}_+)},\ 0<arcsht<2. \tag{4.9}$$

Analogous we have

$$\|I_G^\alpha f_2\|_{L_{q,\lambda}(0,arcsht)} \tag{4.10}$$

$$\lesssim \int_{arcsht}^{\infty} s^{\alpha-\frac{4\lambda}{p}-1}\|f\|_{L_{p,\lambda}(0,arcshs)}\,ds\cdot\left\|\chi_{(0,arcsht)}(\cdot)\right\|_{L_{q,\lambda}(\mathbb{R}_+)},\ 2\le arcsht<\infty. \tag{4.11}$$

Since

$$\left\|\chi_{(0,arcsht)}(\cdot)\right\|_{L_{q,\lambda}(0,arcsht)} \approx \begin{cases}(sharcsht)^{\frac{2\lambda+1}{q}}, & 0<arcsht<2,\\ (sharcsht)^{\frac{4\lambda}{q}}, & 2\le arcsht<\infty.\end{cases}$$
$$= \begin{cases}t^{\frac{2\lambda+1}{q}}, & 0<arcsht<2,\\ t^{\frac{4\lambda}{q}}, & 2\le arcsht<\infty.\end{cases}$$

that from (4.8) and (4.10) we obtain

$$\|I_G^\alpha f_2\|_{L_{q,\lambda}(0,arcsht)} \lesssim \begin{cases}t^{\frac{2\lambda+1}{q}}\int_{arcsht}^{\infty}s^{-\frac{2\lambda+1}{q}-1}\|f\|_{L_{p,\lambda}(0,arcsht)}\,ds, & 0<arcsht<2,\\ t^{\frac{4\lambda}{q}}\int_{arcsht}^{\infty}s^{-\frac{4\lambda}{q}-1}\|f\|_{L_{p,\lambda}(0,arcsht)}\,ds, & 2\le arcsht<\infty.\end{cases} \tag{4.12}$$

Combining (4.5) and (4.12) we obtain (4.1).

Let $p=1$. It is obvious that for any interval $(0,arcsht)$

$$\|I_G^\alpha f\|_{WL_{1,\lambda}(0,arcsht)} \le \|I_G^\alpha f_1\|_{WL_{1,\lambda}(0,arcsht)} + \|I_G^\alpha f_2\|_{WL_{1,\lambda}(0,arcsht)} \tag{4.13}$$

By boundedness of the operator I_G^α from $L_{1,\lambda}(R_+)$ to $WL_{q,\lambda}(\mathbb{R}_+)$ (see [12]) we have

$$\|I_G^\alpha f_1\|_{WL_{1,\lambda}(0,arcsht)} \lesssim \|f\|_{L_{q,\lambda}(0,arcsht)}. \tag{4.14}$$

Note that inequality (4.12) also true in the case $p=1$. Then by (4.12), we get inequality (4.2).

The following Spanne-Guliyev type theorem for the $G-$Riesz potential in generalalzied $G-$Morrey spaces (see [11]) is valid.

Theorem 4.2 *Let $1\le p<\infty, 0<\alpha<\frac{2\lambda+1}{p}, \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$ and the function $\omega_1(x,r)$ and $\omega_2(x,r)$ fulfill the condition*

$$\int_{arcsht}^{\infty} t^\alpha \omega_1(x,r)\frac{dt}{t} \lesssim \omega_2(x,r). \tag{4.15}$$

Then for $p>1$ the operators M_G^α and I_G^α are bounded from $\mathcal{M}_{p,\omega_1,\lambda}(\mathbb{R}_+)$ to $\mathcal{M}_{p,\omega_2,\lambda}(\mathbb{R}_+)$ and for $p=1$ M_G^α and I_G^α are bounded from $\mathcal{M}_{1,\omega_1,\lambda}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega_2,\lambda}(\mathbb{R}_+)$.

