Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **40** (4), 1-9 (2020).

The behavior of solutions of nonlinear degenerate parabolic equations in nonregular domains and removability of singularity on boundary

Tahir S. Gadjiev ⋆ · **Konul Mamedova**

Received: 02.11.2019 / Revised: 28.07.2020 / Accepted: 11.09.2020

Abstract. *In this paper the behavior of solutions of the initial-boundary problem of nonlinear degenerated parabolic equations of higer order in irregular domains and removability of singularity on boundary are studied. Analogies of those known as Saint-Venant's principle in theory of elasticity are obtained.*

Keywords. Nonlinear parabolic equations, degenerate, Saint-Venant's principle, removability of singularity.

Mathematics Subject Classification (2010): 35J25, 35B40

1 Introduction

The Saint-Venant's principle is well-known in theory of elasticity. In many papers analogies of Saint-Venant's principle is used for investigation of the behavior of solutions partial differential equations. The analogies of Saint-Venant's principle are some of energy estimations for partial differential equations.

This method firstly is used in [1,2] and energy estimations for generalized solutions of second order linear equations were studied. We mention also papers [3–27].

Later in [3,4,14] quality properties of solutions nonlinear equations are investigated – a method was devoloped for obtaining "increasing lemma" depending on the geometry of a domain. Otherwise these "increasing lemma" are energy apriori estimates of type of Saint-Venant's principle.

In the study of quality properties of solutions of problems the role of the Saint-Venant's estimates is close to role of "increasing lemmas" of E.M. Landis [14] in analysis of properties continuously solutions of second order equations.

These estimates of type of Saint-Venant's principle allow us to study the behavior of solution in bounded domains with nonsmooth boundary, the behavior of solution in unbounded domains. The last gives of type of the Phragmén-Lindelöf theorems for behavior of solution in unbounded domains with noncompact boundary are obtained.

⋆ Corresponding author

T.S. Gadjiev
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az-1141, V. Bahabzade 9, Baku, Azerbaijan
E-mail: tgadiev@mail.az

K. Mamedova
Nakhchivan State University, Nakhchivan, Azerbaijan
E-mail: tgadiev@mail.az

The Saint-Venant's estimates also allow us to get uniqueness theorems of generalized solutions.

With help of the Saint-Venant's estimates we can also get some theorems about removability singularity on baundary.

Therefore, it is very important to obtain Saint-Venant's estimates based on the geometry of domains.

We investigate equations in bounded domains with nonsmooth boundary, in unbounded domains with noncompact boundaries, also removable singularity at boundary.

2 The behavior of solutions in domains with nonsmooth boundary

Let us consider bounded domain $Q_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $0 < T < \infty$, $\partial Q_T = \Gamma_0 \cup \Gamma_T \cup \Gamma$, $\Gamma_0 = \Omega \times \{0\}$, $\Gamma_T = \Omega \times \{T\}$, $\Gamma = \partial\Omega \times (0,T)$, Ω– has a nonsmooth boundary. Nonsmoothness conditions will be given using a nonlinear frequency. Using a nonlinear frequency non-smooth domains are divided into two classes: narrow and wide domains.

The space $L_p(0,T,W^m_{q,\omega}(\Omega_t))$ is determined as $\left\{u(x,t) : \int\limits_0^T \left(\|u\|_{W^m_{q,\omega}(\Omega_t)}\right)^p dt < \infty\right\}$, where $\Omega_t = Q_T \cap \{(x,\tau) : \tau = t\}$ and $W^m_{q,\omega}(\Omega_t)$- weighted Sobolev space such that weight ω belongs to the Muckenhoupt class (see [8]). The generalized solution is determined from the space $L_p(0,T,W^m_{q,\omega}(\Omega_t)) \cap W^1_2(0,T;L_2(\Omega))$.

