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# Some new beta integral inequalities

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**Abstract.** *In this paper we consider the derivative with a new parameter called beta-derivative. Using this calculus, we prove Chebyschev's integral inequality and obtain a new type of Hermite-Hadamard inequality.*

**Keywords.** Beta-integral, Chebyschev's inequality, Hermite-Hadamard inequality.

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## 1 Introduction

Hermit-Hadamard inequality has attracted the attention of many researchers. Many remarkable generalizations, extensions, variants and applications have been provided. Using this inequality one can find upper and lower bounds and estimations for the mean value of a convex function. Recently several Hermite-Hadamard type inequalities were obtained for various classes of functions using fractional integrals. Researchers especially used the Riemann-Liouville fractional integrals, Caputo fractional integrals and conformable fractional integrals to extend Hermit-Hadamard inequality. For the history and main results on fractional calculus and Hermite-Hadamard inequality, we refer the reader to [1, 11].

In spite of their valuable contributions to mathematical analysis, the Riemann-Liouville, Caputo, conformable fractional calculus have some deficiencies. All of the definitions satisfy the property that the fractional derivative is linear. However there are some setbacks of the definitions. In instance, if $\alpha$ is not a natural number, most of the defined fractional derivatives do not satisfy $D_a^{\alpha}(1) = 0$. Some of the fractional derivatives do not satisfy product rule for two functions. The conformable fractional derivative satisfies the common properties of the standart rules but it has some limitations. We can see the weakness of the defined fractional derivatives in [5].

A. Atangana et al in [4] proposed a suitable derivative called the Beta-derivative that allowed us to escape the lack of the fractional derivatives. We use beta-derivative introduced by Abdon Atangana in [3] to obtain our results.

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**Definition 1.1** *Let $a \in \mathbb{R}$ and let $g$ be a function such that $g : [a, \infty) \to \mathbb{R}$. Then the $\beta - derivative$ of $g$ is defined as*

$$ {}_{a}^{A}D_{t}^{\beta}\left(g(t)\right) = \begin{cases} \lim\limits_{\varepsilon \to 0} \frac{g\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-g(t)}{\varepsilon} & , for\ all\ t \geq a,\ \beta \in (0,1], \\ g(t) & , \quad for\ all\ t \geq a,\ \beta = 0, \end{cases} $$

*where $\Gamma$ is the gamma function. If the limit exists, $g$ is said to be $\beta$ -differentiable.*

For the case $\beta = 1$, we have ${}_{a}^{A}D_{t}^{\beta}\left(g\left(t\right)\right) = \frac{d}{dt}g(t)$. There is a relation between $\beta$-derivative and usual derivative.

$$ {}_{a}^{A}D_{x}^{\beta}\left(f(x)\right) = \left(x + \frac{1}{\Gamma\left(\beta\right)}\right)^{1-\beta} f'(x) $$

where $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

Beta-fractional derivative is linear

$$ {}_{a}^{A}D_{t}^{\beta}\left(\alpha f(t) + \mu g(t)\right) = \alpha {}_{a}^{A}D_{t}^{\beta}\left(f(t)\right) + + \mu {}_{a}^{A}D_{t}^{\beta}\left(g(t)\right) $$

and satisfies the properties mentioned in the followings.

$i)$ ${}_{a}^{A}D_{t}^{\beta}\left(c\right) = 0,\ c \in \mathbb{R},$

$ii)$ ${}_{a}^{A}D_{t}^{\beta}\left(f(t)g(t)\right) = g(t){}_{a}^{A}D_{t}^{\beta}\left(f(t)\right) + f(t){}_{a}^{A}D_{t}^{\beta}\left(g(t)\right),$

$iii)$ ${}_{a}^{A}D_{t}^{\beta}\left(f(t)/g(t)\right) = \frac{g(t){}_{a}^{A}D_{t}^{\beta}(f(t)) - f(t){}_{a}^{A}D_{t}^{\beta}(g(t))}{g^{2}(t)},$

$iv)$ ${}_{a}^{A}D_{t}^{\beta}\left(\left(f \circ g\right)(t)\right) = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} f'(t)g'\left(f(t)\right).$

**Definition 1.2** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the interval $(a, b)$, then the $\beta$-integral of $f$ is given as:*

$$ {}_{a}^{A}I_{t}^{\beta}\left(f(t)\right) = \int_{a}^{t}\left(x + \frac{1}{\Gamma\left(\beta\right)}\right)^{\beta-1} f(x)dx. $$

*This integral was recently reffered to as the Atangana beta-integral.*

**Definition 1.3** *The fundamental theorem of local $\beta$-calculus states that*

$$ {}_{a}^{A}I_{t}^{\beta}\left({}_{a}^{A}D_{t}^{\beta}\left(f(t)\right)\right) = f(t) - f(a) \tag{1.1} $$

*for all $t \geq a$, with given a differentiable function $f$.*

We set

$$ \int_{a}^{b} d_{\beta}x = \frac{\left(b + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \left(a + \frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta} \tag{1.2} $$

in the paper.

