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Some notes on the gh-lifts of affine connections

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Abstract. *Let M be a differentiable manifold with dimension n and T^*M be its cotangent bundle. In this paper, we determine the $gh - lift$ of the affine connection via the musical isomorphism on the cotangent bundle T^*M. We obtain the torsion tensor, curvature tensor and geodesic curve of the $gh - lift$ of Levi-Civita connection.*

Keywords. Horizontal lift, connection, curvature tensor, musical isomorphism, geodesic.

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1 Introduction

The connections were studied from an infinitesimal perspective in Riemannian geometry. Several authors investigated the lifts of connections on the different type bundle. Pogoda [8] defined the horizontal lift of basic connections of order r to a transverse naturel vector bundle and studied its properties. Salimov and Fattayev [9] determined the horizontal lift and complete lifts of the linear connection from a smooth manifold to its coframe bundle. Several authors investigated the lifts of connections on the cotangent bundle T^*M. Yano and Patterson [10] applied the horizontal lifts which was defined by them to the symmetric connections on the cotangent bundle T^*M. Kures [6] determined all natural operators transforming classical torsion-free linear connections on a manifold M into classical linear connections on the cotangent bundle T^*M. Also the Riemannian manifolds and the cotangent bundles have been studied by many authors [3,7,4,5]. In this paper, calculating the coefficients of $gh-$lift ${}^{GH}\overset{*}{\nabla}$ of the affine connection via the musical isomorphism we have determined the $gh-$lift ${}^{GH}\overset{*}{\nabla}$ of the affine connection ∇. After using the coefficients of the $gh-$lift ${}^{GH}\overset{*}{\nabla}$ we have determined the torsion tensor and the curvature tensor of the $gh-$lift ${}^{GH}\overset{*}{\nabla}$ of Levi-Civita connection. Finally we have investigated properties of the geodesic of the $gh-$lift ${}^{GH}\overset{*}{\nabla}$ to the cotangent bundle T^*M.

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2 Preliminaries

Let M be a pseudo-Riemannian manifold with n dimension. The tangent bundle on M is denoted by $TM = \cup_{x\in M} T_x M$. The local coordinates on TM are $(x^i, x^{\bar{i}}) = (x^i, y^i)$ where (x^i) are local coordinates on M and (y^i) are vector space coordinates according to the basis $\partial/\partial x^i$, i.e. $y_x = y^i \frac{\partial}{\partial x^i} \in T_x M$. The cotangent bundle on M is denoted by $T^*M = \cup_{x\in M} T^*_x M$. The local coordinates on T^*M are $(x^i, \tilde{x}^{\bar{i}}) = (x^i, p_i)$ where (x^i) are local coordinates on M and p_i are vector space coordinates according to the basis dx^i, i.e. $p_x = p_i dx^i \in T^*_x M$. We denote by $\Im^r_s(M)$ the set of all tensor fields of type (r, s) on M. Throughout this paper we assume the manifolds, tensor fields and connections to be diferentiable of class C^∞. We use the ranges of the index i being $\{1, ..., n\}$ and the index $\bar{i}$ being $\{n+1, ..., 2n\}$.

Let g be a pseudo Riemannian metric. $g^\sharp : T^*M \to TM$ is the musical isomorphism associated with g pseudo Riemannian metric with inverse given by $g^\flat : TM \to T^*M$.

The musical isomorphism $g^\sharp$ described by

$$g^\sharp : \tilde{x}^M = (x^m, \tilde{x}^{\overline{m}}) = (x^m, p_m) \to x^J = (x^j, x^{\bar{j}}) = (\delta^j_m x^m, y^j = g^{jm} p_m). \tag{2.1}$$

In music notation, the sharp symbol $\sharp$ increase a note by a half step. And similar to this musical notation, the musical isomorphism $g^\sharp$ increase the indice of the vector space coordinate.

The musical isomorphism $g^\flat$ is described by

$$g^\flat : x^I = (x^i, x^{\bar{i}}) = (x^i, y^i) \to \tilde{x}^K = (x^k, \tilde{x}^{\bar{k}}) = (\delta^k_i x^i, p_k = g_{ki} y^i). \tag{2.2}$$

In another music notation, the flat symbol $\flat$ lowers a note by a half step. Similar to this musical notation, the musical isomorphism $g^\flat$ low the indice of the vector space coordinate.