We consider initial-boundary problem

$$\frac{\partial u}{\partial t} - \sum_{|\alpha|\leq 2m} (-1)^{|\alpha|} D^\alpha A_\alpha (x,t,u,Du,\ldots,D^m_x u) = 0, \tag{2.1}$$

$$u|_{t=0} = 0 \tag{2.2}$$

$$D^\alpha_x u\,|_\Gamma = 0,\ |\alpha| \leq m-1, \tag{2.3}$$

where $D^\alpha_x = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $n \geq 1$.

Assume that the coefficients $A_\alpha(x,t,\xi)$ are measurable with respect to $(x,t) \in Q_T$, continuously with respect to $\xi \in \mathbb{R}^M$, where M is the number of different multi-indices of length no more than m and satisfying the conditions

$$\sum_{|\alpha|\leq m} A_\alpha(x,t,\xi)\,\xi^m_\alpha \geq C\,\omega(x)\,|\xi^m|^p - C_1\omega(x)\sum_{i=1}^{m-1}|\xi_i|^p - f_1(x,t) \tag{2.4}$$

$$|A_\alpha(x,t,\xi)| \leq C_2\,\omega(x)\sum_{i=0}^{m}|\xi_i|^p + f_2(x,t), \tag{2.5}$$

where $\xi = \left(\xi^0,...,\xi^m\right), \xi^i = \left(\xi^i_\alpha\right),\ |\alpha| = i,\ \ p > 1,$

$$f_1 \in L_p\left(0,T,L_{p,loc}(\Omega_t)\right),$$

$$f_2 \in L_{1,loc}(\Omega_T).$$

The space $W^m_{q,\omega}(\Omega_t)$ is the closure of Ω_t the functions from $C^m\left(\overline{\Omega}\right)$ with respect to the norm

$$\|u\|_{W^m_{q,\omega}(\Omega_t)} = \left(\int\limits_{\Omega_t}\omega(x)\sum_{|\alpha|\leq m}|D^\alpha_x u(x,t)|^q\,dxdt\right)^{1/q}.$$

Now we describe the geometry of Q_T with nonlinear frequency $\lambda_p^p(r,\tau)$ of section $\sigma(r,\tau) = S(r) \cap \Omega_\tau$, where $S(r) = Q_T \cap \partial Q_T(r)$, $Q_T(r) = Q_T \cap B_r \times (0,T)$, $B_r = B(0,r)$ is the ball with radius r centered at 0. Thus

$$\lambda_p^p(r,\tau) = \inf \left(\int\limits_{\sigma(r,\tau)} \omega(x)|\nabla_s v|^p d\tau \right) \left(\int\limits_{\sigma(r,\tau)} \omega(x)|v|^p d\tau \right)^{-1},$$

where the lower bound is taken by all continuously differentiable functions in the vicinity of $\sigma(r,\tau)$ that vanish on ∂Q_T; $\nabla_s v(x)$ is a projection of the vector $\nabla v(x)$ on a tangential plane to $\sigma(r,\tau)$ at the point x. The function $u \in L_p(0,T; \overset{\circ}{W}{}^m{}_{p,\omega,loc}(\Omega_t)) \cap W_2^1(0,T; L_{2,loc}(\Omega_t))$ is said to be a generalized solution of the problem (2.1)-(2.2) if the integral identity

$$\int\limits_{Q_T} \frac{\partial u}{\partial t} \varphi dxdt + \int\limits_{Q_T} \sum_{|\alpha| \le m} A_\alpha(x,t,u,Du,\ldots,D^m u) D^\alpha \varphi dxdt = 0 \tag{2.6}$$

is fulfilled for the arbitrary function $\varphi \in L_p(0,T; \overset{\circ}{W}{}^m{}_{p,\omega,loc}(\Omega_t)) \cap L_2(Q_T)$.