## 2 Main result

In this section, we give the main theorem of the paper and obtain some results close to the results in classical calculus. We first introduce Chebyschev's integral inequality with a new parameter.

**Theorem 2.1** *We assume that the functions $f$and $g$ are increasing on $[a, b]$. Then the following inequality*

$$\int_a^b f(x)g(x)d_\beta x \geq \frac{\beta}{\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta - \left(a+\frac{1}{\Gamma(\beta)}\right)^\beta} \int_a^b f(x)d_\beta x \int_a^b g(x)d_\beta x \tag{2.1}$$

*holds for $\beta \in (0, 1]$ and $a\Gamma(\beta) \geq -1$.*

**Proof.** Since the functions $f$and $g$ are both increasing on $[a, b]$, we have

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for all $x,\ y \in [a, b]$ . Integrating this inequality, we obtain

$$\begin{aligned}
S \ :&= \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y))d_\beta x d_\beta y \geq 0 \\
&= \int_a^b \int_a^b f(x)g(x)d_\beta y d_\beta x - \int_a^b \int_a^b f(x)g(y)d_\beta x d_\beta y \\
&\quad - \int_a^b \int_a^b f(y)g(x)d_\beta x d_\beta y + \int_a^b \int_a^b f(y)g(y)d_\beta x d_\beta y \\
&= \frac{1}{\beta}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta - \left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right] \int_a^b f(x)g(x)d_\beta x - \int_a^b g(y)\left(\int_a^b f(x)d_\beta x\right) d_\beta y \\
&\quad - \int_a^b f(y)\left(\int_a^b g(x)d_\beta x\right) d_\beta y + \frac{1}{\beta}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta - \left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right] \int_a^b f(y)g(y)d_\beta y \\
&= \frac{2}{\beta}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta - \left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right] \int_a^b f(x)g(x)d_\beta x - 2\int_a^b f(x)d_\beta x \int_a^b g(x)d_\beta x \geq 0.
\end{aligned}$$

This ends the proof.

**Remark 2.1** If the functions in Theorem 2.1 are both decreasing on $[a, b]$, the inequality (2.1) also holds. If one of the function is increasing and the other one is decreasing, then the inequality (2.1) is reversed.

**Theorem 2.2** *Let ${}_{a}^{A}D_{t}^{(n+1)\beta}f$ is monotonic on $[a,b]$. If ${}_{a}^{A}D_{t}^{(n+1)\beta}f$ is increasing then the following inequalities*

$$\begin{aligned}
&\int_a^b \frac{{}_{a}^{A}D_{t}^{(n+1)\beta}f(b)}{(n+1)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\left(a+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]^{n+1}d_{\beta}t \\
&\geq \int_a^b R_{n.f}(a,t)\,d_{\beta}t \qquad (2.2)\\
&\geq \int_a^b \frac{{}_{a}^{A}D_{t}^{(n+1)\beta}f(a)}{(n+1)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\left(a+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]^{n+1}d_{\beta}t
\end{aligned}$$

*hold for $0<\beta\leq 1$, where*

$$R_{n.f}(s,t)=\frac{\beta^{-n}}{n!}\int_a^b\left[\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]^{n+1}{}_{a}^{A}D_{\tau}^{(n+1)\beta}f(\tau)d_{\beta}\tau$$

*is the Taylor remainder function defined in [11].*

**Proof.** We assume that ${}_{a}^{A}D_{t}^{(n+1)\beta}f$ is increasing. Then we define the functions

$$F(t):={}_{a}^{A}D_{t}^{(n+1)\beta}f(t),\ G(t):=\frac{1}{n!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]^{n+1}.$$

We can see easily that the function $G$ is decreasing. Using Theorem 2.1, we have

$$\int_a^b F(t)G(t)d_{\beta}t\leq\frac{\beta}{\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\left(a+\frac{1}{\Gamma(\beta)}\right)^{\beta}}\int_a^b F(t)d_{\beta}t\int_a^b G(t)d_{\beta}t. \qquad (2.3)$$

Integrating the left hand side of the inequality (2.3), we obtain

$$\begin{aligned}
\int_a^b F(t)G(t)d_{\beta}t&=\int_a^b\frac{{}_{a}^{A}D_{\tau}^{(n+1)\beta}f(\tau)}{n!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]^{n+1}d_{\beta}\tau\\
&=\int_a^b R_{n.f}(a,\tau)\,d_{\beta}\tau. \qquad (2.4)
\end{aligned}$$

