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Some differential properties of grand generalized Sobolev-Morrey spaces

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Abstract. In this paper we introduce a grand generalized Sobolev-Morrey spaces. Also, with the help of integral representations we study differential properties of functions from this spaces.

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1 Introduction and preliminary notes

This paper is devoted to investigations of embedding theorems for the grand generalized Sobolev-Morrey spaces

$$\bigcap_{i=0}^{n} L^{<l^i>}_{p^{i)},\varkappa,a}(G), \tag{1.1}$$

where $G \subset \mathbb{R}^n$ is a bounded domain, $1 < p^i < \infty$; $l^i \in N_0^n$ $(i = 0, 1, 2, \dots, n)$, $l_j^0 \geq 0$,$(j = 1, 2, \dots, n)$; $l_j^i \geq 0$, $(i = 0, 1, 2, \dots, n)$, $l_i^i > 0$ are non-negative integers $(i = 0, 1, 2, \dots, n)$; $a \in [0, 1]$; $\varkappa \in (0, \infty)^n$. First we introduce a grand generalized Sobolev-Morrey, and on the based of the method of integral representation we study some differential properties of functions, defined on n-dimensional domains satisfying flexible φ- horn condition (see, [12]).

Note that the grand Lebesgue spaces $L_{p)}(G)$ for a measurable set $G \subset \mathbb{R}^n$ of finite Lebesgue measure were introduced in the work of T. Iwaniec and C. Sbordone in [3]. After a vast amount of research about small Lebesgue-Morrey space, grand and small Sobolev spaces, grand grand Lebesgue-Morrey space, grand grand and small small Sobolev-Morrey spaces, grand grand Nikolskii-Morrey spaces, and grand Sobolev-Morrey spaces with dominant mixed derivatives (with different norms) has been done by many mathematicians (see,e.g. [2,4–8,10,11,13–21]) .

Let $G \subset \mathbb{R}^n$ is a bounded domain, $t > 0$, and let for all $x \in \mathbb{R}^n$

$$G_{t^{\varkappa}}(x) = G \cap \Big\{ y : |y_j - x_j| < \frac{1}{2} t^{\varkappa_j}, j = 1, 2, \dots, n \Big\}.$$

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Definition 1.1 *We denote by* $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varkappa,a}(G)$ *the space of locally summable functions* f *on* G *having weak derivatives* $D^{l^i}f$ $(i=0,1,\dots,n)$ *with the finite norm*

$$\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a}(G)} = \sum_{i=0}^{n} \|D^{l^i}f\|_{p^i),\varkappa,a;G}. \tag{1.2}$$

Here

$$\|f\|_{p^i),\varkappa,a;G} = \|f\|_{L_{p^i),\varkappa,a}(G)}$$

$$= \sup_{\substack{0<t\le d_0,\\ x\in G,\\ 0<\in\varepsilon<p-1}} \left(\frac{1}{t^{|\varkappa|a}}\frac{\varepsilon}{|G_{t^\varkappa}(x)|}\int_{G_{t^\varkappa}(x)} |f(y)|^{p^{i-\varepsilon}}dy\right)^{\frac{1}{p^i-\varepsilon}}, \tag{1.3}$$

and $d_0 = diam\,G$ *.*

Note that the space $L_{p),a,\varkappa}(G)$ defined and studied in [13], and $\bigcap\limits_{i=0}^{n} L_{p^i),\varkappa,a}(G)$, in the case $l^0=(0,\dots,0)$, $l^i=(0,\dots,0,l_i,0,\dots,0)$, $p^i=p(i=0,1,\dots,n)$ coincides with the grand Sobolev-Morrey space $W^l_{p),a,\varkappa}(G)$ introduced and studied in [13].

We give some properties of the spaces $\bigcap\limits_{i=0}^{n} L^{<l>}_{p^i),a,\varkappa}(G)$.