Generally we will consider classes of domains, for which the following estimate

$$\int\limits_{S_r} \omega(x)|u(x,t)|^p dxdt \le \lambda_p^{-p}(r) \int\limits_{S_r} \omega(x)|\nabla u(x,t)|^p dxdt \tag{2.7}$$

holds. The necessary and sufficiently conditions on domains for holds estimate (2.7) is given in [9]. Using a nonlinear frequency we divide non-smooth domains into two classes:

1. The first class is the narrow domain class, i.d. complement of neighborhood of a point is sufficiently massive. For example, this class contains some cone with a vertex at this point. In terms of nonlinear frequency this class of domains satisfies the condition
$$r\lambda_p(r) > d_1 > 0, \forall r \in (0,r_0), r_0 > 0. \tag{A}$$
2. The second class is the broad domain class, i.d. those points, whose complement of neighborhood of a point is rather narrow. In terms of the nonlinear frequency, this class of regions satisfies the condition
$$r\lambda_p(r) < d_2 < \infty, \forall r \in (0,r_0). \tag{B}$$

Also the function $\varphi(r)$ is determined at $(0,r_0)$ by inequality

$$\inf_{r\psi(r)<|x|<r} \lambda_p(|x|)(r - r\psi(r))\omega(x) \ge \mu > 0, \tag{2.8}$$

where $0 < 1 - c_0 < \mu < 1$.

Let

$$J(r) = \int\limits_{\Omega_r} \omega(x)|D^m u(x,t)|^p dxdt,$$

$$G(r) = \int\limits_{\Omega_r} \Big(\sum_{|\alpha| \le m} \omega(x)\big(|F_\alpha(x,t)| + |f_2(x,t)|\big)^{\frac{p}{q}} \lambda_p^{-\frac{m-|\alpha|}{p-1}p}(|x|) + |f_1(x,t)|\Big) dxdt.$$

As mentioned above, Saint-Venant type estimates are needed first to get the required estimates. Further using various lemmas we get estimates in bounded domains with non-smooth boundary, in unbounded domains with noncompact boundary and removable singularity on boundary in bounded domains. In unbounded domains we get theorems of Phragmén-Lindelöf type.

Therefore, we give the following lemma for the case of bounded domains.

Let $0 < \varphi_1(r) < c_0 < 1$ be a measurable function on interval $(0, r_0)$ and r_0 sufficiently small.

Lemma 2.1 *Assume that $J(r)$ is a continuous nondecreasing function on $(0, r_0)$ and satisfies the inequality*

$$J(r(1-\varphi_1(r))) \leq \lambda J(r) + h(r), \forall r \in (0, r_0),$$

where $0 < \lambda < 1$ and $h(r)$ is bounded.

Then the estimation

$$J(r) \leq C \exp\left(-\delta \ln \frac{1}{\lambda} \int_r^{r_0} \frac{d\tau}{\tau \varphi_1(\tau)}\right)(J(r_0) + h(r_0)) \tag{2.9}$$

is valid for $J(r)$. Here $0 < \delta < 1 - c_0$.

The proof of this lemma can be found in [3,4,11].

We define a function $\psi(r)$ on $(0, r_0)$ by the inequlaity

$$\inf_{r\psi(r)<|x|<r} \lambda_p(|x|)(r - r\psi(r))\omega(x) \geq \mu > 0, \tag{2.10}$$

where μ is such that $0 < 1 - c_0 < \psi(r) < 1$.

Theorem 2.1 *Let $u \in L_p(0, T; \overset{\circ}{W}{}^m_{p,\omega,loc}(\Omega_t)) \cap W_2^1(0, T; L_2(\Omega_T))$ be the generalized solution of problem* (2.1)-(2.3). *Assume that coefficients satisfy the conditions* (2.4) *and* (2.5), *the domain satisfies the condition (A), the weight satisfies the Muckenhoupt condition, the function $\psi(r)$ satisfies condition* (2.10) *and $G(r)$ is bounded.*

Then for $J(r)$ the following estimate holds

$$J\left(r \exp\left(-\frac{1-\psi(r)}{1-c_0-\theta}\right)\right) \leq C \exp\left(-\theta \ln \frac{1}{\nu} \int_r^{r_0} \frac{d\tau}{\tau 1 - \psi(r)}\right)(J(r_0) + G(r_0)) \tag{2.11}$$

for every $\nu > 0$ and $\theta < 1 - c_0$.

