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On embeddings into the Morrey and modified Morrey spaces in the Dunkl setting

Fatma A. Muslumova ⋆

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Abstract. *On the $\mathbb{R}^d$ the Dunkl operators $\{D_{k,j}\}_{j=1}^d$ are the differential-difference operators associated with the reflection group $\mathbb{Z}_2^d$ on $\mathbb{R}^d$. We study some embeddings into the Morrey space (D_k-Morrey space) $L_{p,\lambda}(\mu_k)$, $0 \le \lambda < d + 2\gamma_k$ and modified Morrey space (modified D_k-Morrey space) $\widetilde{L}_{p,\lambda}(\mu_k)$ associated with the Dunkl operator on $\mathbb{R}^d$. As applications we get boundedness of the fractional maximal operator $M_{k,\beta}$, $0 \le \beta < d + 2\gamma_k$, associated with the Dunkl operator (fractional D_k-maximal operator) from the spaces $L_{p,\lambda}(\mu_k)$ to $L_\infty(\mathbb{R}^d)$ for $p = \frac{d+2\gamma_k-\lambda}{\beta}$ and from the spaces $\widetilde{L}_{p,\lambda}(\mu_k)$ to $L_\infty(\mathbb{R}^d)$ for $\frac{d+2\gamma_k-\lambda}{\beta} \le p \le \frac{d+2\gamma_k}{\beta}$.*

Keywords. Dunkl operator, generalized translation operator, D_k-Morrey space, modified D_k-Morrey space, fractional D_k-maximal operator

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1 Introduction

Dunkl operators are differential reflection operators associated with finite reflection groups which generalize the usual partial derivatives as well as the invariant differential operators of Riemannian symmetric spaces. They play an important role in harmonic analysis and the study of special functions of several variables. These operators are associated with the differential-difference Dunkl operators on $\mathbb{R}^d$. Rosler in [21] shows that the Dunkl kernel verify a product formula. This allows us to define the Dunkl translations τ_x, $x \in \mathbb{R}^d$.

In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^d)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^d)$ play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [17]). Later, Morrey spaces found important applications to Navier-Stokes ([16,24]) and Schrödinger ([18–20]) equations, elliptic problems with discontinuous coefficients ([3,11]), and potential theory ([1,2,4]). An exposition of the Morrey spaces can be found in the book [13].

In the present work, we study some embeddings into the D_k-Morrey and modified D_k-Morrey spaces. As applications we give boundedness of the fractional D_k-maximal operator in the D_k-Morrey and modified D_k-Morrey spaces.

⋆ Corresponding author

F.A. Muslumova
Nakhchivan Teacher-Training Institute, Nakhchivan, Azerbaijan
E-mail: fmuslumova@gmail.com

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In section 3, we give some embeddings into the D_k-Morrey and modified D_k-Morrey spaces. In section 4, we prove the boundedness of the fractional D_k-maximal operator $M_{k,\beta}$ from the spaces $L_{p,\lambda}(\mu_k)$ to $L_\infty(\mathbb{R}^d)$ for $p = \frac{d+2\gamma_k-\lambda}{\beta}$ and from the spaces $\widetilde{L}_{p,\lambda}(\mu_k)$ to $L_\infty(\mathbb{R}^d)$ for $\frac{d+2\gamma_k-\lambda}{\beta} \le p \le \frac{d+2\gamma_k}{\beta}$.

Finally, we make some conventions on notation. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle \cdot\,,\,\cdot \rangle$ and its associated norm $\|x\| := \sqrt{\langle x, x\rangle}$ for any $x \in \mathbb{R}^d$. For any $v \in \mathbb{R}^d \setminus \{0\}$ let σ_v be the reflection in the hyperplane $H_v \subset \mathbb{R}^d$ orthogonal to v:

$$\sigma_v(x) := x - \Big(\frac{2\langle x, v\rangle}{\|v\|^2}\Big)v, \quad x \in \mathbb{R}^d.$$

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a *root system*, if $\sigma_v R = R$ for all $v \in R$. We assume that it is normalized by $\|v\|^2 = 2$ for all $v \in R$.