The Jacobian matrice of $g^\flat$ is obtained by

$$(g^\flat_*) = \left(\widetilde{A}^M_J\right) = \left(\frac{\partial \tilde{x}^M}{\partial x^J}\right) = \begin{pmatrix} \delta^m_j & 0 \\ y^s \partial_j g_{ms} & g_{mj} \end{pmatrix} \tag{2.3}$$

and the Jacobian matrice of $g^\sharp$ is obtained by

$$(g^\sharp_*) = \left(A^J_M\right) = \left(\frac{\partial x^J}{\partial \tilde{x}^M}\right) = \begin{pmatrix} \delta^j_m & 0 \\ p_s \partial_m g^{js} & g^{jm} \end{pmatrix}, \tag{2.4}$$

where δ is the Kronecker delta [1].

Yano and Patterson [10] defined the horizontal lift and applied this notion to connection in M. The $gh-$lift is newly defined in [2]. We obtain gh-lift ${}^{GH}\overset{*}{\nabla}$ of the affine connection ∇ by transferring the horizontal lift of affine connection ∇ from tangent bundle to cotangent bundle via the musical isomorphism. The problems of transferring the lifts to the cotangent bundle were studied in [1,2].

3 The $gh-$lift of affine connection

Let ∇ be an affine connection on M manifold. The horizontal lift ${}^H\nabla$ of the affine connection ∇ to the tangent bundle TM is formed with equation

$${}^H\nabla = {}^C\nabla - {}^V R \tag{3.1}$$

where ${}^{V}R$ is vertical lift of curvature tensor of ∇ and ${}^{C}\nabla$ is complete lift of affine connection to the tangent bundle TM. And also the horizontal lift ${}^{H}\nabla$ of the affine connection ∇ provides the conditions

$$\begin{aligned} &{}^{H}\nabla_{{}^{V}X}\,{}^{V}Y = 0, && {}^{H}\nabla_{{}^{V}X}\,{}^{H}Y = 0, \\ &{}^{H}\nabla_{{}^{H}X}\,{}^{V}Y = {}^{V}\left(\nabla_{X}Y\right), && {}^{H}\nabla_{{}^{H}X}\,{}^{H}Y = {}^{H}\left(\nabla_{X}Y\right) \end{aligned} \tag{3.2}$$

for any $X, Y \in \Im_0^1(M)$.

Let Γ_{ab}^{c} be coefficient of ∇ according to the local coordinates $\left(x^h\right)$ on M. Let ${}^{H}\Gamma_{AB}^{C}$ be coefficient of ${}^{H}\nabla$ according to the induced coordinates $\left(x^h, y^h\right)$ to the tangent bundle TM. The non-zero coefficients ${}^{H}\Gamma_{AB}^{C}$ of the horizontal lift ${}^{H}\nabla$ to the tangent bundle TM are given by

$${}^{H}\Gamma_{ab}^{\ c} = \Gamma_{ab}^{\ c}\,,\ {}^{H}\Gamma_{a\bar{b}}^{\ \bar{c}} = \Gamma_{ab}^{\ c}\,,\ {}^{H}\Gamma_{\bar{a}b}^{\ \bar{c}} = \Gamma_{ab}^{\ c},\ {}^{H}\Gamma_{ab}^{\ \bar{c}} = y^{s}\partial_{s}\Gamma_{ab}^{\ c} - y^{s}R_{sab}^{c} \tag{3.3}$$

where R_{sab}^{c} is the curvature tensor of ∇.

The horizontal lift ${}^{H}\overset{*}{\nabla}$ of the affine connection ∇ to the cotangent bundle $T^{*}M$ is formed with equation

$${}^{H}\overset{*}{\nabla} = {}^{C}\overset{*}{\nabla} - {}^{V}\overset{*}{R} \tag{3.4}$$

where ∇ is symmetric. ${}^{V}\overset{*}{R}$ is vertical lift of curvature tensor of ∇ and ${}^{C}\overset{*}{\nabla}$ is complete lift of ∇ to the cotangent bundle $T^{*}M$. Let ${}^{H}\overset{*}{\Gamma}{}_{AB}^{C}$ be coefficient of ${}^{H}\overset{*}{\nabla}$ according to the induced coordinates $\left(x^h, p_h\right)$ to the cotangent bundle $T^{*}M$. The non-zero coefficients ${}^{H}\overset{*}{\Gamma}{}_{AB}^{C}$ of the horizontal lift ${}^{H}\overset{*}{\nabla}$ are given by