Proof. This theorem is Saint -Venant's type estimate. For proof we substitute special test functions to integral identity (2.6) and do some calculations (see also [4]).

Corollary 2.1 *This estimate is new in the case of linear equations of following type*

$$\frac{\partial u}{\partial t} - \sum_{|\alpha| \leq m} a_\alpha(x,t) D^\alpha u = \sum_{|\alpha| \leq m} D^\alpha F_\alpha(x,t),$$

$$c_1 \omega(x) |\xi|^{2m} \leq \sum_{|\alpha|=m} a_\alpha \xi^\alpha \leq c_2 \omega(x) |\xi|^{2m}, \quad x \in \Omega, \ \xi \in \mathbb{R}^n,$$

$c_1, c_2 > 0, \ F_\alpha \in L_2(\Omega_t), \ a_\alpha \in C^{|\alpha|-m}(\overline{Q_T})$ *at $|\alpha| > m$ and $a_\alpha(x,t)$ is a measurable function such that $|\alpha| \leq m$.*

Corollary 2.2 *Let* $u \in L_p(0,T; \overset{\circ}{W}{}^m_{p,\omega,loc}(\Omega_t)) \cap W_2^1(0,T;L_2(\Omega_t))$ *be generalized solution of problem* (2.1)-(2.3) *and* $0 \in \partial\Omega$. *The domain* Ω *in the neighborhood of the point has a boundary such that* $\lambda_p(r) > \lambda^{(0)} r^{-1}$ *for any* $r \in (0, r_0)$, $\lambda^{(0)} > 0$. *Then there exists* $\gamma_0 > 0$ *such that if for* $G(r)$ *the following estimate holds*

$$G(r) \le C r^{\gamma_0 + \varepsilon} G(r_0),\ \forall r \in (0, r_0)$$

for sufficiently small ε, *the following estimate holds*

$$\omega(x) \left| D^j u(x,t) \right| \le C \left| x \right|^{m - \frac{n}{p} - j + \gamma_0} \left(J(r) + G(r) \right)^{\frac{1}{p}}, \tag{2.12}$$

$j = 0, 1, ..., \left[m - \frac{n}{p} \right]$.

The proof of this corollary follows from estimate (2.11) and embedding theorems.

3 The removable singularity of solutions

First we state an auxiliary lemma.

Lemma 3.1 *Let* $I(r)$ *be a non-negative and non-growing function on interval* $(0, r_0)$, $r_0 > 0$, *function satisfying condition*

$$I(r) \le \theta I(r\varepsilon(r)) + G(r\varepsilon(r)),\ 0 < \theta < 1, \tag{3.1}$$

where $\varepsilon(r)$ *is a measurable function,* $0 < c_0 < \varepsilon(r) < 1$ *such that*

$$k(r) = (\varphi(r))^{-1} \inf_{r\varepsilon(r) < \tau < r} \varphi(\tau) \ge \nu > 0,\ \varphi(r) \equiv 1 - \varepsilon(r) \tag{3.2}$$

and function $G(r)$ *is a measurable and locally bounded.*

Then the following holds:

1) $I(r_i) < c\, G(r_i)$ *for some sequence* $r_i \to 0$;
2) or $I(r)$ *is growing fast enough as* $r \to 0$, *namely*

$$I(r) \ge C \exp \left(ln(\theta + \delta)^{-1} \int_r^{r_0} \frac{d\tau}{\tau(1 - \varepsilon(r))} \right) I(r_0), \tag{3.3}$$

where $0 < \delta < 1 - \theta$.

See [10] for the proof of this lemma.

We will define

$$I(r) = \int_{\Omega \setminus \Omega_r} \omega(x) \left| D^m u \right|^p dxdt.$$

For small r, for the behavior $I(r)$ following theorem holds. This theorem gives apriori estimates of energy integrals.