The finite group G generated by the reflections $\{\sigma_v\}_{v\in R}$ is called the *reflection group* (or the *Coxeter-Weyl group*) of the root system. Then, we fix a G-invariant function $k : R \to \mathbb{C}$ called the *multiplicity function of the root system* and we consider the family of commuting operators $D_{k,j}$ defined for any $f \in C^1(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$ by

$$D_{k,j}f(x) := \frac{\partial}{\partial x_j}f(x) + \sum_{v\in R_+} k_v \frac{f(x) - f(\sigma_v(x))}{\langle x, v\rangle}\,\langle v, e_j\rangle, \quad 1 \le j \le d,$$

where $C^1(\mathbb{R}^d)$ denotes the set of all functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\left\{\frac{\partial f}{\partial x_j}\right\}_{j=1}^d$ are continuous on $\mathbb{R}^d$, $\{e_i\}_{i=1}^d$ are the standard unit vectors of $\mathbb{R}^d$ and R_+ is a positive subsystem. These operators, defined by Dunkl [9], are independent of the choice of the positive subsystem R_+ and are of fundamental importance in various areas of mathematics and mathematical physics.

Throughout this paper, we assume that $k_v \ge 0$ for all $v \in R$ and we denote by h_k the weight function on $\mathbb{R}^d$ given by

$$h_k(x) := \prod_{v\in R_+} |\langle x, v\rangle|^{k_v}, \quad x \in \mathbb{R}^d.$$

The function h_k is G-invariant and homogeneous of degree γ_k, where $\gamma_k := \sum_{v\in R_+} k_v$.

Closely related to them is the so-called intertwining operator V_κ (the subscript means that the operator depends on the parameters κ_i, except in the rank-one case where the subscript is then a single parameter). The *intertwining operator* V_κ is the unique linear isomorphism of $\oplus_{n\ge 0}P_n$ such that

$$V(P_n) = P_n, V_k(1) = 1, D_iV_k = V_k\,\frac{\partial}{\partial x_i} \text{ for any } i \in \{1, ..., d\}$$

with P_n being the subspace of homogeneous polynomials of degree n in d variables. The explicit formula of V_k is not known in general (see [22]). For the group $G := \mathbb{Z}_2^d$ and $h_k(x) := \prod_{i=1}^d |x_i|^{k_i}$ for all $x \in \mathbb{R}^d$, it is an integral transform

$$V_k f(x) := b_k \int\limits_{[-1,1]^d} f\big(x_1 t_1, \cdots, x_d t_d\big) \prod_{i=1}^d (1+t_i)\Big(1-t_i^2\Big)^{k_i-1} dt, \;\; x \in \mathbb{R}^d. \tag{2.1}$$

We denote by μ_k the measure on $\mathbb{R}^d$ given by $d\mu_k(x) := h_k(x)dx$, and we introduce the Mehta-type constant c_k, by

$$c_k^{-1} := \int_{\mathbb{R}^d} e^{-|x|^2/2} d\mu_k(x).$$

For $y \in \mathbb{R}^d$, the initial problem $D_{k,j} u(\cdot, y)(x) = y_j u(x,y)$, $j = 1, \ldots, d$, with $u(0,y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x,y)$ and called Dunkl kernel (see e.g., [9,12]). This kernel has the Laplace-type representation [22]:

$$E_k(x,z) = \int_{\mathbb{R}^d} e^{<y,z>} d\Gamma_x(y), \; x \in \mathbb{R}^d, \; z \in \mathbb{C}^d, \tag{2.2}$$

where $< y, z > := \sum_{i=1}^d y_i z_i$ and Γ_x is a probability measure on $\mathbb{R}^d$, such that

$$\operatorname{supp}(\Gamma_x) \subset \{y \in \mathbb{R}^d : |y| \le |x|\}.$$

This kernel possesses the following properties: for $x, y \in \mathbb{R}^d$, we have

$$E_k(x,y) = E_k(y,x), E_k(x,0) = 1, \; E_k(-ix,y) = \overline{E_k(ix,y)}, \; |E_k(\pm ix, y)| \le 1. \tag{2.3}$$

Let $B(x,r) := \{y \in \mathbb{R}^d : |x-y| < r\}$ denote the ball in $\mathbb{R}^d$ that centered in $x \in \mathbb{R}^d$ and having radius $r > 0$, $B_r = B(0,r)$. Then having

$$\mu_k(B_r) = \int_{B_r} d\mu_k(x) = b_k \, r^{d+2\gamma_k}, \tag{2.4}$$

where

$$b_k = \left(\frac{a_k}{d+2\gamma_k}\right) \text{ and } a_k := \left(\int_{S^{d-1}} h_k^2(x)\, d\sigma(x)\right)^{-1},$$

S^{d-1} is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$.