$${}^{H}\overset{*}{\Gamma}{}_{ab}^{\ c} = \Gamma_{ab}^{\ c},\ {}^{H}\overset{*}{\Gamma}{}_{a\bar{b}}^{\ \bar{c}} = -\Gamma_{ac}^{\ b},\ {}^{H}\overset{*}{\Gamma}{}_{\bar{a}b}^{\ \bar{c}} = -\Gamma_{cb}^{\ a},\ {}^{H}\overset{*}{\Gamma}{}_{ab}^{\ \bar{c}} = p_{s}\left(-\partial_{a}\Gamma_{bc}^{s} + \Gamma_{kc}^{s}\Gamma_{ab}^{k} + \Gamma_{bt}^{s}\Gamma_{ac}^{t}\right) \tag{3.5}$$

according to the induced coordinates $\left(x^h, p_h\right)$ to the cotangent bundle $T^{*}M$ [11].

We obtain the $\frac{\partial^2 \tilde{x}^K}{\partial x^A \partial x^B}$ that have components

$$\begin{aligned} &\frac{\partial^2 \tilde{x}^k}{\partial x^a \partial x^b} = 0,\quad \frac{\partial^2 \tilde{x}^k}{\partial x^a \partial x^{\bar{b}}} = 0,\quad \frac{\partial^2 \tilde{x}^k}{\partial x^{\bar{a}} \partial x^b} = 0,\quad \frac{\partial^2 \tilde{x}^k}{\partial x^{\bar{a}} \partial x^{\bar{b}}} = 0 \\ &\frac{\partial^2 \tilde{x}^{\bar{k}}}{\partial x^a \partial x^b} = \frac{p_s \partial^2 g^{ks}}{\partial x^a \partial x^b},\quad \frac{\partial^2 \tilde{x}^{\bar{k}}}{\partial x^a \partial x^{\bar{b}}} = \partial_a g^{kb},\quad \frac{\partial^2 \tilde{x}^{\bar{k}}}{\partial x^{\bar{a}} \partial x^b} = \partial_b g^{ka},\quad \frac{\partial^2 \tilde{x}^{\bar{k}}}{\partial x^{\bar{a}} \partial x^{\bar{b}}} = 0. \end{aligned} \tag{3.6}$$

And using (2.3), (2.4) , (3.3) and (3.6), the coefficients ${}^{GH}\overset{*}{\Gamma}{}_{AB}^{C}$ of ${}^{GH}\overset{*}{\nabla}$ are obtained from equation

$$g_{*}^{\flat}\,{}^{H}\Gamma_{AB}^{C} = \left({}^{GH}\overset{*}{\Gamma}{}_{AB}^{C}\right) = \left(\frac{\partial \tilde{x}^M}{\partial x^A}\frac{\partial \tilde{x}^S}{\partial x^B}\frac{\partial x^C}{\partial \tilde{x}^K}\,{}^{H}\Gamma_{MS}^{K} + \frac{\partial x^C}{\partial \tilde{x}^K}\frac{\partial^2 \tilde{x}^K}{\partial x^A \partial x^B}\right). \tag{3.7}$$

The non-zero coefficients ${}^{GH}\overset{*}{\Gamma}{}_{AB}^{C}$ of ${}^{GH}\overset{*}{\nabla}$ are obtained as follows where $A, B, ... = 1, ..., 2n$:

$$
{}^{GH}\overset{*}{\Gamma}{}^{c}_{ab} = \delta^m_a \delta^s_b \delta^c_k \Gamma^k_{ms} = \Gamma^c_{ab}
$$