Theorem 3.1 *Let* $u \in L_p(0,T; \overset{\circ}{W}{}^m_{p,\omega,loc}(\Omega, \Gamma)) \cap W_2^1(0,T;L_2(\Omega_T))$ *be generalized solution of problem* (2.1)-(2.3). *Assume that coefficients satisfy the conditions* (2.4) *and* (2.5), *the domain satisfies the condition* (A), *the weight satisfies the Muckenhoupt condition, the function* $\psi(r)$ *satisfies condition* (2.10) *and* $k(r)$ *satisfies condition* (3.2). *Then for* $I(r)$ *the following holds:*

1) $I(r_i) \le c(1+G(r_i))$ for some sequence $r_i \to 0$;
2) or $I(r)$ is growing fast enough as $r \to 0$, namely

$$I(r) \le C \exp\Big(ln \frac{1}{k_0+\delta} \int\limits_r^{r_0} \frac{d\tau}{\tau\psi(r)} \Big), \tag{3.4}$$

where $k_0 = const.$

Also the following theorem holds.

Theorem 3.2 *Let $u \in L_p(0,T; \overset{\circ}{W}{}^m_{p,\omega,loc}(\Omega,\Gamma)) \cap W_2^1(0,T; L_2(\Omega_T))$ be generalized solution of problem* (2.1)-(2.3)*. In the conditions of Theorem 3.1 the following inequality holds*

$$I(r) \le C \exp\left(-c \int\limits_r^{r_0} \frac{\psi(\tau)\tau^{-1}}{1+\psi(\tau)} d\tau \right), \ \forall r < r_0. \tag{3.5}$$

Then singularity set Γ of solution $u(x,t)$ removable, i.d. $u \in L_p(0,T; \overset{\circ}{W}{}^m{}_{p,\omega}(\Omega)) \cap W_2^1(0,T; L_2(\Omega))$.

The proof of Theorems 3.1 and 3.2 similarly to correspondingly theorems from [4, 10, 11].
These theorems generalize the corresponding results in [12, 14, 16–19].

4 The behavior solutions in unbounded domains. Theorem of types Phragmén-Lindelöf

Now we consider unbounded domains. First we define a measurable function $\psi(r) : 1 < \psi(r) < \infty, \forall r > r_0 > 0$ and φ_0 is continuous, non-growing function such that $\varphi_0(r) \ge r^{-1}\sup(h(r)) - 1$ at $r > r_0$. Here the upper bound is taken over all non-decreasing function $h(r) : h(r) \le r\psi(r)$ at $r_0 < r < \infty$.

Lemma 4.1 *Assume that on (r_0, ∞) non-negative, continuous function $I(r)$ satisfies inequality*

$$I(r) \le \theta I(r\psi(r)), \ \psi(r) = 1 + \varphi_0(r) \tag{4.1}$$

for all $r \in (r_0, \infty)$ and $0 < \theta < 1$. Then for any $r \in (r_0, \infty)$ the following estimate holds

$$I\left(r \exp\left(-\frac{\varphi_0(r)}{1-\nu} \right) \right) \ge \theta \exp\left(\nu \ln \theta^{-1} \int\limits_{r_0}^{r} \frac{d\tau}{\tau\varphi_0(\tau)} \right) I(r_0) \tag{4.2}$$

for all $\nu \in (0,1)$.

The proof of Lemma 4.1 is similar to the proof of the corresponding result in [11].

The unbounded domains which satisfy isoperimetric conditions divided into two classes, depending on behavior nonlinear frequency function $\lambda_p(r)$ at $r \to \infty$. First class is narrow domains and in our terms this condition is

$$r\lambda_p(r) > c > 0, \forall r > r_0 > 0. \tag{A_1}$$

Second class is wide domains and in our terms this condition is

$$r\lambda_p(r) \le c < \infty, \forall r > r_0 > 0. \tag{B_1}$$

Also we define the function $\psi(r)$ for $h_0 > 0$ by inequality

$$\inf_{r<\tau<r\psi(r)} r\lambda_p(\tau)(\psi(r)-1) \geq h_0, \psi(r) > 1, \forall r > r_0. \tag{4.3}$$

Let

$$J(r) = \int_{\Omega_r} \omega(x) |D^m u(x,t)|^p \, dxdt,$$

$$G(r) = \int_{\Omega_r} \Big(\sum_{|\alpha|\leq m} (|F_\alpha(x,t)| + |f_2(x,t)|)^{\frac{p}{p-1}} \lambda_p^{-\frac{m-|\alpha|}{p-1}p}(|x|) + |f_1(x,t)| \Big) dxdt.$$

The following theorem is true.