We denote by $L_p(\mu_k) \equiv L_p(\mathbb{R}^d, d\mu_k)$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L_p(\mu_k)} := \Big(\int_{\mathbb{R}^d} |f(x)|^p d\mu_k(x)\Big)^{1/p} < \infty, \; 1 \le p < \infty,$$
$$\|f\|_{L_\infty(\mathbb{R}^d)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < \infty.$$

For $f \in L_1(\mu_k)$ the Dunkl transform is defined (see [10]) by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix,y) f(y) d\mu_k(x), \; x \in \mathbb{R}^d.$$

The Dunkl transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L_2(\mu_k)$ onto itself. In particular,

$$\|\mathcal{F}_k f\|_{L_2(\mu_k)} = \|f\|_{L_2(\mu_k)}. \tag{2.5}$$

The Dunkl transform allows us to define a generalized translation operator on $L_2(\mu_k)$ by setting $\mathcal{F}_k(\tau_x f)(y) = E_k(ix, y)\mathcal{F}_k(f)(y)$, $y \in \mathbb{R}^d$. It is the definition of Thangavelu and Xu given in [25]. It plays the role of the ordinary translation $\tau_x f(\cdot) = f(x + \cdot)$ in $\mathbb{R}^d$, since the Euclidean Fourier transform satisfies $\mathcal{F}(\tau_x f)(y) = e^{ixy}\mathcal{F}(f)(y)$.

Note that from (2.3) and (2.5), the definition makes sense and

$$\|\tau_x f\|_{L_2(\mu_k)} \leq \|f\|_{L_2(\mu_k)}.$$

Rösler [23] introduced the Dunkl translation operators for radial functions. If f are radial functions, $f(x) = F(|x|)$, then

$$\tau_x f(y) = \int_{\mathbb{R}^d} F(\sqrt{|z|^2 + |y|^2 + 2 < y, z >})\, d\Gamma_x(y),\ x \in \mathbb{R}^d,$$

where Γ_x is the representing measure given by (2.2).

This formula allows us to establish the following result, see [25,26].

For all $1 \leq p \leq 2$ and for all $x \in \mathbb{R}^d$, the Dunkl translation $\tau_x : L_p^{\mathrm{rad}}(\mu_k) \to L_p(\mu_k)$ is a bounded operator, and for $f \in L_p^{\mathrm{rad}}(\mu_k)$,

$$\|\tau_x f\|_{L_p(\mu_k)} \leq \|f\|_{L_p(\mu_k)}. \tag{2.6}$$

If $G = \mathbb{Z}_2^d$, then for all $1 \leq p \leq \infty$ and for all $x \in \mathbb{R}^d$, the Dunkl translation $\tau_x : L_p(\mu_k) \to L_p(\mu_k)$ is a bounded operator, and for $f \in L_p(\mu_k)$,

$$\|\tau_x f\|_{L_p(\mu_k)} \leq C_0 \|f\|_{L_p(\mu_k)}. \tag{2.7}$$

In the analysis of this generalized translation a particular role is played by the space (cf. [22,23,25,27])

$$A_k(\mathbb{R}^d) = \{f \in L_1(\mu_k) : \mathcal{F}_k f \in L_1(\mu_k)\}.$$

The operator τ_x satisfies the following properties:

Proposition 2.1 *Assume that $f \in A_k(\mathbb{R}^d)$ and $g \in L_1(\mu_k)$, then*

$$(i)\ \int_{\mathbb{R}^d} \tau_x f(y)\, g(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y)\, \tau_{-x} g(y) d\mu_k(y);$$
$$(i)\ \tau_x f(y) = \tau_{-y} f(-x).$$