1.The following embedding holds:

$$\bigcap_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a}(G) \hookrightarrow \bigcap_{i=0}^{n} L^{<l^i>}_{p^i}(G),$$

i.e.,

$$\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i)}(G)} \le C\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a}(G)}. \tag{1.4}$$

Norm in grand generalized Sobolev space $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i)}(G)$ defined as:

$$\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i)}(G)} = \sum_{i=0}^{n} \|D^{l^i}f\|_{p^i),G},$$

$$\|f\|_{p^i),G} = \|f\|_{L_{p^i)}(G)} = \sup_{0<\varepsilon<p^i-1}\left(\frac{\varepsilon}{|G|}\int_G |f(x)|^{p^i-\varepsilon}dx\right)^{\frac{1}{p^i-\varepsilon}};$$

2. The spaces $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a}(G)$ is complete;

3. $\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,0}(G)} = \|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i)}(G)}$.

To prove the main theorems, we need some auxiliary inequalities given in the lemma below. Let $\psi(\cdot,y,z)\in C_0^\infty(\mathbb{R}^n)$ be such that

$$S(\psi) = supp\ \psi \subset I_1 = \left\{x : |x_j| < \frac{1}{2}, j=1,2,\dots,n\right\},$$

and let $0 < T \leq 1$, $\lambda = (\lambda_1, \ldots, \lambda_n), \lambda_j > 0 (j = 1, 2, \ldots, n)$ and put

$$V = \bigcup_{0<t\leq T} \Big\{ y : \Big(\frac{y}{t^\lambda}\Big) \in S(\psi) \Big\}.$$

Clearly, $V \subset I_{T^\lambda} = x; |x_j| < \frac{1}{2} T_j^\lambda, j = 1, 2, \ldots, n$, and let U be an open set contained in the domain G; henceforth we always assume that $U + V \subset G$, and put $G_{T^\varkappa}(U) = U + I_{T^\varkappa(x)} \cap G$.

Obviously, if $0 < \varkappa_j \leq \lambda_j (j = 1, 2, \ldots, n)$, then $I_{T^\lambda} \subset I_{T^\varkappa}$ and therefore $U + V \subset G_{T^\varkappa}(U) = Z$.

Lemma 1.1 *Let* $1 < p^i < p \leq r \leq \infty$*;* $0 < |\varkappa| \leq \frac{|\lambda|}{1+a}$*;* $0 < t, \eta \leq T \leq 1$*;* $0 < \gamma < \gamma_0$*;* $\nu = (\nu_1, \nu_2, \ldots, \nu_n), \nu_j \geq 0$ *are integers* $(j = 1, 2, \ldots, n)$*;* $\varphi \in L_{p^i), \varkappa, a}(G)$ *and*

$$m^i = (l^i, \lambda) - (\nu, \lambda) - (|\lambda| - |\varkappa| - |\varkappa| a) \left(\frac{1}{p^i - \varepsilon} - \frac{1}{p - \varepsilon} \right), \tag{1.5}$$

$$(\nu, \lambda) = \sum_{j=1}^n \nu_j \lambda_j, |\lambda| = \sum_{i=1}^n \lambda_j,$$

$$R_\eta^i(x) = \int_0^\eta t^{-1-|\lambda|-(\nu,\lambda)+(l^i,\lambda)} dt \int_{\mathbb{R}^n} \varphi(x+y) \psi(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)) dy, \tag{1.6}$$

$$R_{\eta,T}^i(x) = \int_\eta^T t^{-1-|\lambda|-(\nu,\lambda)+(l^i,\lambda)} dt \int_{\mathbb{R}^n} \varphi(x+y) \psi(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)) dy. \tag{1.7}$$

Then

$$\sup_{\bar{x} \in U} \|R_\eta^i\|_{p-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} \leq C^1 \|\varphi\|_{p^i), \varkappa, a, Z} \, \varepsilon^{-\frac{1}{p^i - \varepsilon}} \gamma^{|\varkappa| \frac{a+1}{p^i - \varepsilon}} \eta^{m_i} \; (m^i > 0), \tag{1.8}$$

$$\sup_{\bar{x} \in U} \|R_{\eta,T}^i\|_{p-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} \leq C^2 \|\varphi\|_{p^i), \varkappa, a, Z} \varepsilon^{-\frac{1}{p^i - \varepsilon}} \gamma^{|\varkappa| \frac{a+1}{p^i - \varepsilon}}$$

$$\times \begin{cases} T^{m^i} \textit{ for } m^i > 0, \\ \ln \frac{T}{\eta} \textit{ for } m^i = 0, \\ \eta^{m^i} \textit{ for } m^i < 0. \end{cases} \tag{1.9}$$