$$
\begin{aligned}
{}^{GH}\overset{*}{\Gamma}{}^{\bar{c}}_{a\bar{b}} &= \delta^m_a g^{sb} g_{ck} \Gamma^k_{ms} + g_{ck} \partial_a g^{kb} \\
&= \Gamma^b_{ac} - g^{kb} \partial_a g_{ck} \\
&= -\Gamma^b_{ac} + g^{bk} \partial_c g_{ak} - g^{bk} \partial_k g_{ac} \\
&= -\Gamma^b_{ac} + g^{bk} \left(\bigtriangledown_c g_{ak} + \Gamma^t_{ca} g_{tk} + \Gamma^t_{ck} g_{at} \right) - g^{bk} \left(\bigtriangledown_k g_{ac} + \Gamma^t_{ka} g_{tc} + \Gamma^t_{kc} g_{ta} \right) \\
&= -\Gamma^b_{ac} + g^{bk} \bigtriangledown_c g_{ak} + g^{bk} \Gamma^t_{ca} g_{tk} - g^{bk} \bigtriangledown_k g_{ac} - g^{bk} \Gamma^t_{ka} g_{tc} \\
&= -\Gamma^b_{ac}
\end{aligned}
$$

$$
\begin{aligned}
{}^{GH}\overset{*}{\Gamma}{}^{\bar{c}}_{\bar{a}b} &= g^{ma} \delta^s_b g_{ck} \Gamma^k_{ms} + g_{ck} \partial_b g^{ka} \\
&= \Gamma^a_{cb} - g^{ka} \partial_b g_{ck} \\
&= -\Gamma^a_{cb} + g^{ak} \partial_c g_{bk} - g^{ak} \partial_k g_{cb} \\
&= -\Gamma^a_{cb} + g^{ak} \left(\nabla_c g_{bk} + \Gamma^t_{ck} g_{bt} + \Gamma^t_{cb} g_{tk} \right) - g^{ak} \left(\nabla_k g_{cb} + \Gamma^t_{kc} g_{tb} + \Gamma^t_{kb} g_{ct} \right) \\
&= -\Gamma^a_{cb} + g^{ak} \nabla_c g_{bk} + g^{ak} \Gamma^t_{cb} g_{tk} - g^{ak} \nabla_k g_{cb} - g^{ak} \Gamma^t_{kb} g_{ct} \\
&= -\Gamma^a_{cb}
\end{aligned}
$$

