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Operator-valued Fourier multipliers in vector-valued function spaces and application

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Abstract. The operator-valued Fourier multiplier theorems in E-valued weighted Lebesgue and Besov spaces are studied. These results permit us to show embedding theorems in weighted Besov-Lions type spaces $B^{l,s}_{p,q,\gamma}(\Omega; E_0, E)$, where E_0, E are two Banach spaces and $E_0 \subset E$. The most regular class of interpolation space E_α, between E_0 and E are found such that the mixed differential operator D^α is bounded from $B^{l,s}_{p,q,\gamma}(\Omega; E_0, E)$ to $B^{s}_{p,q,\gamma}(\Omega; E_\alpha)$ and Ehrling-Nirenberg-Gagliardo type sharp estimates are established. By using these results the $B^{s}_{p,q,\gamma}-$separability properties of degenerate differential operators are studied.

Keywords. Banach space -valued functions; Operator-valued multipliers; embedding of abstract weighted spaces; Abstract differential equations; Interpolation of Banach spaces

1 Introduction

Fourier multipliers in vector-valued function spaces has been studied e.g. in [12], [17], [28], [31]. Operator-valued Fourier multipliers in abstract function spaces have been investigated in [1], [8 − 11], [13]. Mikhlin type Fourier multipliers in scalar weighted spaces have been studied e.g. in [14] and [30]. Moreover, operator-valued Fourier multipliers in weighted abstract L_p spaces were investigated e.g. in [2], [7], [13], [16]. Regularity properties of abstract differential equations have been studied e.g. in [1], [3], [9], [21 − 26], [30]. A comprehensive introduction to DOEs and historical references may be found in [1] and [30] .

In the paper operator-valued multiplier theorems in $E-$valued Besov space

$$X = B^{s}_{p,q,\gamma}(\mathbb{R}^n; E)$$

are shown. Then we consider the $E-$valued anisotropic Sobolev-Besov spaces

$$Y = B^{l,s}_{p,q,\gamma}(\Omega; E_0, E),$$

here E_0, E are two Banach spaces, E_0 is continuously and densely embedded into E, and $\gamma = \gamma(x)$ is a weighted function from A_p, $p \in (1, \infty)$ class. We prove the boundedness and compactness of embedding operators in these spaces. This result generalized and improved the results [4, § 9, 27, § 1.7] for scalar Sobolev space,

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the result [15] for one dimensional Sobolev-Lions spaces and the results [22, 23] for Hilbert-space valued class. Finally, we consider anisotropic abstract elliptic equation

$$Lu = \sum_{|\alpha:l|=1} a_\alpha D^\alpha u + Au + \sum_{|\alpha:l|<1} A_\alpha D^\alpha u = f, \tag{1.1}$$

where a_α are complex numbers, A, $A_\alpha(x)$ are linear operators in a Banach space E. Here, $l = (l_1, l_2, ..., l_n)$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, l_k, α_k are integer numbers and $|\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k}$.

We say that the problem (1.1) is $X-$separable, if there exists a unique solution $u \in Y$ of the problem (1.1) for all $f \in X$ and there exists a positive constant C independent of f such that the following coercive estimate holds

$$\sum_{|\alpha:l|\leq 1} \|D^\alpha u\|_X + \|Au\|_X \leq C \|f\|_X . \tag{1.2}$$

The estimate (1.2) implies that if $f \in X$ and u is a solution of (1.1), then all terms of the equation (1.1) belong to X (i.e. all terms are separable in X).

The paper is organized as follows. In Section 2 the necessary tools from Banach space theory and some background materials are given. In Sections 3-5 the multiplier theorems in vector-valued weighted Lebesgue and Besov spaces are proved. In Section 6 by using these multiplier theorems, embedding theorems in weighted abstract Besov spaces are shown. Finally, in Section 7 the separability properties of problem (1.1) is established.

2 Notations and background

Let E be a Banach space and $\gamma = \gamma(x)$, $x = (x_1, x_2, ..., x_n)$ be a positive measurable function on the measurable subset $\Omega \subset \mathbb{R}^n$. Let $L_{p,\gamma}(\Omega; E)$ denote the space of strongly measurable $E-$valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \gamma(x) dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty,$$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)} = \operatorname*{ess\,sup}_{x\in\Omega} \|f(x)\|_E \gamma(x), \ p = \infty.$$

For $\gamma(x) \equiv 1$, the space $L_{p,\gamma}(\Omega; E)$ will be denoted by $L_p = L_p(\Omega; E)$.