The *maximal operator* M_k associated with the Dunkl operator on $\mathbb{R}^d$ is given by

$$M_k f(x) := \sup_{r>0} \big(\mu_k(B_r)\big)^{-1} \int_{B_r} \tau_x |f|(y)\, d\mu_k(y),\ x \in \mathbb{R}^d$$

and the *maximal commutator* $M_{b,k}$ associated with the Dunkl operator on $\mathbb{R}^d$ and with a locally integrable function $b \in L_1^{\mathrm{loc}}(\mu_k)$ is defined by (see [7,8,15])

$$M_{b,k} f(x) := \sup_{r>0} \big(\mu_k(B_r)\big)^{-1} \int_{B_r} |b(x) - b(y)|\, \tau_x |f|(y)\, d\mu_k(y),\ x \in \mathbb{R}^d.$$

Theorem 2.1 *[8] 1. If $f \in L_1(\mu_k)$, then for every $\beta > 0$*

$$\mu_k \{x \in \mathbb{R} : M_k f(x) > \beta\} \le \frac{C}{\beta} \int_{\mathbb{R}^d} |f(x)| \, d\mu_k(x),$$

where $C > 0$ is independent of f.

2. If $f \in L_p(\mu_k), \quad 1 < p \le \infty$, then $M_k f \in L_p(\mu_k)$ and

$$\|M_k f\|_{L_p(\mu_k)} \le C_p \, \|f\|_{L_p(\mu_k)},$$

where $C_p > 0$ is independent of f.

Corollary 2.1 *If $f \in L_1^{loc}(\mu_k)$, then*

$$\lim_{r \to 0} \frac{1}{\mu_k(B(0,r))} \int_{B(0,r)} \big|\tau_x f(y) - f(x)\big| \, d\mu_k(y) = 0$$

for a. e. $x \in \mathbb{R}^d$.

Corollary 2.2 *If $f \in L_1^{loc}(\mu_k)$, then*

$$\lim_{r \to 0} \frac{1}{\mu_k B(0,r)} \int_{B(0,r)} \tau_x f(y) d\mu_k(y) = f(x)$$

for a. e. $x \in \mathbb{R}^d$.

3 Some embeddings into the D_k-Morrey and modified D_k-Morrey spaces

Definition 3.1 *[14] Let $1 \le p < \infty$, $0 \le \lambda \le d + 2\gamma_k$ and $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L_{p,\lambda}(\mu_k)$ Morrey space ($\equiv D_k$-Morrey space) and by $\widetilde{L}_{p,\lambda}(\mu_k)$ the modified Morrey space ($\equiv$ modified D_k-Morrey space), associated with the Dunkl operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with the finite norms*

$$\|f\|_{L_{p,\lambda}(\mu_k)} := \sup_{x \in \mathbb{R}^d, \, t > 0} \left(t^{-\lambda} \int_{B_t} \tau_x |f|^p(y) \, d\mu_k(y) \right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)} := \sup_{x \in \mathbb{R}^d, \, t > 0} \left([t]_1^{-\lambda} \int_{B_t} \tau_x |f|^p(y) \, d\mu_k(y) \right)^{1/p},$$

respectively.

If $\lambda < 0$ or $\lambda > d + 2\gamma_k$, then $\widetilde{L}_{p,\lambda}(\mu_k) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}$.

Note that

$$L_p(\mu_k) \subset\!\!\succ \widetilde{L}_{p,0}(\mu_k) = L_{p,0}(\mu_k),$$
$$\|f\|_{\widetilde{L}_{p,0}(\mu_k)} = \|f\|_{L_{p,0}(\mu_k)} \le C_0 \|f\|_{L_p(\mu_k)}.$$

$$\widetilde{L}_{p,\lambda}(\mu_k) \subset\!\!\succ L_p(\mu_k) \quad \text{and} \quad \|f\|_{L_p(\mu_k)} \le \|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)}, \tag{3.1}$$

$$\widetilde{L}_{p,\lambda}(\mu_k) \subset\!\!\succ L_{p,\lambda}(\mu_k) \quad \text{and} \quad \|f\|_{L_{p,\lambda}(\mu_k)} \le \|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)}. \tag{3.2}$$

Definition 3.2 *[5] Let $1 \le p < \infty$, $0 \le \lambda \le d + 2\gamma_k$. We denote by $WL_{p,\lambda}(\mu_k)$ weak D_k-Morrey space and by $W\widetilde{L}_{p,\lambda}(\mu_k)$ the modified weak D_k-Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^d$ with finite norms*

$$\|f\|_{WL_{p,\lambda}(\mu_k)} := \sup_{r>0} r \sup_{x\in\mathbb{R}^d,\, t>0} \Big(t^{-\lambda}\mu_k\left\{y \in B_t :\ \tau_x|f|(y) > r\right\}\Big)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mu_k)} := \sup_{r>0} r \sup_{x\in\mathbb{R}^d,\, t>0} \Big([t]_1^{-\lambda}\mu_k\left\{y \in B_t :\ \tau_x|f|(y) > r\right\}\Big)^{1/p},$$

respectively.