$$
\begin{aligned}
{}^{GH}\overset{*}{\Gamma}{}^{\bar{c}}_{ab} &= \delta^m_a \delta^s_b y^t \partial_k g_{tc} \Gamma^k_{ms} + \delta^m_a \delta^s_b g_{ck} \left(y^t \partial_t \Gamma^k_{ms} - y^t R^k_{tms} \right) + p_t \partial_a g^{mt} \delta^s_b g_{ck} \Gamma^k_{ms} \\
&\quad + \delta^m_a p_t \partial_b g^{st} g_{ck} \Gamma^k_{ms} + g_{ck} p_t \partial^2_{ab} g^{tk} \\
&= y^t \left(\partial_k g_{tc} \right) \Gamma^k_{ab} + g_{ck} y^t \partial_t \Gamma^k_{ab} - g_{ck} y^t R^k_{tab} + p_t \partial_a g^{mt} g_{ck} \Gamma^k_{mb} + p_t \partial_b g^{st} g_{ck} \Gamma^k_{as} \\
&\quad + g_{ck} p_t \partial^2_{ab} g^{tk} \\
&= y^t \left(\partial_k g_{tc} \right) \Gamma^k_{ab} + g_{ck} y^t \partial_t \Gamma^k_{ab} - g_{ck} y^t R^k_{tab} + p_t \partial_a g^{mt} g_{ck} \Gamma^k_{mb} + p_t \partial_b g^{st} g_{ck} \Gamma^k_{as} \\
&\quad - p_t \left(\partial_a \Gamma^t_{bc} \right) - g_{ck} p_t \Gamma^t_{bs} \partial_a g^{sk} - g_{ck} p_t \left(\partial_a \Gamma^k_{bs} \right) g^{ts} - g_{ck} p_t \Gamma^k_{bs} \partial_a g^{ts} \\
&= -p_t \left(\partial_a \Gamma^t_{bc} \right) + y^t \left(\partial_k g_{tc} \right) \Gamma^k_{ab} + g_{ck} y^t \partial_t \Gamma^k_{ab} - g_{ck} y^t R^k_{tab} + p_t \left(\partial_b g^{st} \right) g_{ck} \Gamma^k_{as} \\
&\quad - g_{ck} p_t \Gamma^t_{bs} \partial_c g^{sk} - g_{ck} p_t \left(\partial_a \Gamma^k_{bs} \right) g^{ts} \\
&= -p_t \left(\partial_a \Gamma^t_{bc} \right) + y^t \left(\partial_k g_{tc} \right) \Gamma^k_{ab} + g_{ck} y^t \partial_t \Gamma^k_{ab} - g_{ck} y^t R^k_{tab} - g_{ck} p_t \left(\partial_a \Gamma^k_{bs} \right) g^{ts} \\
&\quad + p_t \left(\partial_b g^{st} \right) g_{ck} \Gamma^k_{as} - g_{ck} p_t \Gamma^t_{bs} \partial_a g^{sk} \\
&= -p_t \left(\partial_a \Gamma^t_{bc} \right) + y^t \left(\partial_k g_{tc} \right) \Gamma^k_{ab} + g_{ck} \left(g^{ts} p_s \partial_t \Gamma^k_{ab} - p_t g^{ts} \partial_a \Gamma^k_{bs} - y^t R^k_{tab} \right) \\
&\quad + p_t \left(-\Gamma^s_{bm} g^{mt} - \Gamma^t_{bm} g^{sm} \right) g_{ck} \Gamma^k_{as} - g_{ck} p_t \Gamma^t_{bs} \left(-\Gamma^s_{am} g^{mk} - \Gamma^k_{am} g^{ms} \right) \\
&= -p_t \left(\partial_a \Gamma^t_{bc} \right) + p_s \Gamma^s_{ck} \Gamma^k_{ab} + g^{ts} g_{mc} p_s \Gamma^m_{kt} \Gamma^k_{ab} + g_{ck} g^{ts} p_s \left(\partial_t \Gamma^k_{ab} - \partial_a \Gamma^k_{bt} \right) \\
&\quad - g_{ck} g^{ts} p_s R^k_{tab} - g_{ck} p_s \left(\Gamma^t_{bm} \Gamma^k_{at} g^{ms} + \Gamma^s_{bm} \Gamma^k_{at} g^{tm} \right) \\
&\quad + g_{hk} p_s \left(\Gamma^s_{bt} \Gamma^t_{am} g^{mk} + \Gamma^s_{bt} \Gamma^k_{am} g^{mt} \right) \\
&= -p_t \left(\partial_a \Gamma^t_{bc} \right) + p_s \Gamma^s_{ck} \Gamma^k_{ab} + p_s \Gamma^s_{kc} \Gamma^k_{ab} + p_s \partial_c \Gamma^s_{ab} - p_s \partial_a \Gamma^s_{bc} - p_s \partial_c \Gamma^s_{ab} \\
&\quad + p_s \partial_a \Gamma^s_{cb} - p_s \Gamma^s_{cm} \Gamma^m_{ab} + p_s \Gamma^s_{am} \Gamma^m_{cb} - p_s \Gamma^s_{at} \Gamma^t_{bc} - p_s \Gamma^s_{bm} \Gamma^m_{ac} + p_s \Gamma^s_{bt} \Gamma^t_{ac} \\
&\quad + p_s \Gamma^s_{bt} \Gamma^t_{ac} \\
&= -p_s \partial_a \Gamma^s_{bc} + p_s \Gamma^s_{kc} \Gamma^k_{ab} + p_s \Gamma^s_{bt} \Gamma^t_{ac} \\
&= p_s \left(-\partial_a \Gamma^s_{bc} + \Gamma^s_{kc} \Gamma^k_{ab} + \Gamma^s_{bt} \Gamma^t_{ac} \right)
\end{aligned}
$$

After these results we get the following theorem

Theorem 3.1 *Let (M, g) be a pseudo Riemannian manifold with dimension n. Let ∇ be an affine connection on manifold M and ${}^H\nabla$ be the horizontal lift of ∇ to the tangent bundle TM. Then the differential of ${}^H\nabla$ by $g^\flat$, i.e., a gh−lift ${}^{GH}\overset{*}{\nabla}$ to the cotangent bundle T^*M coincides with the horizontal lift ${}^{H}\overset{*}{\nabla}$ to the cotangent bundle T^*M if ∇ is a Levi-Civita connection.*