The weight γ is said to be satisfy an A_p condition [18], i.e., $\gamma \in A_p$, $1 < p < \infty$ if there is a positive constant C such that

$$\left(\frac{1}{|Q|} \int_Q \gamma(x) dx \right) \left(\frac{1}{|Q|} \int_Q \gamma^{-\frac{1}{p-1}}(x) dx \right)^{p-1} \leq C,$$

for all rectangles $Q \subset \mathbb{R}^n$.

The Banach space E is called a UMD-space and written as $E \in$ UMD if only if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in the space $L_p(\mathbb{R}, E)$, $p \in (1, \infty)$ (see e.g. [6]). UMD spaces include e.g. L_p, l_p spaces and Lorentz spaces L_{pq}, p, $q \in (1, \infty)$. Let $\mathbb{C}$ be the set of complex numbers and

$$S_\varphi = \{\xi;\ \xi \in \mathbb{C},\ \ |\arg \xi| \leq \varphi\} \cup \{0\},\ 0 \leq \varphi < \pi.$$

Let E_1 and E_2 be two Banach spaces. We denote that the space of linear bounded operators from E_1 to E_2 by $L(E_1, E_2)$. For $E_1 = E_2 = E$ it will be denoted by $L(E)$. A linear operator A is said to be positive in E, with bound M if $D(A)$ is dense on E and

$$\left\| (A + \xi I)^{-1} \right\|_{L(E)} \leq M (1 + |\xi|)^{-1}$$

with $\xi \in S_\varphi$, $\varphi \in [0,\ \pi)$, where M is a positive constant and I is an identity operator in E. Sometimes instead of $A + \xi I$ will be written $A + \xi$ and it is denoted by A_ξ. It is known [28, §1.15.1] there exist fractional powers $A^\theta$of the positive operator A.

Definition 2.1. A positive operator A is said to be $R-$positive in a Banach space E if there exists $\varphi \in [0,\ \pi)$ such that the set

$$\left\{ (\xi)(A + \xi I)^{-1} : \xi \in S_\varphi \right\}$$

is $R-$bounded (see e.g. [9]).

We denote the space of compact operators in E by $\sigma_\infty(E)$. Let $E\left(A^\theta\right)$ denote the space $D\left(A^\theta\right)$ with the graphical norm defined as

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\| A^\theta u \right\|^p \right)^{\frac{1}{p}},\ 1 \leq p < \infty,\ -\infty < \theta < \infty.$$

By $(E_1, E_2)_{\theta,p}$ will be denoted interpolation spaces obtained from $\{E_1, E_2\}$ by the $K-$method [28, §1.3.1], where $\theta \in (0, 1)$, $p \in [0, 1]$. We denote by $D(\mathbb{R}^n; E)$ the space of $E-$valued $C^\infty-$ function with compact support, equipped with the usual inductive limit topology and $S(E) = S(\mathbb{R}^n; E)$ denote the $E-$valued Schwartz space of rapidly decreasing, smooth functions. For $E = \mathbb{C}$ we will denoted their $D(\mathbb{R}^n)$ and $S = S(\mathbb{R}^n)$, respectively. $D'(\mathbb{R}^n; E) = L(D(\mathbb{R}^n), E)$ denote the space of $E-$valued distributions and $S'(E) = S'(\mathbb{R}^n; E)$ is a space of linear continued mapping from $S(\mathbb{R}^n)$ into E. The Fourier transform for $u \in S'(\mathbb{R}^n; E)$ is defined by

$$F(u)(\varphi) = u(F(\varphi)),\ \varphi \in S(R^n).$$

Let γ be such that $S(\mathbb{R}^n; E_1)$ is dense in $L_{p,\gamma}(\mathbb{R}^n; E_1)$. A function $\Psi \in C^{(l)}(\mathbb{R}^n; L(E_1, E_2))$ is called a multiplier from $L_{p,\gamma}(\mathbb{R}^n; E_1)$ to $L_{q,\gamma}(\mathbb{R}^n; E_2)$ if there exists a positive constant C such that

$$\left\| F^{-1} \Psi(\xi) F u \right\|_{L_{q,\gamma}(\mathbb{R}^n; E_2)} \leq C \|u\|_{L_{p,\gamma}(\mathbb{R}^n; E_1)}$$

for all $u \in S(\mathbb{R}^n; E_1)$.