We note that

$$L_{p,\lambda}(\mu_k) \subset WL_{p,\lambda}(\mu_k) \ \text{ and } \ \|f\|_{WL_{p,\lambda}(\mu_k)} \le \|f\|_{L_{p,\lambda}(\mu_k)}$$

and

$$\widetilde{L}_{p,\lambda}(\mu_k) \subset W\widetilde{L}_{p,\lambda}(\mu_k) \ \text{ and } \ \|f\|_{W\widetilde{L}_{p,\lambda}(\mu_k)} \le \|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)}.$$

Lemma 3.1 *Let $1 \le p < \infty$. Then*

$$L_{p,d+2\gamma_k}(\mu_k) = L_\infty(\mathbb{R}^d),$$

and

$$\|f\|_{L_{p,d+2\gamma_k}(\mu_k)} \approx \|f\|_{L_\infty(\mathbb{R}^d)}.$$

Proof. Let $f \in L_\infty(\mathbb{R}^d)$. Then by (2.4) we have

$$L_\infty(\mathbb{R}^d) \subset\!\!\!\succ L_{p,d+2\gamma_k}(\mu_k)$$

and

$$\|f\|_{L_{p,d+2\gamma_k}(\mu_k)} \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}.$$

Let $f \in L_{p,d+2\gamma_k}(\mu_k)$. By the Lebesgue's Theorem we have (see section 2, Corolarry 2)

$$\lim_{t\to 0} \mu_k(B_t)^{-1} \int_{B_t} \tau_x|f(y)|^p \, d\mu_k(y) = |f(x)|^p.$$

Then

$$\begin{aligned} |f(x)| &= \left(\lim_{t\to 0} \mu_k(B_t)^{-1} \int_{B_t} \tau_x|f(y)|^p \, d\mu_k(y)\right)^{1/p} \\ &\lesssim \|f\|_{L_{p,d+2\gamma_k}(\mu_k)}. \end{aligned}$$

Therefore $f \in L_\infty(\mathbb{R}^d)$ and

$$\|f\|_{L_\infty(\mathbb{R}^d)} \lesssim \|f\|_{L_{p,d+2\gamma_k}(\mu_k)}.$$

Thus $L_{p,d+2\gamma_k}(\mu_k) = L_\infty(\mathbb{R}^d)$ and $\|f\|_{L_{p,d+2\gamma_k}(\mu_k)} \approx \|f\|_{L_\infty(\mathbb{R}^d)}$.

Lemma 3.2 *Let* $1 \le p < \infty$, $0 \le \lambda \le d + 2\gamma_k$. *Then*

$$\widetilde{L}_{p,\lambda}(\mu_k) = L_{p,\lambda}(\mu_k) \cap L_p(\mu_k)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)} = \max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, \|f\|_{L_p(\mu_k)}\right\}.$$

Proof. Let $f \in \widetilde{L}_{p,\lambda}(\mu_k)$. Then by (3.1) and (3.2) we have

$$\widetilde{L}_{p,\lambda}(\mu_k) \subset\!\!\succ L_{p,\lambda}(\mu_k) \cap L_p(\mu_k)$$

and

$$\max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, \|f\|_{L_p(\mu_k)}\right\} \le \|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)}.$$