Here $U_{\gamma^\varkappa} \bar{x} = \{x : |x_j - \bar{x}_j| < \frac{1}{2} \gamma^{\varkappa_j}, j = 1, 2, \ldots, n$ *and* C^1 , C^2 *are constants independent of* φ, γ, η, T *and* ε .

Proof. Applying the generalized Minkowskii inequality for any $\bar{x} \in U$, we get

$$\|R_\eta^i\|_{p-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} \leq C \int_0^{\eta_0} t^{-1-|\lambda|-(\nu,\lambda)+(l^i,\lambda)} \|\phi\|_{p-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} dt, \tag{1.10}$$

where

$$\phi(x, t) = \int_{\mathbb{R}^n} \varphi(x+y) \Psi \left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho^1(t^{\lambda, x}) \right) dy. \tag{1.11}$$

From Hölder's inequality $(p \leq r)$, we have

$$\|\phi(.,t)\|_{p-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})} \leq \|\phi(.,t)\|_{r-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}\gamma^{|\varkappa|\left(\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}\right)}. \tag{1.12}$$

Now we estimate the norm $\|\phi(.,t)\|_{p-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}$.

Let χ be the characteristic function of $S(\Psi)$ and let $\Psi(x,y,z)$ satisfy condition $|\Psi(x,y,z)| \leq C|\Psi_1(x)|$, for a.e. $(y,z) \in \mathbb{R}^n \times \mathbb{R}^n$.

Account into that $1 < p^i < r \leq \infty$, $s \leq r \left(\frac{1}{s} = 1 - \frac{1}{p^i-\varepsilon} + \frac{1}{r-\varepsilon}\right)$, represent the integrand function (1.11) as the form

$$\|\varphi\Psi\| = (|\varphi|^{p^i-\varepsilon}|\Psi|^{\frac{1}{s}})^{\frac{1}{p^i-\varepsilon}})(|\Psi|^{p^i-\varepsilon}\chi)^{\frac{1}{p^i-\varepsilon}-\frac{1}{r-\varepsilon}} - (|\Psi|^s)^{\frac{1}{s}-\frac{1}{r-\varepsilon}}$$

and applying Hölders inequality $\left(\frac{1}{r-\varepsilon} + \left(\frac{1}{p^i-\varepsilon} - \frac{1}{r-\varepsilon}\right) + \left(\frac{1}{s} - \frac{1}{r-\varepsilon}\right) = 1\right)$, we have

$$\|\phi(\cdot,t)\|_{r-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})} \leq \sup_{\bar{x}\in U_{\gamma^{\varkappa}}(\bar{x})} \left(\int_{\mathbb{R}^n} |\varphi(x+y)|^{p^i-\varepsilon}\chi\left(\frac{y}{t^{\lambda}}\right)dy\right)^{\frac{1}{p^i-\varepsilon}-\frac{1}{r-\varepsilon}}$$

$$\times \sup_{\bar{y}\in V}\left(\int_{U_{\gamma^{\varkappa}}(\bar{x})} |\varphi(x+y)^{p^i-\varepsilon}dx\right)^{\frac{1}{p^i-\varepsilon}} \left(\int_{\mathbb{R}^n} |\Psi_1\left(\frac{y}{t^{\lambda}}\right)|^s dy\right)^{\frac{1}{s}}. \tag{1.13}$$