In a similar way, we can define the multiplier from $B^s_{p,q,\gamma}(\mathbb{R}^n; E_1)$ to $B^s_{p,q,\gamma}(\mathbb{R}^n; E_2)$. We denote the set of all multipliers from $B^s_{p,q,\gamma}(\mathbb{R}^n; E_1)$ to $B^s_{p,q,\gamma}(\mathbb{R}^n; E_2)$ by $M^{s,\gamma}_{p,q,\gamma}(E_1, E_2)$. For $E_1 = E_2 = E$ we denote $M^{s,\gamma}_{p,p,\gamma}(E_1, E_2)$ by $M^{s,\gamma}_{p,q,\gamma}(E)$.

Definition 2.2. Let $\gamma \in A_q$ for $q \in [1, \infty]$. Assume that E is a Banach space and $p \in [1, 2]$. Suppose that there exists a positive constant $C_0 = C_0(p, \gamma, E)$ so that

$$\|Fu\|_{L_{p',\gamma'}(\mathbb{R}^n; E)} \leq C_0 \|Fu\|_{L_{p,\gamma}(\mathbb{R}^n; E)} \tag{2.1}$$

for $\frac{1}{p}+\frac{1}{p'}=1$, $\gamma'(\cdot)=\gamma^{-\frac{1}{p-1}}(\cdot)$ and each $u\in S(\mathbb{R}^n;E)$. Then E is called weighted Fourier type γ and p. It is called Fourier type $p\in[1,2]$ if $\gamma(x)\equiv 1$(see e.g [19])..

Remark 2.1. The estimate (2.1) shows that each Banach space E has weighted Fourier type γ and 1. By Bourgain [6] has shown that each $B-$convex Banach space (thus, in particular, each uniformly convex Banach space) has some non-trivial Fourier type $p\in[1,2]$, i.e. UMD spaces are Fourier type for some $p\in[1,2]$.

In order to define abstract Besov spaces we consider the dyadic-like subsets $\{J_k\}_{k=0}^{\infty}$, $\{I_k\}_{k=0}^{\infty}$ of $\mathbb{R}^n$ and partition of unity $\{\varphi_k\}_{k=0}^{\infty}$ defined e.g. in [28, § 2.3.2].

Remark 2.2. Note the following useful properties are satisfied:

supp $\varphi_k\subset\bar{I}_k$ for each $k\in\mathbb{N}_0$; $\sum\limits_{k=0}^{\infty}\varphi_k(s)=1$ for each $s\in\mathbb{R}^n$; $I_m\cap$ supp $\varphi_k=\emptyset$ if $|m-k|>1$; $\varphi_{k-1}(s)+\varphi_k(s)+\varphi_{k+1}(s)=1$ for each $s\in$ supp φ_k and $k\in\mathbb{N}_0$.

Among the many equivalent descriptions of Besov spaces, the most useful one for us is given in terms of the so called Littlewood-Paley decomposition. This means that we consider $f\in S'(E)$ as a distributional sum $f=\sum\limits_k f_k$ analytic functions f_k whose Fourier transforms have support in dyadic-like I_k and then define the Besov norm in terms of the f_k's.

Definition 2.3. Let $\gamma\in A_q$, $1\leq r$, $q\leq\infty$ and $s\in\mathbb{R}$. Let $l_r(E)$ denotes $E-$valued sequence space [28, § 1.18].

The Besov space $Y^s=B_{q,r,\gamma}^s(\mathbb{R}^n;E)$ is the space of all $f\in S'(\mathbb{R}^n;E)$ for which

$$\|f\|_{B_{q,r,\gamma}^s(\mathbb{R}^n;E)}=\left\|\left\{2^{ks}(\check{\varphi}_k*f)\right\}_{k=0}^{\infty}\right\|_{l_r(L_{q,\gamma}(\mathbb{R}^n;E))} \tag{2.2}$$

$$=\begin{cases}\left[\sum\limits_{k=0}^{\infty}2^{ksr}\|\check{\varphi}_k*f\|_{L_{q,\gamma}(\mathbb{R}^n;E)}^r\right]^{\frac{1}{r}}<\infty,\text{ if }1\leq r<\infty,\\ \sup\limits_{k\in\mathbb{N}_0}\left[\sum\limits_{k=0}^{\infty}2^{ks}\|\check{\varphi}_k*f\|_{L_{q,\gamma}(\mathbb{R}^n;E)}\right]<\infty,\text{ if }r=\infty.\end{cases}$$

$B_{q,r,\gamma}^s(\mathbb{R}^n;E)$-together with the norm in (2.1) is a Banach space. $\mathring{B}_{q,r,\gamma}^s(\mathbb{R}^n;E)$ is the closure of $S(\mathbb{R}^n;E)$ in $B_{q,r,\gamma}^s(\mathbb{R}^n;E)$ with the induced norm. For $E=\mathbb{C}$ and $\gamma(x)\equiv 1$ the space $B_{q,r,\gamma}^s(\mathbb{R}^n;E)$ states to be the usual Besov spasce (see e.g. [4], [13]).