Let $f \in L_{p,\lambda}(\mu_k) \cap L_p(\mu_k)$. Then by Proposition 2.1 we have

$$\begin{aligned}\|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)} &= \sup_{x\in\mathbb{R}^d, t>0}\left([t]_1^{-\lambda}\int_{B_t}\tau_x|f|^p(y)\,d\mu_k(y)\right)^{1/p} \\ &= \max\left\{\sup_{x\in\mathbb{R}^d, 0<t\le 1}\left(t^{-\lambda}\int_{B_t}\tau_x|f|^p(y)\,d\mu_k(y)\right)^{1/p},\right. \\ &\quad\left.\sup_{x\in\mathbb{R}^d, t>1}\left(\int_{B_t}\tau_x|f|^p(y)\,d\mu_k(y)\right)^{1/p}\right\} \le \max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, C_0\|f\|_{L_p(\mu_k)}\right\}.\end{aligned}$$

Therefore, $f \in \widetilde{L}_{p,\lambda}(\mu_k)$ and the embedding $L_{p,\lambda}(\mu_k) \cap L_p(\mu_k) \subset\!\!\succ \widetilde{L}_{p,\lambda}(\mu_k)$ is valid.

Thus $\widetilde{L}_{p,\lambda}(\mu_k) = L_{p,\lambda}(\mu_k) \cap L_p(\mu_k)$ and

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)} = \max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, \|f\|_{L_p(\mu_k)}\right\}.$$

From Lemmas 3.1 and 3.2 for $1 \le p < \infty$ we have

$$\widetilde{L}_{p,d+2\gamma_k}(\mu_k) = L_\infty(\mu_k) \cap L_p(\mu_k). \tag{3.3}$$

Lemma 3.3 *Let* $G = \mathbb{Z}_2^d$ *and* $0 \le \lambda \le d + 2\gamma_k$. *Then*

$$L_{\frac{d+2\gamma_k}{d+2\gamma_k-\lambda}}(\mu_k) \subset\!\!\succ L_{1,\lambda}(\mu_k) \quad and \quad \|f\|_{L_{1,\lambda}(\mu_k)} \le C_0 b_k^{\frac{\lambda}{d+2\gamma_k}}\,\|f\|_{L_{\frac{d+2\gamma_k}{d+2\gamma_k-\lambda}}(\mu_k)}.$$

Proof. The embedding is a consequence of Hölder's inequality and (2.7). Indeed,

$$\begin{aligned}\|f\|_{L_{1,\lambda}(\mu_k)} &= \sup_{x\in\mathbb{R}^d, t>0} t^{-\lambda}\int_{B_t}\tau_x|f|(y)\,d\mu_k(y) \\ &\le \sup_{x\in\mathbb{R}^d, t>0} t^{-\lambda}\,(\mu_k B_t)^{\frac{\lambda}{d+2\gamma_k}}\left(\int_{B_t}\left(\tau_x|f|(y)\right)^{\frac{d+2\gamma_k}{d+2\gamma_k-\lambda}}d\mu_k(y)\right)^{\frac{d+2\gamma_k-\lambda}{d+2\gamma_k}} \\ &\le C_0 b_k^{\frac{\lambda}{d+2\gamma_k}}\,\|f\|_{L_{\frac{d+2\gamma_k}{d+2\gamma_k-\lambda}}(\mu_k)}.\end{aligned}$$

On the D_k-Morrey spaces the following embedding is valid.

Lemma 3.4 *Let $G=\mathbb{Z}_2^d$, $0\le\lambda<d+2\gamma_k$ and $0\le\beta<d+2\gamma_k-\lambda$. Then for $p=\frac{d+2\gamma_k-\lambda}{\beta}$*

$$L_{p,\lambda}(\mu_k)\subset L_{1,d+2\gamma_k-\beta}(\mu_k) \quad and \quad \|f\|_{L_{1,d+2\gamma_k-\beta}(\mu_k)} \le b_k^{1/p'}\,\|f\|_{L_{p,\lambda}(\mu_k)},$$

where $1/p+1/p'=1$.

Proof. The embedding is a consequence of Hölder's inequality and (2.7). Indeed,

$$\begin{aligned}\|f\|_{L_{1,d+2\gamma_k-\beta}(\mu_k)} &= \sup_{x\in\mathbb{R}^d,t>0} t^{\beta-d-2\gamma_k}\int_{B_t}\tau_x|f|(y)\,d\mu_k(y)\\ &\le b_k^{\frac{1}{p'}}\sup_{x\in\mathbb{R}^d,t>0} t^{\beta-\frac{d+2\gamma_k}{p}}\left(\int_{B_t}\big(\tau_x|f|(y)\big)^p\,d\mu_k(y)\right)^{\frac{1}{p}}\\ &\le b_k^{\frac{1}{p'}}\,\|f\|_{L_{p,\lambda}(\mu_k)}.\end{aligned}$$

On the modified D_k-Morrey spaces the following embedding is valid.