4 The Torsion and Curvature Tensors of $gh-$ lift of Levi-Civita Connection

Let $\widetilde{T}$ be torsion tensor of horizontal lift of the symmetric affine connection ∇ on M to the cotangent bundle $T^{*}M$. The torsion tensor $\widetilde{T}$ of ${}^{H}\overset{*}{\nabla}$ is determined by

$$\widetilde{T}\left({}^{V}\theta,{}^{V}\omega\right)=0,\qquad \widetilde{T}\left({}^{H}X,{}^{V}\omega\right)=0,\qquad \widetilde{T}\left({}^{H}X,{}^{H}Y\right)=-\gamma R\left(X,Y\right)\tag{4.1}$$

where R is the curvature tensor of ∇. ${}^{H}X,\ {}^{H}Y$ are horizontal lift of $X,\ Y\in\Im_{0}^{1}\left(M\right)$ and ${}^{V}\theta,\ {}^{V}\omega$ are vertical lift of $\theta,\omega\in\Im_{1}^{0}\left(M\right)$ to the cotangent bundle $T^{*}M$. The non zero component $\widetilde{T}_{AB}^{C}$ of $\widetilde{T}$ is given

$$\widetilde{T}_{ab}^{\bar{c}}=-p_{t}R_{abc}^{t}\tag{4.2}$$

according to the induced coordinates $\left(x^{h},p_{h}\right)$ to the cotangent bundle $T^{*}M$ [11].

Let $\widehat{T}$ be torsion tensor of the gh-lift ${}^{GH}\overset{*}{\nabla}$ of the Levi-Civita connection on M to the cotangent bundle $T^{*}M$. Using the coefficient ${}^{GH}\overset{*}{\Gamma}{}_{AB}^{C}$ of ${}^{GH}\overset{*}{\nabla}$ to the cotangent bundle $T^{*}M$ the components $\widehat{T}{}_{AB}^{C}$ of $\widehat{T}$ are obtained with the equation

$$\widehat{T}{}_{AB}^{C}={}^{GH}\overset{*}{\Gamma}{}_{AB}^{C}-{}^{GH}\overset{*}{\Gamma}{}_{BA}^{C}\tag{4.3}$$

according to the induced coordinates $\left(x^{h},p_{h}\right)$ to the cotangent bundle $T^{*}M$. We have

$$\begin{array}{llll}\widehat{T}{}_{ab}^{c}=0, & \widehat{T}{}_{\bar{a}b}^{c}=0, & \widehat{T}{}_{a\bar{b}}^{c}=0, & \widehat{T}{}_{ab}^{\bar{c}}=-p_{t}R_{abc}^{t}\\ \widehat{T}{}_{\bar{a}b}^{\bar{c}}=0, & \widehat{T}{}_{a\bar{b}}^{\bar{c}}=0, & \widehat{T}{}_{\bar{a}\bar{b}}^{c}=0, & \widehat{T}{}_{\bar{a}\bar{b}}^{\bar{c}}=0.\end{array}\tag{4.4}$$

After we have

Corollary 4.1 *The torsion tensor $\widehat{T}$ of the gh$-$lift ${}^{GH}\overset{*}{\nabla}$ of Levi-Civita connection coincides with the torsion tensor $\widetilde{T}$ of horizontal lift ${}^{H}\overset{*}{\nabla}$ of the symmetric affine connection.*

Let $\widetilde{R}$ be curvature tensor of horizontal lift of the symmetric affine connection ∇ on M to the cotangent bundle $T^{*}M$. For any $X,\ Y,\ Z\in\Im_{0}^{1}\left(M\right)$ and $\omega,\ \theta,\ \psi\in\Im_{1}^{0}\left(M\right)$, the curvature tensor $\widetilde{R}$ of ${}^{H}\overset{*}{\nabla}$ provides the conditions

$$\begin{gathered}\widetilde{R}\left({}^{V}\theta,{}^{V}\omega\right)=0,\qquad \widetilde{R}\left({}^{H}X,{}^{V}\omega\right)=0\\ \widetilde{R}\left({}^{H}X,{}^{H}Y\right){}^{V}\psi=-{}^{V}\left(\psi\circ R\left(X,Y\right)\right)\\ \widetilde{R}\left({}^{H}X,{}^{H}Y\right){}^{H}Z={}^{H}\left(R\left(X,Y\right)Z\right)\end{gathered}\tag{4.5}$$