Let Ω be a domain in $\mathbb{R}^n$. Here, $B_{q,r,\gamma}^s(\Omega;E)$ denotes the space of restrictions to Ω of all functions in $B^s=B_{q,r,\gamma}^s(\mathbb{R}^n;E)$ with the norm given by

$$\|u\|_{B_{q,r,\gamma}^s(\Omega;E)}=\inf_{g\in B^s,g|_{\Omega}=u}\|g\|_{B_{q,r,\gamma}^s(\mathbb{R}^n;E)}.$$

Let $l=(l_1,l_2,...,l_n)$, l_k are positive integers, $s\in\mathbb{R}$ and $1\leq q$, $r\leq\infty$. Here, $B_{q,r,\gamma}^{l,s}(\Omega;E)$ denote a E-valued Sobolev-Besov weighted space of functions $u\in B_{q,\theta,\gamma}^s(\Omega;E)$ that have weak derivatives $D_k^{l_k}u=\frac{\partial^{l_k}}{\partial x_k^{l_k}}u\in B_{q,r,\gamma}^s(\Omega;E)$ with the norm

$$\|u\|_{B_{q,\theta,\gamma}^{l,,s}(\Omega;E)}=\|u\|_{B_{q,r,\gamma}^s(\Omega;E)}+\sum_{k=1}^n\left\|D_k^{l_k}u\right\|_{B_{q,\theta r\gamma}^s(\Omega;E)}<\infty.$$

Let E_0 be continuously and densely belongs into E. $B_{q,\theta,\gamma}^{l,s}\left(\Omega;E_0,E\right)$ denotes the space $B_{q,\theta,\gamma}^{s}\left(\Omega;E_0\right)\cap B_{q,\theta,\gamma}^{l,s}\left(\Omega;E\right)$ with the norm

$$\|u\|_{B_{q,\theta,\gamma}^{l,s}}=\|u\|_{B_{q,\theta,\gamma}^{l,s}(\Omega;E_0,E)}=\|u\|_{B_{q,\theta,\gamma}^{s}(\Omega;E_0)}+\sum_{k=1}^{n}\left\|D_k^{l_k}u\right\|_{B_{q,\theta,\gamma}^{s}(\Omega;E)}<\infty.$$

Let $(E(X);E^*(X^*))$ be one of the pairs. There is an embedding of $E^*(X^*)\subset[E(X)]^*$ as a norming subspace for $E(X)$. This embedding is given by the duality map

$$\langle .,.\rangle_{E(X)}:E^*(X^*)\times E(X)\to\mathbb{C},$$

where

$$\langle g,f\rangle_{L_{q,\gamma}(X)}=\int_{\mathbb{R}^n}\langle g\left(t\right),f\left(t\right)\rangle_X dt=\int_{\mathbb{R}^n}g\left(t\right)f\left(t\right)dt$$

in weighted Lebesgue space setting with $E=L_{q,\gamma}$ and

$$\langle g,f\rangle_{B_{q,r,\gamma}^{s}(X)}=\sum_{n,m\in\mathbb{N}_0}\langle\check{\varphi}_n*g,\check{\varphi}_m*f\rangle_{L_{q,\gamma}(X)} \tag{2.3}$$

in Besov space setting with $E=B_{q,r,\gamma}^{s}\left(X\right)$. One can check that this definition of duality is independent of the choice of the $\{\varphi_k\}_{k=0}^{\infty}$.