Lemma 3.5 *Let $G=\mathbb{Z}_2^d$, $0\le\lambda<d+2\gamma_k$ and $0\le\beta<d+2\gamma_k-\lambda$. Then for $\frac{d+2\gamma_k-\lambda}{\beta}\le p\le\frac{d+2\gamma_k}{\beta}$*

$$\widetilde{L}_{p,\lambda}(\mu_k)\subset\!\!\succ \widetilde{L}_{1,d+2\gamma_k-\beta}(\mu_k) \quad and \quad \|f\|_{\widetilde{L}_{1,d+2\gamma_k-\beta}(\mu_k)} \le C_0 b_k^{\frac{\lambda}{d+2\gamma_k}}\,\|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)}.$$

Proof. Let $0<\lambda<d+2\gamma_k$, $0<\beta<d+2\gamma_k-\lambda$, $f\in\widetilde{L}_{p,\lambda}(\mu_k)$ and $\frac{d+2\gamma_k-\lambda}{\beta}\le p\le\frac{d+2\gamma_k}{\beta}$. By the Hölder's inequality we have

$$\begin{aligned}\|f\|_{\widetilde{L}_{1,d+2\gamma_k-\beta}(\mu_k)} &= \sup_{x\in\mathbb{R}^d,\,t>0}[t]_1^{\beta-d-2\gamma_k}\int_{B_t}\tau_x|f|(y)\,d\mu_k(y)\\ &\lesssim \sup_{x\in\mathbb{R}^d,\,t>0}\big([t]_1\,t^{-1}\big)^{-(d+2\gamma_k)/p'}\,[t]_1^{\beta-\frac{d+2\gamma_k-\lambda}{p}}\\ &\times\left([t]_1^{-\lambda}\int_{B_t}\tau_x|f|^p(y)\,d\mu_k(y)\right)^{1/p}\\ &\approx \sup_{x\in\mathbb{R}^d,\,t>0}\big([t]_1\,t^{-1}\big)^{d+2\gamma_k-\beta}\big([t]_1\,t^{-1}\big)^{-(d+2\gamma_k)/p'}\,[t]_1^{\beta-\frac{d+2\gamma_k-\lambda}{p}}\\ &\times\left([t]_1^{-\lambda}\int_{B_t}\tau_x|f|^p(y)\,d\mu_k(y)\right)^{1/p}\\ &\lesssim \|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)}\sup_{t>0}\big([t]_1\,t^{-1}\big)^{\frac{d+2\gamma_k}{p}-\beta}\,[t]_1^{\beta-\frac{d+2\gamma_k-\lambda}{p}}.\end{aligned}$$

Note that

$$\sup_{t>0}\big([t]_1\,t^{-1}\big)^{\frac{d+2\gamma_k}{p}-\beta}\,[t]_1^{\beta-\frac{d+2\gamma_k-\lambda}{p}} = \max\{\sup_{0<t\le1}t^{\beta-\frac{d+2\gamma_k-\lambda}{p}},\sup_{t>1}t^{\beta-\frac{d+2\gamma_k}{p}}\}<\infty$$

$$\text{if and only if}\quad \frac{d+2\gamma_k-\lambda}{\beta}\le p\le\frac{d+2\gamma_k}{\beta}.$$

Therefore $f\in\widetilde{L}_{1,d+2\gamma_k-\beta}(\mu_k)$ and

$$\|f\|_{\widetilde{L}_{1,d+2\gamma_k-\beta}(\mu_k)}\lesssim\|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)}.$$

Remark 3.1 Note that, in the case of $d=1$, Lemmas 3.1, 3.2, 3.3, 3.4, 3.5 were proved in [6].

4 Some applications

In this section, using the results of section 3, we get the boundedness of the fractional D_k-maximal operator in the D_k-Morrey and modified D_k-Morrey spaces.