We have from (1.1) that

$$\int_a^b F(t)d_{\beta}t={}_{a}^{A}D_{t}^{n\beta}f(b)-{}_{a}^{A}D_{t}^{n\beta}f(a) \qquad (2.5)$$

and

$$\begin{aligned}
\int_a^b G(t)d_{\beta}t&=\int_a^b\frac{1}{(n+1)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]^{n+1}d_{\beta}\tau\\
&=\frac{1}{(n+2)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\left(a+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]^{n+2}. \qquad (2.6)
\end{aligned}$$

We replace (2.4), (2.5) and (2.6) in (2.3), we obtain

$$\int_a^b R_{n.f}(a,\tau)\, d_\beta\tau$$

$$\leq \frac{\beta}{\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta}\frac{{}_a^AD_t^{n\beta}f(b)-{}_a^AD_t^{n\beta}f(a)}{(n+2)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right]^{n+2}$$

$$\frac{{}_a^AD_t^{n\beta}f(b)-{}_a^AD_t^{n\beta}f(a)}{(n+2)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right]^{n+1}.$$

Since ${}_a^AD_t^{(n+1)\beta}f$ is increasing on $[a,b]$, we can write

$$\frac{{}_a^AD_t^{(n+1)\beta}f(a)}{(n+2)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right]^{n+2}$$

$$\leq \frac{{}_a^AD_t^{n\beta}f(b)-{}_a^AD_t^{n\beta}f(a)}{(n+2)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right]^{n+1}$$

$$\leq \frac{D^{(n+1)\beta}f(b)}{(n+2)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right]^{n+2}.$$

So we have

$$\int_a^b R_{n.f}(a,\tau)\, d_\beta\tau-\frac{{}_a^AD_t^{n\beta}f(b)-{}_a^AD_t^{n\beta}f(a)}{(n+2)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right]^{n+1}$$

$$\geq \int_a^b R_{n.f}(a,\tau)\, d_\beta\tau-\frac{{}_a^AD_t^{(n+1)\beta}f(b)}{(n+2)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right]^{n+2}.$$

Since ${}_a^AD_t^{(n+1)\beta}f$ is increasing, we have

$$\int_a^b \frac{{}_a^AD_t^{(n+1)\beta}f(b)}{(n+1)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right]^{n+1} d_\beta\tau$$

$$\geq \int_a^b R_{n.f}(a,\tau)\, d_\beta\tau$$

$$\geq \int_a^b \frac{{}_a^AD_t^{(n+1)\beta}f(a)}{(n+1)!\beta^{n+1}}\left[\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta\right]^{n+1} d_\beta\tau.$$

We obtain the right hand side of the inequality (2.2) and this ends the proof.

Using Theorem 2.2, we obtain a new type Hermite-Hadamard inequality.

**Corollary 2.1** *Let* $\beta\in(0,1]$. *If* ${}_a^AD_t^\beta f$ *is increasing on* $[a,b]$, *then*

$$\frac{\beta}{\left(b+\frac{1}{\Gamma(\beta)}\right)^\beta-\left(a+\frac{1}{\Gamma(\beta)}\right)^\beta}\int_a^b f(t)d_\beta t\leq\frac{f(a)+f(b)}{2}. \tag{2.7}$$

*If* ${}_a^AD_t^\beta f$ *is decreasing on* $[a,b]$, *then the inequality is reversed.*

We have different type of Hermite-Hadamard inequality from the paper [11].

**Lemma 2.1** *Let* $\beta \in (0,1]$ *and the function* $f : [a,b] \to \mathbb{R}$ *be* $\beta - differentiable$. *If* ${}_a^A D_t^\beta f$ *is increasing and* $f$ *is increasing on* $[a,b]$, *then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\beta}{\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \left(a+\frac{1}{\Gamma(\beta)}\right)^{\beta}} \int_a^b f(t) d_\beta t \leq f(a) + f(b) - f\left(\frac{a+b}{2}\right). \tag{2.8}$$

If we combine the two types of Hermite-Hadamard inequality given in (2.7) and (2.8), we can state the following remark.

**Remark 2.2** Let $\beta \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a $\beta - differentiable$ function. If ${}_a^A D_t^\beta f$ is increasing and $f$ is increasing on $[a,b]$, then we have the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{\beta}{\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \left(a+\frac{1}{\Gamma(\beta)}\right)^{\beta}} \int_a^b f(t) d_\beta t \leq \frac{f(a)+f(b)}{2}. \tag{2.9}$$

If we choose $\beta = 1$ in inequalities (2.9), we obtain the well-known Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\, dt \leq \frac{f(a)+f(b)}{2}.$$

**Conclusion 1** *In this paper, we have studied Taylor formula with a new parameter and have obtained new results for beta-derivative. The main theorem improves previously results and this presents a new approach to* $\beta-$ *version of Steffensen inequality and well known Hermite-Hadamard inequality.*

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