Proof. Since (see [19] proof of Theorem 4.4) $M_G^{\alpha} f(chx) \lesssim I_G^{\alpha}(|f|(chx))$, it suffices to tread only the case of the operator I_G^{α}. Let $1 < p < \infty$ and $f \in \mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$. By Theorem 4.1 we have

$$\begin{aligned}
\|I_G^{\alpha} f\|_{\mathcal{M}_{p,\omega_1,\lambda}} &\lesssim \sup_{\substack{x\in\mathbb{R}_+ \\ 0<arcsht<2}} \omega_2(x,t)^{-1} \int_{arcsht}^{\infty} r^{-\frac{2\lambda+1}{q}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr \\
&+ \sup_{\substack{x\in\mathbb{R}_+ \\ 2\le arcsht<\infty}} \omega_2(x,t)^{-1} \int_{arcsht}^{\infty} r^{-\frac{4\lambda}{q}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr \\
&\lesssim \|f\|_{\mathcal{M}_{p,\omega_1,\lambda}} \left(\sup_{\substack{x\in\mathbb{R}_+ \\ arcsht>0}} \omega_2(x,t)^{-1} \int_{arcsht}^{\infty} r^{\alpha} \omega_1(x,t) \frac{dr}{r} \right) \\
&\lesssim \|f\|_{\mathcal{M}_{p,\omega_1,\lambda}}, \qquad (4.16)
\end{aligned}$$

by (5.13).

Let $p = 1$ and $f \in \mathcal{M}_{1,\omega_1,\lambda}(\mathbb{R}_+)$. By Theorem 4.1 we obtain

$$\begin{aligned}
\|I_G^{\alpha} f\|_{W\mathcal{M}_{q,\omega_2,\lambda}} &= \sup_{\substack{x\in\mathbb{R}_+ \\ 0<arcsht<2}} \frac{t^{-\frac{2\lambda+1}{q}}}{\omega_2(x,t)} \|I_G^{\alpha} f\|_{WL_{q,\lambda}(0,arcsht)} \\
&+ \sup_{\substack{x\in\mathbb{R}_+ \\ 2<arcsht<\infty}} \frac{t^{-\frac{4\lambda}{q}}}{\omega_2(x,t)} \|I_G^{\alpha} f\|_{WL_{q,\lambda}(0,arcsht)} \\
&\lesssim \sup_{\substack{x\in\mathbb{R}_+ \\ 0<arcsht<2}} \frac{1}{\omega_2(x,t)} \int_{arcsht}^{\infty} r^{-\frac{2\lambda+1}{q}-1} \|f\|_{L_{1,\lambda}(0,arcshr)}\, dr \\
&+ \sup_{\substack{x\in\mathbb{R}_+ \\ 2<arcsht<\infty}} \frac{1}{\omega_2(x,t)} \int_{arcsht}^{\infty} r^{-\frac{4\lambda}{q}-1} \|f\|_{L_{1,\lambda}(0,arcshr)}\, dr \\
&\lesssim \|f\|_{\mathcal{M}_{1,\omega_1,\lambda}} \sup_{\substack{x\in\mathbb{R}_+ \\ arcsht>0}} \frac{1}{\omega_2(x,t)} \int_{arcsht}^{\infty} r^{\alpha} \omega_1(x,t) \frac{dr}{r} \\
&\lesssim \|f\|_{\mathcal{M}_{1,\omega_1,\lambda}}
\end{aligned}$$

by (4.15).

From this and (4.16) we obtain the assertion of the Theorem 4.2.

42 Adams-Guliyev type result

The following pointwise Guliyev type estimate (see [11]) plays a key role where we prove our main results.