For any $x \in U$, we have

$$\int |\varphi(x+y)|^{p^i-\varepsilon}\chi\left(\frac{y}{t^{\lambda}}\right)dy \leq \int_{Z_{t^{\lambda}}(x)} |\varphi(y)|^{p^i-\varepsilon}dy$$

$$\leq \int_{Z_{t^{\varkappa}}(x)} |\varphi(y)|^{p^i-\varepsilon}dy \leq \|\varphi\|^{p^i-\varepsilon}_{p^i-\varepsilon,Z_{t^{\lambda}}(x)} \leq \|\varphi\|^{p^i-\varepsilon}_{p),\varkappa,a;Z}\,\varepsilon^{-1}\,\gamma^{|\varkappa|(a+1)}. \tag{1.14}$$

Also, for $y \in U$

$$\int_{U_{\gamma^{\varkappa}}(\bar{x})} |\varphi(x+y)|^{p^i-\varepsilon}dx \leq \int_{U_{\gamma^{\varkappa}}(\bar{x}+y)} |\varphi(x)|^{p^i-\varepsilon}dx \leq \|\varphi\|^{p^i-\varepsilon}_{p),\varkappa,a;Z}\varepsilon^{-1}\gamma^{|\varkappa|(a+1)}, \tag{1.15}$$

$$\int_{\mathbb{R}^n} |\Psi(\frac{y}{t^{\lambda}})|^s dy = t^{|\lambda|}\|\Psi_1\|^s_s. \tag{1.16}$$

From inequalities (1.12)-(1.16) for $(r = p)$ it is implies that

$$\|\phi(\cdot,t)\|_{p-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}$$

$$\leq C_1\|\Psi_1\|_s\|\varphi\|_{p^i),\varkappa,a;Z}\varepsilon^{-\frac{1}{p^i-\varepsilon}}\,\gamma^{|\varkappa|\frac{a+1}{p^i-\varepsilon}}\,t^{|\lambda|-(|\lambda|-|\varkappa|-|\varkappa|a)(\frac{1}{p^i-\varepsilon}-\frac{1}{p-\varepsilon})}.$$

Take into account last inequality in (1.10) for all $x \in U$, we have

$$\|R^i_{\eta}\|_{p-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})} \leq C^1\|\varphi\|_{p^i),\varkappa,a;Z}\,\varepsilon^{-\frac{1}{p^i-\varepsilon}}\,\gamma^{|\varkappa|\frac{a+1}{p^i-\varepsilon}}\eta^{m^i}\ (m^i > 0). \tag{1.17}$$

Similarly, we can prove (1.9).

This complete the proof.

2 Main results

Now we proved theorems for functions from grand generalized Sobolev-Morrey spaces $\bigcap_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a}(G)$.

Theorem 2.1 *Let $G \subset \mathbb{R}^n$ be an open bounded set satisfying the flexible λ-horn condition (see [1]); $1 < p^i < p \leq \infty$;$|\varkappa| \leq \frac{|\lambda|}{1+a}$; $\nu = (\nu_1, \nu_2, \dots, \nu_n)$, $\nu_j \geq 0$ are integers $(j = 1, 2, \dots, n)$; $m^i > 0\, (i = 1, 2, \dots, n)$, and $f \in \bigcap_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a}(G)$.*

Then the following embeddings hold

$$D^\nu : \bigcap_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a}(G) \hookrightarrow L_{p-\varepsilon}(G).$$

More precisely, for $f \in \bigcap_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a}(G)$ there exists a generalized (weak) derivatives $D^\nu f$ and

$$\|D^\nu f\|_{p-\varepsilon,G} \leq C^1(\varepsilon) \sum_{i=0}^{n} T^{m^i} \|D^{l^i} f\|_{p^i),\varkappa,a;G}. \tag{2.1}$$

In particular, if $m^{i,0} = (l^i, \lambda) - (\nu, \lambda) - (|\lambda| - |\varkappa| - |\varkappa|a)\frac{1}{p^i-\varepsilon} > 0$ $(i = 0, 1, 2, \dots, n)$, then $D^\nu f$ are continuous on G and

$$\sup_{x\in G} |D^\nu f(x)| \leq C^1(\varepsilon) \sum_{i=0}^{n} T^{m^{i,0}} \|D^{l^i} f\|_{p^i),\varkappa,a;G}, \tag{2.2}$$

where $0 < T \leq d_0$, $C^1(\varepsilon) = C^1 \varepsilon^{-\frac{1}{p^i-\varepsilon}}$ and C^1 is a constant independent of f, T and ε .