Theorem 4.1 *Let* $u \in L_p(0,T; \overset{\circ}{W}{}^{m}_{p,\omega(x),loc}(\Omega_t)) \cap W_2^1(0,T; L_2(\Omega_T))$ *be generalized solution of problem* (2.1)-(2.3)*,* $\omega(x)$ *be in Muckenhoupt classes. Assume that coefficients satisfy the conditions* (2.4) *and* (2.5)*. Moreover domain* Ω *is narrow enough in the sense that* $\lambda_p(r) > \delta^{-1}$ *at* (r_0, ∞)*. Let* $\psi(r)$ *be any function which satisfy condition* (4.3) *and* $\varphi_0(r)$ *be the function under construction by* $\psi(r)$ *as in Lemma 4.1. Then for* $J(r)$ *the following hold:*

1.

$$\lim_{r\to\infty} \frac{J(r)}{G(r)} < \infty; \tag{4.4}$$

2. Or

$$J\Big(r\exp\Big(\frac{\varphi_0(r)}{1-\nu}\Big)\Big) \geq \theta \exp\Big(\nu \ln\frac{1}{\theta}\int_{r_0}^{r} \frac{d\tau}{\tau\varphi_0(\tau)}\Big) J(r_0) \tag{4.5}$$

for all $\nu \in (0,1)$ *and for all* $r > r_0$ *at big enough* r_0*.*

The proof of Theorem 4.1 is similar the proof of the corresponding result in [11].

Corollary 4.1 *This is Phragmén-Lindelöf type theorem. Also we can give this type theorems for wide classes of domains and for integrals of functions.*

Corollary 4.2 *Many examples can be given showing the sharpness of the results obtained in different domains.*

Acknowledgements
The authors thank the referee(s) for careful reading the paper and useful comments.

References

1. Oleinik, O.A., Iosif'yan, G.A.: *Boundary value problems for second order elliptic equations in unbounded domains and Saint-Venant's principle*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (2), 269-290 (1977)
2. Oleinik, O.A., Iosif'yan, G.A.: *Removable boundary singularities and uniqueness of solutions of boundary value problems for second order elliptic and parabolic equations*, (Russian) Funkcional. Anal. i Priloen. **11** (3), 54-67 (1977)
3. Tedeev, A.F., Shishkov, A.E.: *Qualitative properties of solutions and subsolutions of quasilinear elliptic equations*, Izv. Vyssh. Uchebn. Zaved. Mat. (Russian) **1**, 62-68 (1984)