3 The Fourier transform in weighted Besov spaces

Let

$$X_{q,\gamma}=L_{q,\gamma}\left(\mathbb{R}^n;E\right),\ B_{q,r,\gamma}^{s}=B_{q,r,\gamma}^{s}(E)=B_{q,r,\gamma}^{s}\left(\mathbb{R}^n;E\right).$$

By applying the Hausdorff-Young inequality we get the following estimates for the Fourier transform on Besov spaces

Theorem 3.1. Assume that $\gamma\in A_p$. Let E be a Banach space with weighted Fourier type γ and $p\in(1,\,2]$. Let $1\le q\le p'$ and $s\ge n\left(\frac{1}{q}-\frac{1}{p'}\right)$ and $1\le r\le\infty$. Then there exists constant C, depending only on $C_0\left(p,\gamma,E\right)$ so that if $f\in B_{q,r,\gamma}^{s}$ then

$$\left\|\left\{\widehat{f}\chi_{J_m}\right\}_{m=0}^{\infty}\right\|_{l_r(X_{q,\gamma'})}\le C\left\|f\right\|_{B_{p,r,\gamma}^{s}}, \tag{3.1}$$

where $C_0\left(p,\gamma,E\right)$ is a positive constant defined in the Definition 2.1 and $\gamma'=\gamma^{-\frac{1}{p-1}}$.

An immediate corollary of Theorem 3.1 follows by choosing for $q=r=1$ and $r=q=p'$ we obtain respectively

Corollary 3.1. Assume that $\gamma\in A_p$. Let E be a Banach space with Fourier type γ and $p\in(1,\,2]$. Then the Fourier transform F defines the following bounded operator

$$F:B_{p,1,\gamma}^{\frac{n}{p}}\to L_{1,\gamma'}\left(\mathbb{R}^n;E\right),\ F:B_{p,p',\gamma}^{0}\to L_{p',\gamma'}\left(\mathbb{R}^n;E\right). \tag{3.2}$$

The norms of the above maps F are bounded by a constant depending only on $C_0\left(n,E\right)$.

Theorem 3.1 and Corollary 3.1 remain valid if F is replaced with F^{-1}.

Proof of Theorem 3.1. Let $f \in B^s_{p,r,\gamma}$. Then, for each $k \in \mathbb{N}_0$, since $\check{\varphi}_k * f \in X_{q,\gamma}$ and E has weighted Fourier type γ and p,

$$\varphi_k \cdot \widehat{f} = F\left(\check{\varphi}_k * f\right) \in X_{p',\gamma'}.$$

Thus by Remark 2.2,

$$\widehat{f}\chi_{J_m} = \left(\sum_{k=m-1}^{m+1} \varphi_k \cdot \widehat{f}\right)\chi_{J_m} \in X_{q,\gamma'} \text{ for each } m \in \mathbb{N}_0. \tag{3.3}$$

Moreover, by Definition 2.2 we get

$$\left\|\varphi_k \widehat{f}\right\|_{X_{p',\gamma'}} = \left\|F\left(\check{\varphi}_k * f\right)\right\|_{X_{p',\gamma'}} \leq C_0 \left\|\check{\varphi}_k * f\right\|_{X_{p,\gamma}},$$

i.e.

$$\sum_{k=m-1}^{m+1} 2^{ks} \left\|\varphi_k \widehat{f}\right\|_{X_{p',\gamma'}} \leq C_0 \sum_{k=m-1}^{m+1} 2^{ks} \left\|\check{\varphi}_k * f\right\|_{X_{p,\gamma}}. \tag{3.4}$$

In view of (3.4), it suffices to show that there exists the positive constant C_1 so that the following holds

$$\left\|\widehat{f}\chi_{J_m}\right\|_{X_{q,\gamma'}} \leq C_1 \sum_{k=m-1}^{m+1} 2^{ks} \left\|\varphi_k \widehat{f}\right\|_{X_{p',\gamma'}}. \tag{3.5}$$

We consider the case when $q \neq p'$. Choose $1 \leq \sigma < p$, that $\frac{1}{q} = \frac{1}{p'} + \frac{1}{\sigma}$; so, $\frac{n}{\sigma} \leq s$. By the generalized Hölder's inequality for each $m \in \mathbb{N}_0$,

$$\left\|\widehat{f}\chi_{J_m}\right\|_{X_{q,\gamma'}} \leq \sum_{k=m-1}^{m+1} \left\|\varphi_k \widehat{f}\chi_{J_m}\right\|_{L_{q,\gamma}(J_m;E)} \tag{3.6}$$