Proof. At first note that in the conditions of Theorem 2.1 there exists a generalized derivatives $D^\nu f$ on G. Indeed, from the condition $m^i > 0(i = 1, \dots, n)$ it follows that for $p^i < p, |\varkappa| \leq \frac{|\lambda|}{1+a}$, and

$$\bigcap_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a(G)} \hookrightarrow \bigcap_{i=0}^{n} L^{<l^i>}_{p^i)(G)} \hookrightarrow \bigcap_{i=0}^{n} L^{<l^i>}_{p^i-\varepsilon}(G)\ (p^i - \varepsilon > 1).$$

Then $D^\nu f$ exists on G and belongs to $L_{p^i-\varepsilon}(G)$ and for almost each $x \in G$ the integral identity is hold

$$D^\nu f(x) = f^{(\nu)}_{T^\lambda}(x) + \int_0^T \int_{\mathbb{R}^n} \sum_{i=1}^{n} t^{-1-|\lambda|-(\nu,\lambda)+(l^i,\lambda)} D^{l^i} f(x+y)$$

$$\times M^{(\nu)}_i\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)\right) dy dt, \tag{2.3}$$

$$f^{(\nu)}_{T^\lambda}(x) = (-1)^{|\nu|} T^{-|\lambda|-(\nu,\lambda)+(l^0,\lambda)}$$

$$\times \int_{\mathbb{R}^n} D^{l^0} f(x+y) \Omega^{(\nu)}\left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x)\right) dy, \tag{2.4}$$

where $0 < T \leq \min(d_0, T_0)$, $\Omega(\cdot, y, z)$ and $M_i(\cdot, y, z) \in C_0^\infty(\mathbb{R}^n)$(see, [9]). Recall that the support of the integral representations (2.3) and (2.4) is $V(\lambda)$. Applying the Minkowskii inequality, from (2.3) and (2.4), we get

$$\|D^\nu f\|_{p-\varepsilon,G} \leq \|f^\nu_{T^\lambda}\|_{p-\varepsilon,G} + \sum_{i=1}^n \|R^i_T\|_{p-\varepsilon,G} \tag{2.5}$$

By (1.8) for $U = G$, $D^{l^0} f = \phi$, $\Psi = \Omega^{(\nu)}$, $\eta = T$, one has

$$\|f^\nu_{T^\lambda}\|_{p-\varepsilon,G} \leq C_1(\varepsilon)\|D^{l^0} f\|_{p^0),\varkappa,a,G}\, T^{m^0} \tag{2.6}$$

and for $U = G$, $D^{l^i} = \phi$,$\Psi = M_i^{(\nu)}$, $\eta = T$, we have

$$\|R^i_T\|_{p-\varepsilon,G} \leq C_2(\varepsilon)\|D^{l^i} f\|_{p^i),\varkappa,a;G}\, T^{m^i}. \tag{2.7}$$

Substituting (2.6) and (2.7) in (2.5), we get (2.5).

Let $m^{i,0} > 0 (i = 1, 2, \dots, n)$. By (2.3), (2.4) from inequality (2.5) for $p = \infty$ and $m^i(p = \infty) = m^{i,0}(i = 1, \dots, n)$, we obtain

$$\|D^\nu f - f^\nu_{T^\lambda}\|_{\infty,G} \leq \sum_{i=1}^n T^{m^{i,0}} \|D^{l^i} f\|_{p^i),\varkappa,a;G}.$$

It follows that the left hand side of the last inequality tends to zero as $T \to 0$. Since $f^\nu_{T^\lambda}$ is continuous on G, the convergence in $L_\infty(G)$ coincides with usually uniform convergence. Then the limit function $D^\nu f$ is continuous on G .