where R is the curvature tensor of ∇. The non zero components $\widetilde{R}{}_{ABD}^{C}$ of $\widetilde{R}$ are given

$$\begin{aligned}&\widetilde{R}_{abd}^{c}=R{}_{abd}^{c}\\ &\widetilde{R}_{abd}^{\bar{c}}=p_{s}\left(\Gamma_{ct}^{s}\,R{}_{abd}^{t}+\Gamma_{dt}^{s}\,R{}_{abc}^{t}\right)\\ &\widetilde{R}_{ab\bar{d}}^{\bar{c}}=-R{}_{abc}^{d}\end{aligned}\tag{4.6}$$

according to the induced coordinates $\left(x^{h},p_{h}\right)$ to the cotangent bundle $T^{*}M$.

Let $\widehat{R}$ be curvature tensor of the gh-lift ${}^{GH}\overset{*}{\nabla}$ of Levi-Civita connection on M to the cotangent bundle $T^{*}M$. Using the coefficient ${}^{GH}\overset{*}{\Gamma}{}_{AB}^{C}$ of ${}^{GH}\overset{*}{\nabla}$ to the cotangent bundle $T^{*}M$ the components $\widehat{R}{}_{ABD}^{C}$ of $\widehat{R}$ are obtained with the equation

$$\widehat{R}{}_{ABD}^{C}= \partial_{A}\,{}^{GH}\overset{*}{\Gamma}{}_{BD}^{C}-\partial_{B}\,{}^{GH}\overset{*}{\Gamma}{}_{AD}^{C}+\,{}^{GH}\overset{*}{\Gamma}{}_{AT}^{C}\,{}^{GH}\overset{*}{\Gamma}{}_{BD}^{T}-\,{}^{GH}\overset{*}{\Gamma}{}_{BT}^{C}\,{}^{GH}\overset{*}{\Gamma}{}_{AD}^{T} \tag{4.7}$$

according to the induced coordinates $\left(x^{h},p_{h}\right)$. We have

$$\begin{aligned}
\widehat{R}_{abd}^{c}&=R\,{}_{abd}^{\;c}\\
\widehat{R}_{abd}^{\overline{c}}&=p_{s}\left(\Gamma_{ct}^{s}\,R\,{}_{abd}^{\;t}+\Gamma_{dt}^{s}\,R\,{}_{abc}^{\;t}\right)\\
\widehat{R}_{ab\overline{d}}^{\overline{c}}&=-R\,{}_{abc}^{\;d}\quad .
\end{aligned} \tag{4.8}$$

After we have

Corollary 4.2 *The curvature tensor* $\widehat{R}$ *of the* $gh-$*lift* ${}^{GH}\overset{*}{\nabla}$ *of Levi-Civita connection coincides with the curvature tensor* $\widetilde{R}$ *of horizontal lift* ${}^{H}\overset{*}{\nabla}$ *of the symmetric affine connection.*

5 Geodesics of $gh-$lift of Levi-Civita Connection

Let $\widetilde{C}$ be a geodesic curve of the gh-lift ${}^{GH}\overset{*}{\nabla}$ of Levi-Civita connection on M to the cotangent bundle $T^{*}M$. The geodesic $\widetilde{C}$ is determined with the equations

$$\frac{d^{2}x^{C}}{dt^{2}}+\,{}^{GH}\overset{*}{\Gamma}{}_{AB}^{C}\frac{dx^{A}}{dt}\frac{dx^{B}}{dt}=0 \tag{5.1}$$

according to the induced coordinates $\left(x^{c},x^{\overline{c}}\right)=\left(x^{c},p_{c}\right)$.

Using the coefficient of gh-lift ${}^{GH}\overset{*}{\nabla}$ we obtain following equations from (5.1):