$$\leq \sum_{k=m-1}^{m+1} \left\|\varphi_k \left(\frac{1+|.|}{4}\right)^{\frac{n}{\sigma}} \widehat{f}\gamma^{\frac{1}{p'}}(\cdot)\right\|_{L_{p',\gamma'}(J_m;E)} \left\|\gamma^{\frac{1}{p}}\left(\frac{1+|\cdot|}{4}\right)^{-\frac{n}{\sigma}}\right\|_{L_\sigma(J_m)}$$

$$\leq \sum_{k=m-1}^{m+1} \left\|\widehat{f}\varphi_k\right\|_{L_{p',\gamma'}(J_m;E)} \left\|\left(\frac{1+|\cdot|}{4}\right)^{\frac{n}{\sigma}} \chi_{J_m}\right\|_{L_\infty} \leq C_2 \sum_{k=m-1}^{m+1} 2^{ks} \left\|\widehat{f}\varphi_k\right\|_{L_{p',\gamma'}(J_m;E)},$$

where C_2 is a positive constant defined by

$$C_2 = \left\|\gamma^{\frac{1}{p}}\left(\frac{1+|\cdot|}{4}\right)^{-\frac{n}{\sigma}}\right\|_{L_\sigma(J_m)} \leq \left\|\left(\frac{1+|\cdot|}{4}\right)^{-n}\right\|_{L_\infty(J_m)} \left\|\gamma^{\frac{\sigma}{p}}\right\|_{L(J_m)}$$

$$\leq 4^n \left[\sup_{m \in \mathbb{N}_0} 2^{-(m-1)n} \int_{J_m} \gamma^{\frac{\sigma}{p}}(s)ds\right]^{\frac{1}{\sigma}}. \tag{3.7}$$

In view of $\gamma \in A_p$, we have

$$\sup_{m \in \mathbb{N}_0} 2^{-(m-1)n} \int_{J_m} \gamma^{\frac{\sigma}{p}}(s)ds < \infty.$$

For $q = p'$ and for each $m \in \mathbb{N}$ we get

$$\left\| \widehat{f} \chi_{J_m} \right\|_{X_{q,\gamma}} \leq \sum_{k=m-1}^{m+1} \left\| \varphi_k \widehat{f} \chi_{J_m} \right\|_{L_{p',\gamma'}(J_m;E)}$$

$$\leq \sum_{k=m-1}^{m+1} 2^{ks} \left\| \varphi_k \widehat{f} \right\|_{X_{p',\gamma'}}. \tag{3.8}$$

So, from (3.6)-(3.8) we obtain (3.5).

Remark 3.1. By using the embedding $W^j_{p,\gamma}(\mathbb{R}^n; E) \subset B^s_{q,r,\gamma}$ for $s < j \in \mathbb{N}$ we get that the statement of Theorem 3.1 remains valid if $B^s_{q,r,\gamma}$ is replaced by $W^j_{p,\gamma}(\mathbb{R}^n; E)$.

Also, it follows from Corollary 3.1 that if E has weighted Fourier type for $\gamma \in A_\nu$, $\nu \in [1, \infty]$ and $j > \frac{n}{p}$ then the Fourier transform F defines bounded operator:

$$W^j_{p,\gamma}(\mathbb{R}^n; E) \to X_{1,\gamma'}.$$

Furthermore, if E has weighted Fourier type for $\gamma \in A_\nu$, $\nu \in [1, \infty]$ and $j > \frac{n}{p}$ then there is a constant C so that

$$\left\| \widehat{f} \right\|_{X_{1,\gamma'}} \leq C \left\| f \right\|^{1-\frac{n}{jp}}_{X_{p,\gamma}} \left[\sum_{|\alpha|=j} \left\| D^\alpha f \right\|_{X_{p,\gamma}} \right]^{\frac{n}{jp}} \tag{3.9}$$

for each $f \in W^j_{p,\gamma}(\mathbb{R}^n; E)$.

4 Fourier multipliers on weighted Lebesgue spaces

Let $m : \mathbb{R}^n \to L(E_1, E_2)$ be a bounded measurable function. In this section, we identify conditions on m, generalizing the classical Mikhlin condition so that the multiplication operator induced by m, i.e. the operator: $u \to T_m = F^{-1} m F u$ is bounded from $L_{q,\gamma}(\mathbb{R}^n; E_1)$ to $L_{q,\gamma}(\mathbb{R}^n; E_2)$. We will first give rather general criteria for Fourier multipliers in terms of the weighted Besov norm of the multiplier function; later we derive from these results analogues of the classical Mikhlin and Hörmander conditions. To simplify the statements of our results, we let

$$M_{p,\gamma}(m) = \inf_{a>0} \left\{ \left\| m(a,.) \right\|_{B^{\frac{n}{p}}_{p,1.\gamma}(\mathbb{R}^n; L(E_1,E_2))} \right\}.$$