Theorem 2.1 is proved.

Let ρ be an n-dimensional vector.

Theorem 2.2 *Let all the conditions of Theorem 2.1 be fulfilled. If $m^i > 0 (i = 1, 2, \dots, n)$, then $D^\nu f$ satisfies the Hölder condition with exponent σ on G in the metric of $L_{p-\varepsilon}$. More exactly,*

$$\|\Delta(\zeta, G)D^\nu f\|_{p-\varepsilon,G} \leq C^2(\varepsilon)\|f\|_{\bigcap\limits_{i=0}^n L^{<l^i>}_{p^i),\varkappa,a}(G)} |\zeta|^\sigma, \tag{2.8}$$

where $C^2(\varepsilon) = C^2 \varepsilon^{-\frac{1}{p^i-\varepsilon}}$ and C^2 is a constant independent of f and ε and σ is an arbitrary number satisfying the inequalities:

$$0 \leq \sigma \leq 1, \text{ if } \frac{m^0}{\lambda_0} > 0,$$

$$0 \leq \sigma < 1, \text{ if } \frac{m^0}{\lambda_0} = 1,$$

$$0 \leq \sigma \leq \frac{m^0}{\lambda_0}, \text{ if } \frac{m^0}{\lambda_0} < 1,$$

where $m^0 = \min_{1\leq i\leq n} m^i$; $\lambda_0 = \max_{1\leq i\leq n} \lambda_i$.

If $m^{i,0} > 0 (i = 1, 2, \dots, n)$, then

$$\sup_{x\in G} |\Delta(\zeta, G)D^\nu f(x)| \leq C^2(\varepsilon)\|f\|_{\bigcap\limits_{i=0}^n L^{<l^i>}_{p^i),\varkappa,a}(G)} |\zeta|^{\sigma_0}, \tag{2.9}$$

where σ_0 satisfy the same conditions as σ with $m^{i,0}$ instead of m^i.

Proof. According to Lemma 8.6 in [1], there exists a domain $G_\sigma \subset G(\sigma = \xi r(x), r = \rho_\lambda(x, \partial G), x \in G)$.

Suppose that $|\zeta|_\lambda < \sigma$, then for any $x \in G_\sigma$ the segment joining the points $x, x+\zeta$ is contained in G. Consequently, for all the points of this segment, identities (2.3) and (2.4) with the same kernels are valid. After same transformations, from (2.3) and (2.4), we get

$$|\Delta(\zeta, G)D^\nu f(x)| \le T^{-|\lambda|-(\nu,\lambda)+(l^0,\lambda)}$$

$$\times \int_{\mathbb{R}^n} D^{l^0} f(x+y) \left| \Omega^{(\nu)}\left(\frac{y-\zeta}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x)\right)\right.$$

$$\left. -\Omega^{(\nu)}\left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x)\right)\right| dy$$

$$+\sum_{i=1}^n \int_0^{|\zeta|^{\frac{1}{\lambda_0}}} t^{-1-|\lambda|-(\nu,\lambda)+(l^i,\lambda)} \int_{\mathbb{R}^n} \left(|D^{l^i} f(x+y+\zeta)| + |D^{l^i} f(x+y)|\right)$$

$$\times |M_i^{(\nu)}(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)| dy dt$$

$$+\int_{|\zeta|^{\frac{1}{\lambda_0}}}^T t^{-1-|\lambda|-(\nu,\lambda)+(l^i,\lambda)} \int_{\mathbb{R}^n} |D^{l^i} f(x+y)| \left| M_i^{(\nu)}\left(\frac{y-\zeta}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)\right)\right.$$

$$\left. -M_i^{(\nu)}\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)\right)\right| dy dt$$

$$= R(x,\zeta) + \sum_{i=0}^n (R_{1,i}(x,\zeta) + R_{2,i}(x,\zeta)), \tag{2.10}$$

where $0 < T \le d_0$. We also assume that $|\zeta|_\lambda < T$, and consequently $|\zeta|_\lambda \le \min\{\sigma, T\}$. If $x \in G \setminus G_\sigma$, then $\Delta(\zeta, G)D^\nu f(x) = 0$. By (2.10)