$$\begin{aligned}
\tfrac{d^{2}x^{c}}{dt^{2}}+\,{}^{GH}\overset{*}{\Gamma}{}_{ab}^{c}\tfrac{dx^{a}}{dt}\tfrac{dx^{b}}{dt}+\,{}^{GH}\overset{*}{\Gamma}{}_{\overline{a}b}^{c}\tfrac{dx^{\overline{a}}}{dt}\tfrac{dx^{b}}{dt}+\,{}^{GH}\overset{*}{\Gamma}{}_{a\overline{b}}^{c}\tfrac{dx^{a}}{dt}\tfrac{dx^{\overline{b}}}{dt}+\,{}^{GH}\overset{*}{\Gamma}{}_{\overline{a}\overline{b}}^{c}\tfrac{dx^{\overline{a}}}{dt}\tfrac{dx^{\overline{b}}}{dt}&=0\\
\tfrac{d^{2}x^{c}}{dt^{2}}+\Gamma_{ab}^{c}\tfrac{dx^{a}}{dt}\tfrac{dx^{b}}{dt}&=0,
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\tfrac{d^{2}x^{\overline{c}}}{dt^{2}}+\,{}^{GH}\overset{*}{\Gamma}{}_{ab}^{\overline{c}}\tfrac{dx^{a}}{dt}\tfrac{dx^{b}}{dt}+\,{}^{GH}\overset{*}{\Gamma}{}_{\overline{a}b}^{\overline{c}}\tfrac{dx^{\overline{a}}}{dt}\tfrac{dx^{b}}{dt}+\,{}^{GH}\overset{*}{\Gamma}{}_{a\overline{b}}^{\overline{c}}\tfrac{dx^{a}}{dt}\tfrac{dx^{\overline{b}}}{dt}+\,{}^{GH}\overset{*}{\Gamma}{}_{\overline{a}\overline{b}}^{\overline{c}}\tfrac{dx^{\overline{a}}}{dt}\tfrac{dx^{\overline{b}}}{dt}&=0\\
\tfrac{d^{2}p_{c}}{dt^{2}}+p_{s}\left(-\partial_{a}\Gamma_{bc}^{s}+\Gamma_{cm}^{s}\Gamma_{ab}^{m}+\Gamma_{bm}^{s}\Gamma_{ca}^{m}\right)\tfrac{dx^{a}}{dt}\tfrac{dx^{b}}{dt}-\Gamma_{cb}^{a}\tfrac{dp_{a}}{dt}\tfrac{dx^{b}}{dt}-\Gamma_{ac}^{b}\tfrac{dx^{a}}{dt}\tfrac{dp_{b}}{dt}&=0\\
\tfrac{d}{dt}\left(\tfrac{dp_{c}}{dt}-\Gamma_{cb}^{a}p_{a}\tfrac{dx^{b}}{dt}\right)-\Gamma_{ac}^{m}\left(\tfrac{dp_{m}}{dt}-\Gamma_{bm}^{s}p_{s}\tfrac{dx^{b}}{dt}\right)\tfrac{dx^{a}}{dt}&=0\\
\tfrac{d}{dt}\left(\tfrac{\delta p_{c}}{dt}\right)-\Gamma_{ac}^{m}\left(\tfrac{\delta p_{m}}{dt}\right)\tfrac{dx^{a}}{dt}&=0\\
\tfrac{\delta}{dt}\left(\tfrac{\delta p_{c}}{dt}\right)&=0\\
\tfrac{\delta^{2}p_{c}}{dt^{2}}&=0
\end{aligned} \tag{5.3}$$

where $\frac{\delta p_{c}}{dt}=\frac{dp_{c}}{dt}-\Gamma_{cb}^{a}p_{a}\frac{dx^{b}}{dt}$. After expressions (5.2) and (5.3) we have

Theorem 5.1 *Let $\widetilde{C}$ be a geodesic according to the gh-lift ${}^{GH}\overset{*}{\nabla}$ of Levi-Civita connecion on M to the cotangent bundle $T^{*}M$. The geodesic $\widetilde{C}$ has the eqations*

$$\frac{d^2x^c}{dt^2} + \Gamma^c_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt} = 0,$$
$$\frac{\delta^2 p_c}{dt^2} = 0$$

according to the induced coordinates (x^c, p_c) to $T^{*}M$.

Theorem 5.2 *The curve $\widetilde{C}$ on the cotangent bundle $T^{*}M$ is a geodesic according to the gh-lift ${}^{GH}\overset{*}{\nabla}$ of Levi-Civita connecion on M if the projection $C = \pi\left(\widetilde{C}\right)$ on M is a geodesic according to the ∇ on M and the second covariant differentiation of $p_c = p_c(t)$ along C vanishes where $\pi : T^{*}M \to M$ is the naturel projection.*

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