Let

$$X_k = X_q(E_k) = L_{q,\gamma}(\mathbb{R}^n; E_k),\ k = 1, 2,$$

$$X_1(L) = L_1(\mathbb{R}^n;\ L(E_1, E_2)),\ Y = B^{\frac{n}{p}}_{p,1.\gamma}(\mathbb{R}^n; L(E_1, E_2)).$$

First we give a multiplier result from X_1 to X_2 in the spirit of Steklin's theorem.

Theorem 4.1. Assume that $\gamma \in A_q$ for $q \in [1, \infty]$. Let E_1, E_2 be Banach spaces with weighted Fourier type γ and $p \in p \in (1,\ 2]$. Suppose that

$$T_m \in B^{\frac{n}{p}}_{p,1,\gamma}(\mathbb{R}^n; L(X_1, X_2)).$$

Then there exists a constant C depending only on $C_{01}(p,\gamma,E_1)$ and $C_{02}(p,\gamma,E_2)$, so that if $m\in Y$, then m is a Fourier multiplier from X_1 to X_2 and

$$\|T_m\|_{L(X_1,X_2)}\leq CM_{p,\gamma}(m)$$

for each $q\in[1,\infty]$.

Let E^* denotes the dual space of E and A^*denotes the conjugate of the operator A.

The proof of Theorem 4.1 uses the following lemma.

Lemma 4.1. Assume that $\gamma\in A_q$ for $q\in[1,\infty]$ and $k\in X_1(L)$. Suppose that there exists constants C_i so that for each $x\in E_1$ and $x^*\in E_2^*$

$$\int_{\mathbb{R}^n}\|k(s)x\|_{E_2}ds\leq M_0\|x\|_{E_1},\ \int_{\mathbb{R}^n}\|k^*(s)x^*\|_{E_1^*}ds\leq M_1\|x^*\|_{E_2^*}. \tag{4.1}$$

Then the convolution operator $K:X_1\to X_2$ defined by

$$(Kf)(t)=\int_{\mathbb{R}^n}k(t-s)f(s)ds \text{ for } t\in\mathbb{R}^n \tag{4.2}$$

satisfies that

$$\|K\|_{L(X_1,X_2)}\leq M_0^{\frac{1}{q}}M_1^{1-\frac{1}{q}}.$$

Proof. Since $k\in X_1(L)$ it is well-known that (4.2) defines a bounded operator on X_1. Indeed, for $f\in X_1\cap X_\infty(E_1)$ we have

$$\int_{\mathbb{R}^n}\|k(t-s)f(s)\|_{E_2}ds=\int_{\mathbb{R}^n}\|k(s)f_s(t)\|_{E_2}ds\leq\|k\|_{X_1(L)}\|f\|_{X_\infty(E_1)} \tag{4.3}$$

for each $t\in\mathbb{R}^n$ and $f_s(t)=f(t-s)$. From (4.3) by applying the Minkowski's inequality for integral with weight [20, § A.1] we get

$$\|Kf(\cdot)\|_{X_2}\leq\int_{\mathbb{R}^n}\|k(s)f_s(t)\|_{X_2}ds\leq\int_{\mathbb{R}^n}\|k(s)\|_{L(E_1,E_2)}\|f_s\|_{X_1}ds$$
$$=\|k\|_{X_1(L)}\|f_s\|_{X_1}.$$

Now, for $q=1$ we have from (4.1)

$$\|Kf\|_{X_1(E_1)}\leq\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\|k(s)f_s(t)\|_{E_1}ds\right)\gamma(t)\,dt$$
$$\leq M_0\int_{\mathbb{R}^n}\|f(t)\|_{E_1}\gamma(t)\,dt=M_0\|f\|_{X_1(E_1)}.$$

Hence,

$$\|K\|_{L(X_1(E_1))}\leq M_0. \tag{4.4}$$

If $q=\infty$, then for each $X_\infty(E_1)$, $x^*\in E_2^*$ and $t\in\mathbb{R}^n$ by using (4.1) we get

$$|\langle x^*,(Kf)(t)\rangle_{E_2}|\leq\int_{\mathbb{R}^n}|\langle k^*(t-s)x^*,f(s)\rangle_{E_1}|\,\gamma(s)ds$$

$$\leq \int_{\mathbb{R}^n} \|k^*(t-s)\,x^*\|_{E_1^*} \|f(s)\|_{E_1} \gamma(s)ds \leq M_1 \|x^*\|_{E_1^*} \|f\|_{X_1(E_1)}.$$