Theorem 4.3 *Let $1 \le p < \infty, 0 < \alpha < \frac{2\lambda+1}{p}$ and $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$. Then*

$$|I_G^{\alpha} f(chx)| \lesssim t^{\alpha} M_G f(chx) + \begin{cases} \int_{arcsht}^{\infty} r^{\alpha-\frac{2\lambda+1}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr, & 0 < arcsht < 2, \\ \int_{arcsht}^{\infty} r^{\alpha-\frac{4\lambda}{p}-1} \|f\|_{L_{p,\lambda}(0,arcshr)}\, dr, & 2 \le arcsht < \infty. \end{cases} \qquad (4.17)$$

Proof. As in the proof of Theorem 3.1, we represent f in form (3.3) and have

$$I_G^\alpha f(chx) = I_G^\alpha f_1(chx) + I_G^\alpha f_2(chx). \tag{4.18}$$

For $I_G^\alpha f_1(chx)$ we have (see proof of the Corollary 3.1 in [18])

$$|I_G^\alpha f_1(chx)| \lesssim \int_0^{arcsht} \frac{A_{chy}^\lambda |f(chx)| sh^{2\lambda} y}{(chy)^{2\lambda+1-\alpha}} dy. \tag{4.19}$$

Let $0 < arcsht < 2$. Then from (4.19) we obtain

$$\begin{aligned} |I_G^\alpha f_1(chx)| &\lesssim \sum_{\nu=0}^\infty \int_{2^{-(\nu+1)} arcsht}^{2^{-\nu} arcsht} \frac{A_{chy}^\lambda |f(chx)| sh^{2\lambda} y}{(shy)^{2\lambda+1-\alpha}} dy \\ &\lesssim \sum_{\nu=0}^\infty \left(sh\frac{arcsht}{2^{\nu+1}} \right)^\alpha \left(sh\frac{arcsht}{2^{\nu+1}} \right)^{-2\lambda-1} \int_0^{2^{-\nu} arcsht} A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \\ &\lesssim (sharcsht)^\alpha \sum_{\nu=0}^\infty 2^{-(\nu+1)\alpha} \left(sh\frac{arcsht}{2^{\nu+1}} \right)^{-2\lambda-1} \int_0^{2^{-\nu} arcsht} A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \\ &\lesssim t^\alpha M_G f(chx) \left(\sum_{\nu=0}^\infty 2^{-(\nu+1)\alpha} \right) \lesssim t^\alpha M_G f(chx). \end{aligned} \tag{4.20}$$

Let $2 \le arcsht < \infty$ and $0 < \alpha < 4\lambda$. Then

$$\begin{aligned} |I_G^\alpha f_1(chx)| &\lesssim \int_0^{arcsht} \frac{A_{chy}^\lambda |f(chx)| sh^{2\lambda} y}{(chy)^{2\lambda+1-\alpha}} dy \\ &\lesssim \int_0^{arcsht} \frac{A_{chy}^\lambda |f(chx)| sh^{2\lambda} y}{(chy)^{4\lambda-\alpha}} dy \lesssim \int_0^{arcsht} \frac{A_{chy}^\lambda |f(chx)| sh^{2\lambda} y}{(shy)^{4\lambda-\alpha}} dy \\ &\lesssim \sum_{\nu=0}^\infty \int_{2^{-(\nu+1)} arcsht}^{2^{-\nu} arcsht} \frac{A_{chy}^\lambda |f(chx)| sh^{2\lambda} y}{(shy)^{4\lambda-\alpha}} dy \\ &\lesssim \sum_{\nu=0}^\infty \left(sh\frac{arcsht}{2^{\nu+1}} \right)^\alpha \left(sh\frac{arcsht}{2^{\nu+1}} \right)^{-4\lambda} \int_0^{2^{-\nu} arcsht} A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \\ &\lesssim t^\alpha M_G f(chx). \end{aligned} \tag{4.21}$$

Now let $4\lambda \le \alpha < 2\lambda + 1$. From (4.19) we have

$$\begin{aligned} |I_G^\alpha f(chx)| &\lesssim \int_0^{arcsht} A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \\ &= \frac{\left(sh\frac{arcsht}{2}\right)^{4\lambda}}{\left(sh\frac{arcsht}{2}\right)^{4\lambda}} \int_0^{arcsht} A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \\ &\lesssim sh(arcsht)^{4\lambda} M_G f(chx) \lesssim t^{4\lambda} M_G f(chx) \lesssim t^\alpha M_G f(chx). \end{aligned} \tag{4.22}$$