$$\|\Delta(\zeta, G)D^\nu f\|_{p-\varepsilon, G} = \|\Delta(\zeta, G)D^\nu f\|_{p-\varepsilon, G_\sigma}$$

$$\le \|R(\cdot,\zeta)\|_{p-\varepsilon, G_\sigma} + \sum_{i=0}^n (\|R_{1,i}(\cdot,\zeta)\|_{p-\varepsilon, G_\sigma} + \|R_{2,i}(\cdot,\zeta)\|_{p-\varepsilon, G_\sigma}). \tag{2.11}$$

Note that

$$\left|\Omega^{(\nu)}\left(\frac{y-\zeta}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x)\right) - \Omega^{(\nu)}\left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x)\right)\right|$$

$$\le \sum_{i=0}^n T^{-\lambda_j} \int_0^{|\zeta|} \left| D_j \Omega^{(\nu)}(\frac{y-\xi e_\zeta}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x))\right| d\xi,$$

$e_\zeta = \frac{\zeta}{|\zeta|}$. Therefore

$$R(x,\zeta) \le \sum_{j=1}^n T^{-\lambda_j-|\lambda|-(\nu,\lambda)+(l^i,\lambda)}$$

$$\times \int_0^{|\zeta|} d\xi \int_{\mathbb{R}^n} \left|D^{l^0} f(x+\xi e_\zeta + y)\right| \left|D_j \Omega^{(\nu)}(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x))\right| dy.$$

Similarly, we get

$$R_{2,i}(x,\zeta) \leq \sum_{j=1}^{n} \int_0^{|\zeta|} t^{-1-|\lambda|-(\nu,\lambda)-\lambda_j+(l^i,\lambda)} dt$$

$$\times \int_{\mathbb{R}^n} \left|D^{l^i} f(x+\xi e_\zeta + y)\right| \left|D_j M_i^\nu (\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x))\right| dy.$$

Taking into account $\xi e_\zeta + G_\sigma \subset G$, by inequality (1.8) for $U = G$, $D^{l^0} f = \phi$, $\Omega^{(\nu)} = \psi$, and $p^i = p^0$ one has

$$\|R(\cdot,\zeta)\|_{p-\varepsilon,G_\delta} \leq C_1(\varepsilon)\|\zeta|\|D^{l^0} f\|_{p^0),\varkappa,a;G}. \tag{2.12}$$

By (1.9) for $U = G$,$t = T$, $D^{l^i} f = \phi$, $M_i^\nu = \Psi$, $\eta = |\zeta|^{\frac{1}{\lambda_0}}$, we have

$$\|R_{1,i}(\cdot,\zeta)\|_{p-\varepsilon,G_\sigma} \leq C_2(\varepsilon)|\zeta|^{\frac{m^i}{\lambda_0}} \|D^{l^i} f\|_{p^i),\varkappa,a;G}. \tag{2.13}$$

Also from (1.10) for $U = G$, $\eta = |\zeta|^{\frac{1}{\lambda_0}}$, $D^{l^i} f = \varphi$, $M^{(\nu)} = \psi$, we have

$$\|R_{2,i}(\cdot,\zeta)\|_{p-\varepsilon,G_\sigma} \leq C_2(\varepsilon)|\zeta|^\sigma \|D^{l^i} f\|_{p^i),\varkappa,a;G}. \tag{2.14}$$

It follows from (2.11) - (2.14) that

$$\|\Delta(\zeta,G)D^\nu f\|_{p-\varepsilon,G_\sigma} \leq C_3(\varepsilon)\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i),\varkappa,a}(G)} |\zeta|^\sigma|.$$

Estimating $\|D^\nu f\|_{p-\varepsilon,G}$ by means of (2.1), in this case we get estimation (2.8).
This complete the proof.