For $0 \le \beta < d + 2\gamma_k$ we define the fractional maximal functions

$$M_{k,\beta} f(x) := \sup_{t>0} \mu_k\big(B_t\big)^{-1+\frac{\beta}{d+2\gamma_k}} \int_{B_t} \tau_x |f|(y)\, d\mu_k(y)$$

and

$$M_{p,k,\beta} f(x) := \left(M_{k,\beta} |f|^p\right)^{1/p}(x).$$

In the case $\beta = 0$, we denote $M_{p,k,0}f$ by $M_{p,k}f$. Note that $M_{1,k}f = M_k f$.

Lemma 4.1 *Let $1 \le p < \infty$, $0 \le \beta < d + 2\gamma_k$ and $f \in L_{p,d+2\gamma_k-\beta}(\mu_k)$. Then $M_{p,k,\beta} f \in L_\infty(\mathbb{R}^d)$ and the following relation*

$$\|M_{p,k,\beta} f\|_{L_\infty(\mathbb{R}^d)} = b_k^{(\frac{\beta}{d+2\gamma_k}-1)\frac{1}{p}} \|f\|_{L_{p,d+2\gamma_k-\beta}(\mu_k)}$$

is valid.

Proof.

$$\|M_{p,k,\beta} f\|_{L_\infty(\mathbb{R}^d)} = b_k^{(\frac{\beta}{d+2\gamma_k}-1)\frac{1}{p}} \sup_{x\in\mathbb{R}^d, t>0} \left(t^{\beta-d-2\gamma_k} \int_{B_t} \tau_x |f|^p(y)\, d\mu_k(y) \right)^{1/p}$$
$$= b_k^{(\frac{\beta}{d+2\gamma_k}-1)\frac{1}{p}} \|f\|_{L_{p,d+2\gamma_k-\beta}(\mu_k)}.$$

Taking $\beta = 0$ in Lemma 4.1 and using Lemma 3.1, we get for $M_{p,k}f$ the following result.

Corollary 4.1 *Let $1 \le p < \infty$. Then*

$$\|M_{p,k} f\|_{L_\infty(\mathbb{R}^d)} \approx \|f\|_{L_\infty(\mathbb{R}^d)}.$$

Lemma 4.2 *Let $1 \le p < \infty$, $0 \le \beta < d + 2\gamma_k$ and $f \in \widetilde{L}_{p,d+2\gamma_k-\beta}(\mu_k)$. Then $M_{p,k,\beta} f \in L_\infty(\mathbb{R}^d)$ and the following equality*

$$\|M_{p,k,\beta} f\|_{L_\infty(\mathbb{R}^d)} = b_k^{(\frac{\beta}{d+2\gamma_k}-1)\frac{1}{p}} \|f\|_{\widetilde{L}_{p,d+2\gamma_k-\beta}(\mu_k)}$$

is valid.

Corollary 4.2 *Let $G = \mathbb{Z}_2^d$, $0 \le \lambda < d + 2\gamma_k$ and $0 \le \beta < d + 2\gamma_k - \lambda$. Then the operator $M_{k,\beta}$ is bounded from $L_{p,\lambda}(\mu_k)$ to $L_\infty(\mathbb{R}^d)$ for $p = \frac{d+2\gamma_k-\lambda}{\beta}$. Moreover*

$$\|M_{k,\beta} f\|_{L_\infty(\mathbb{R}^d)} \le b_k^{\frac{\beta}{d+2\gamma_k}-\frac{1}{p}} \|f\|_{L_{p,\lambda}(\mu_k)}.$$

Corollary 4.3 *Let $G = \mathbb{Z}_2^d$, $1 \le p < \infty$, $0 \le \lambda < d + 2\gamma_k$, $0 \le \beta < d + 2\gamma_k - \lambda$. Then the operator $M_{k,\beta}$ is bounded from $\widetilde{L}_{p,\lambda}(\mu_k)$ to $L_\infty(\mathbb{R}^d)$ for $\frac{d+2\gamma_k-\lambda}{\beta} \le p \le \frac{d+2\gamma_k}{\beta}$. Moreover*

$$\|M_{k,\beta} f\|_{L_\infty(\mathbb{R}^d)} \le b_k^{\frac{\beta}{d+2\gamma_k}-\frac{1}{p}} \|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)}.$$

Remark 4.1 Note that, in the case of $d = 1$, Lemmas 4.1 and 4.2 were proved in [6].

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