4. Gadjiev, T.S.: *On the behavior of solutions of a mixed boundary value problem for higher-order quasilinear elliptic equations of divergence type*, (Russian) Differentsial'nye Uravneniya **26** (4), 703706 (1990)
5. Bonafade, S.: *Quazilinear degenerate elliptic variational inequalities with discontinuous coefficients*, Comment. Math. Univ. Carolinae **34** (1), 55-61 (1993)
6. Lancaster, K., Stanley, J.: *On the asymptotic behavior of solutions of quasilinear elliptic equations*, Ann. Univ. Ferrara-Sez. VII-Sc. Mat., vol. IL, 85-125 (2003)
7. Jorge, Garcia-Melian: *Boundary behavior for large solutions to elliptic equations with singular weights*, Nonlinear Analysis: Theory, Methods and Applications **67** (3), 818-826 (2007)
8. Chanillo, S., Wheeden, R.: *Weighted Poincare and Sobolev inequalities and estimates for weighted Peano maximal functions*, Mer. J. Math. **107**, 1191-1226 (1985)
9. Miklyukov, V.M.: *On asymptotic properties of subsolutions of elliptic type quasilinear equations*, (Russian) Mat. Sbor. **111**(145), No 1, 42-66 (1980)
10. Gadjiev, T.S., Sadigova, N.R., Rasulov, R.A.: *Removable singularities of solutions of degenerate nonlinear elliptic equations on the boundary of a domain*, Nonlinear Anal. **74** (16), 5566-5571 (2011)
11. Gadjiev, T.S., Mamedova, K.N.: *On behavior of solutions degenerate parabolic equations*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **32** (4), Mathematics and Mechanics, 43-50 (2012)
12. Carleson, L.: *Selected Problems on Exceptional Sets*, D. Van Nostrand Company, Toronto, London, Melbourne (1967)
13. Kondratyev, V.A., Oleynik, O.A.: *On the behavior of generalized solutions of the Dirichlet problem for high-order elliptic equations in the neighborhood of a boundary*, (Russian) Boundary value problems of mathematical physics and related questions in the theory of functions, 14. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **115**, 114-125 (1982)
14. Kondratyev, V.A., Landis, E.M.: *Qualitative theory of partial differential equations of second order*, (Russian) (Russian) Partial differential equations, 3 (Russian), 99-215, **220**, Itogi Nauki i Tekhniki, Sovrem. Probl. Mat. Fund. Naprav., 32, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow (1988)
15. Gilbarg, D., Trudinger, N.: *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, New York (1977)
16. Gadjiyev, T.S., Mamedova, V.: *On removable sets of solutions of the second order elliptic and parabolic equations in nondivergent form*, Ukrainian Math. J. **61** (11), 1743-1756 (2009)
17. Diederich, J.: *Removable singularities of solutions of elliptic partial differential equations*, Trans. Amer. Math. Soc. **165**, 333-352 (1972)
18. Harvey, R., Polking, J.: *Removable singularities of solutions of linear partial differential equations*, Acta Math. **125**, 39-56 (1970)
19. Kilpelainen, T., Zhong, X.: *Removable sets for continuous solutions of quasilinear elliptic equations*, Proc. Amer. Math. Soc. **130** (6), 1681-1688 (2002)
20. Chanillo, S., Wheeden, R.: *Weighted Poincare and Sobolev inequalities and estimates for weighted Peano maximal functions*, Amer. J. Math. **107**, 1191-1226 (1985)
21. Miklyukov, V.M.: *Asymptotic properties of subsolutions of quasilinear equations of elliptic type and mappings with bounded distortion*, (Russian) Mat. Sb. (N.S.) **111** (153) (1), 4266 (1980)
22. Gadjiyev, T.S., Mamedova, K.N.: *On behavior of solutions degenerate parabolic equations*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **32** (4), Mathematics and Mechanics, 43-50 (2012)
23. Alikakos, N.D., Rostamian, R.: *Gradient estimates for degenerate diffusion equations*, Math. Ann. **259** (1), 5370 (1982)

24. Guliyev, V.S., Gadjiev, T.S., Galandarova, Sh.: *Dirichlet boundary value problems for uniformly elliptic equations in modified local generalized Sobolev-Morrey spaces*, Electron. J. Qual. Theory Differ. Equ. Paper No. 71, 17 pp. (2017)
25. Gadjiev, T.S., Galandarova, Sh., Guliyev, V.S.: *Regularity in generalized Morrey spaces of solutions to higher order nondivergence elliptic equations with VMO coefficients*, Electron. J. Qual. Theory Differ. Equ. Paper No. **55**, 17 pp. (2019)
26. Gadjiev, T.S., Guliyev, V.S., Suleymanova, K.G.: *The Dirichlet problem for the uniformly elliptic equation in generalized weighted Morrey spaces*, Studia Sci. Math. Hungar. **57** (1), 68-90 (2020)
27. Guliyev, V.S., Gadjiev, T.S., Serbetci, A.: *The Dirichlet problem for the uniformly higher-order elliptic equations in generalized weighted Sobolev-Morrey spaces*, Nonlinear Stud. **26** (4), 831-842 (2019)