Taking into account (4.20)-(4.22) in (4.19) we get

$$|I_G^\alpha f_1(chx)| \lesssim t^\alpha M_G f(chx), \quad t > 0, \quad 0 < \alpha < 2\lambda + 1. \tag{4.23}$$

For $I_G^{\alpha} f_2(chx)$ we have

$$
\begin{aligned}
|I_G^{\alpha} f_2(chx)| &\lesssim \int_t^{\infty} A_{chy}^{\lambda} |f(chx)| \left(\int_{shy}^{\infty} s^{\alpha-2\lambda-2} ds \right) sh^{2\lambda} y dy \\
&\lesssim \int_{arcsht}^{\infty} \left(\int_{\{x\in R_+ : arcsht \le y \le arcshs\}} A_{chy}^{\lambda} |f(chx)| sh^{2\lambda} y dy \right) s^{\alpha-2\lambda-2} ds \\
&\lesssim \int_{arcsht}^{\infty} s^{\alpha-2\lambda-2} \|f\|_{L_{p,\lambda}(0,arcshs)} \cdot \|1\|_{L_{p,\lambda}(0,arcshs)} ds.
\end{aligned}
$$

Using (3.11) we obtain

$$
|I_G^{\alpha} f_2(chx)| \lesssim \begin{cases} \int_{arcsht}^{\infty} s^{\alpha-\frac{2\lambda+1}{p}-1} \|f\|_{L_{p,\lambda}(0,arcsht)} ds, & 0 < arcsht < 2, \\ \int_{arcsht}^{\infty} s^{\alpha-\frac{4\lambda}{p}-1} \|f\|_{L_{p,\lambda}(0,arcsht)} ds, & 2 \le arcsht < \infty. \end{cases} \tag{4.24}
$$

Finally from (4.23), (4.24) and (4.18) we obtain (4.17).

The following Adams-Guliyev type theorem for the $G-$Riesz potential in generalalzied $G-$Morrey spaces (see [11]) is valid.

Theorem 4.4 *Let $1 \le p < \infty, 0 < \alpha < \frac{2\lambda+1}{p}$ and let $\omega(x,t)$ satisfy condition (5.13) and the condition*

$$
t^{\alpha}\omega(x,t) + \int_{arcsht}^{\infty} r^{\alpha}\omega(x,r)\frac{dr}{r} \lesssim \omega(x,r)^{\frac{p}{q}}. \tag{4.25}
$$

where $p \le q$. Suppose also that for almost every $x \in \mathbb{R}_+$, the function $\omega(x,r)$ fulfills the condition

$$
\text{there exist } a = a(x) > 0 \text{ such that } \omega(x,\cdot) : [0,\infty] \to [\alpha,\infty) \text{ is surjective.} \tag{4.26}
$$

Then for $p > 1$ the operators M_G^{α} and I_G^{α} are bounded from $\mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega,\lambda}(\mathbb{R}_+)$ and for $p = 1$ the operators M_G^{α} and I_G^{α} are bounded from $\mathcal{M}_{1,\omega,\lambda}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\omega^{\frac{1}{q}},\lambda}(\mathbb{R}_+)$.

Proof. Since (see [19] proof of Theorem 4.4) $M_G^{\alpha} f(chx) \lesssim I_G^{\alpha}(|f|(chx))$, it suffices to tread only the case of the operator I_G^{α}.

Let $1 \le p < \infty$ and $f \in \mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$. By Theorem 4.3 we have

$$
|I_G^{\alpha} f(chx)| \lesssim r^{\alpha} M_G f(chx) + \|f\|_{\mathcal{M}_{p,\omega,\lambda}} \int_{arcsht}^{\infty} t^{\alpha}\omega(x,t)\frac{dt}{t}. \tag{4.27}
$$

From (4.25) we get $r^{\alpha}\omega(x,r) \lesssim \omega(x,r)^{\frac{p}{q}}$.