References

1. Besov O.V., Ilyin V.P., Nikolskii S.M.: *Integral representations of functions and embeddings theorems*, M. Nauka, 480 p. (1996).
2. Fiorenza A., Karadzhov C.E.: *Grand and small Lebesgue spaces and their analogs*, J. Anal. Appl. **23** (4) 657–681 (2004)
3. Iwaniec T., Sbordone C.: *On the integrability of the Jacobian under minimal hypotheses*, Arch. Ration. Mech. Anal. **119** 129–143 (1992).
4. Kokilashvili V.: *The Riemann boundary value problem for analytic functions in the frame of grand L_p spaces*, Bull. Georgian Nat. Acad. Sci. **4** (1) 5–7 (2010).
5. Kokilashvili V., Meskhi A.: *Trace inequalities for fractional integrals in grand Lebesgue spaces*, Studia Math. **210** (2), 159–176 (2012)
6. Kokilashvili V., Meskhi A., Rafeiro H.: *Estimates for nondivergence elliptic equations with VMO coefficients in generalized grand Morrey spaces*, Complex Var. Elliptic Equ. **8** (59), 1169–1184 (2014)
7. Meskhi A.: *Maximal functions, potentials and singular integrals in grand Morrey spaces*, Complex Var. Elliptic Equ. 534–793 (2011) http://dx.doi.org/10.1080/17476933:2010.
8. Mizuta Y., Ohno T.: *Trudingers exponential integrability for Riesz potensials of function in generalized grand Morrey spaces*, J. Math. Anal. Appl. **420** (1), 268–278 (2014).
9. Najafov A.M., Kadimova L.Sh.: *Theorems on imbedding of functions from the Sobolev-Morrey generalized space*, Proc. A. Razmadze Math. Inst. **154**, 97-109 (2010).

10. Najafov A.M.: *The differential properties of functions from Sobolev-Morrey type spaces of fractional order*, J. Math. Res., **7** (3), 149-158 (2015).
11. Najafov A.M., Orujova A.T.: *On the solution of a class of partial differential equations*, Electron. J. Qual. Theory Differ. Equ. **2017** (44), 1–9 (2017).
12. Najafov A.M., Babayev R.F.: *Some properties of functions from generalized Sobolev-Morrey type spaces*, Math. Aeterna, **7** (3), 301–311 (2017).
13. Najafov A.M., Rustamova N.R.: *Some differential properties of anisotropic grand Sobolev-Morrey spaces*, Trans. A. Razmadze Math. Inst., **172** (1), 82-89 (2018).
14. Najafov A.M.: *On embedding theorems in small small Sobolev-Morrey spaces*, Inter. Conf. "Modern problems of Math. and Mech."devoted to the 60th anniversary of the Inst. Math. and Mech., 23-25 October, Baku, Azerbaijan, p. 403.
15. Najafov A.M., Alekberli S.T.: *On properties functions from grand grand Sobolev-Morrey spaces*, J. Baku Eng. Univ., **2** (1), (2018).
16. Najafov A.M., Gasimova A.M.: *On embedding theorems in grand grand Nikolskii-Morrey spaces*, Eur. J. Pure Appl. Math., **12** (4), 1602-1611 (2019)
17. Najafov A.M., Rustamova N.R., Alekberli S.T.: *On solvability of a quasi-elliptic partial differential equations*. J. Elliptic Parabol. Equ., On line version, 1-13 (2019)
18. Najafov A.M., Alekberli S.T.: *Some properties of grand Sobolev-Morrey spaces with dominant mixed derivatives*, J. Math. Inequal. **13** (4), 1171–1180 (2019).
19. Samko S.G., Umarkhadzhiev S.M., *On Iwaniec - Sbordone spaces on sets which may have infinite measure*, Azerb. J. Math. **1** (1), 67–84 (2011)
20. Samko S.G., Umarkhadzhiev S.M.: *On Iwaniec - Sbordone spaces on sets which may have infinite measure*, Azerb. J. Math. **1** (2), 143–144 (2011)
21. Sbordone C.: *Grand Sobolev spaces and their applications to variational problems*, in: Le Mathematiche, **LI**, 335–347 (1996) Fasc. II.