Making also use of condition (4.25), we obtain

$$
|I_G^{\alpha} f(chx)| \lesssim \omega(x,r)^{\frac{p}{q}-1} M_G f(chx) + \omega(x,r)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\omega,\lambda}}. \tag{4.28}
$$

Since $\omega(x,r)$ is surjective, we can choose $r > 0$ so that $\omega(x,r) = M_G f(chx) \|f\|_{\mathcal{M}_{p,\omega,\lambda}}^{-1}$, assuming that f is not identical 0.

Hence, for every $x \in \mathbb{R}_+$, we have

$$
|I_G^{\alpha} f(chx)| \lesssim (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\omega,\lambda}}^{1-\frac{p}{q}}. \tag{4.29}
$$

Hence the statement of the theorem follows in wive of the boundedness of the maximal operator M_G in $\mathcal{M}_{p,\omega,\lambda}(\mathbb{R}_+)$ provided by Theorem 3.2 in virtue of condition (3.12)

$$
\begin{aligned}
\|I_G^\alpha f\|_{M_{q,\omega^{\frac{p}{q}},\lambda}} &= \sup_{\substack{x\in\mathbb{R}_+\\ 0<arcsht<2}} \frac{t^{-\frac{2\lambda+1}{q}}}{\omega(x,r)^{\frac{p}{q}}} \|I_G^\alpha f\|_{L_{p,q}(0,arcsht)} \\
&+ \sup_{\substack{x\in\mathbb{R}_+\\ 2\le arcsht<\infty}} \frac{t^{-\frac{4\lambda}{q}}}{\omega(x,r)^{\frac{p}{q}}} \|I_G^\alpha f\|_{L_{p,q}(0,arcsht)} \\
&\lesssim \|f\|_{\mathcal{M}_{p,\omega,\lambda}}^{1-\frac{p}{q}} \left(\sup_{\substack{x\in\mathbb{R}_+\\ 0<arcsht<2}} \frac{t^{-\frac{2\lambda+1}{q}}}{\omega(x,r)^{\frac{p}{q}}} \|M_G f\|_{L_{p,\lambda}(0,arcsht)}^{\frac{p}{q}} + \right. \\
&\left. + \sup_{\substack{x\in\mathbb{R}_+\\ 2<arcsht<\infty}} \frac{t^{-\frac{4\lambda}{q}}}{\omega(x,r)^{\frac{p}{q}}} \|M_G f\|_{L_{p,\lambda}(0,arcsht)}^{\frac{p}{q}} \right) \\
&\lesssim \|f\|_{\mathcal{M}_{p,\omega,\lambda}} .
\end{aligned} \tag{4.30}
$$

of $1 < p < q < \infty$ and

$$
\begin{aligned}
\|I_G^\alpha f\|_{W\mathcal{M}_{q,\omega,\lambda}} &= \sup_{\substack{x\in\mathbb{R}_+\\ 0<arcsht<2}} \omega(x,r)^{-\frac{1}{q}} t^{-\frac{2\lambda+1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda}(0,arcsht)} \\
&+ \sup_{\substack{x\in\mathbb{R}_+\\ 2\le arcsht<\infty}} \omega(x,r)^{-\frac{1}{q}} t^{-\frac{4\lambda}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda}(0,arcsht)}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|f\|_{\mathcal{M}_{1,\omega,\lambda}}^{1-\frac{1}{q}} \left(\sup_{\substack{x\in\mathbb{R}_+\\ 0<arcsht<2}} \omega(x,r)^{-\frac{1}{q}} t^{-\frac{2\lambda+1}{q}} \|M_G f\|_{WL_{1,\lambda}(0,arcsht)}^{\frac{1}{q}} \right. \\
&\left. + \sup_{\substack{x\in\mathbb{R}_+\\ 2\le arcsht<\infty}} \omega(x,r)^{-\frac{1}{q}} t^{-\frac{4\lambda}{q}} \|M_G f\|_{WL_{1,\lambda}(0,arcsht)}^{\frac{1}{q}} \right) \\
&\lesssim \|f\|_{\mathcal{M}_{1,\omega,\lambda}} ,
\end{aligned}
$$

if $p = 1 < q < \infty$.

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