Thus,

$$\|K\|_{L(X_1(E_1))} \leq M_1. \tag{4.5}$$

Let $\bar{X}_\infty(E_1)$ denotes the closure in $X_\infty(E_1)$ norm of the simple functions $\sum_{k=1}^{m} x_k \chi_{A_k}$, where $x_k \in E_1$, vol $A_k < \infty$ and $m \in \mathbb{N}$. Then one can check that K maps $\bar{X}_\infty(E_1)$ into $\bar{X}_\infty(E_1)$. Indeed, for $f = \chi_A$, we have

$$Kf(t) = \int_{t-A} k(s)xds \to 0 \text{ for } t \to \infty$$

and Kf is a continuous function from $\mathbb{R}^n$ to E_2. Now, the Riesz-Thorin theorem (cf. [5, Thm 5.1.2]) yields the claim for $1 < p < \infty$.

Proof of Theorem 4.1. First assume in addition that $m \in S(L(E_1, E_2))$. Hence, $\check{m} \in S(L(E_1, E_2))$. Fix $x \in E_1$. For an appropriate choice of $a > 0$, we can apply Corollary 3.1 to the function $t \to m(at)\,x$ in $B_{p,1.\gamma}^{\frac{n}{p}}(\mathbb{R}^n; E_2)$ and use that

$$F^{-1}[m(a.)\,x](s) = a^{-n}\check{m}\left(\frac{s}{a}\right)x$$

to get

$$\|\check{m}(\cdot)x\|_{X_1(E_1)} = \left\|F^{-1}m(a.)\,x\right\|_{X_1(E_1)}$$
$$\leq C_1 \|m(a.)\,x\|_Y \|x\|_{E_1} \leq 2C_1 M_{p,\gamma} \|x\|_{E_1},$$

for some constant C_1 which depends on $C_0(p, \gamma, E_2)$.

By the additional assumption on m we get

$$m^*(\cdot) \in S(L(E_2^*, E_1^*)), \text{ and } F^{-1}m^*(\cdot) = [\check{m}(\cdot)]^* \in S(L(E_2^*, E_1^*)).$$

Let $x^* \in E_2^*$. Similarly, by applying Corollary 3.1 to an appropriate function

$$t \to [m(at)]^*\,x^* \text{ in } B_{p,1.\gamma}^{\frac{n}{p}}(R^n; E_1^*)$$

and using the fact that $M_{p,\gamma}(m) = M_{p,\gamma}(m^*)$, one has

$$\|[\check{m}(\cdot)]^*\,x^*\|_{X_1(E_1^*)} \leq 2C_2 M_{p,\gamma}(m) \|x^*\|_{E_2^*}$$

for some constant C_2 which depends $C_0(p, \gamma, E_1^*)$. By Lemma 4.1, the convolution operator

$$(T_m f)(t) = \int_{\mathbb{R}^n} \check{m}(t-s)\,f(s)ds$$

satisfies

$$\|T_m\|_{B(X_1, X_2)} \leq CM_{p,\gamma}(m),$$

where $C = 2\max\{C_1, C_2\}$. Furthermore, since $m \in L_1(\mathbb{R}^n; L(E_1, E_2))$, then T_m satisfies the following

$$T_m f = F^{-1}m(\cdot)f(\cdot) \text{ for all } f \in S(\mathbb{R}^n; E_1), \tag{4.6}$$

also

$$T_m \in C(\sigma(X_1, X_1^*), \sigma(X_2, X_2^*)), \tag{4.7}$$

where $\sigma\left(X_k, X_k^*\right)$ denote the interpolation spaces of X_k, X_k^*.

For the general case, let $m \in Y$. It is known that $S\left(\mathbb{R}^n; L\left(E_1, E_2\right)\right)$ is dense in Y when $\gamma \in A_\nu$, $\nu \in [1, \infty]$. Now, let we choose a sequence $\{m_n\}_n^\infty \subset S\left(\mathbb{R}^n; L\left(E_1, E_2\right)\right)$ that converges to m in the $Y-$norm and obtain operators $T_{m_n} \in L\left(X_1, X_2\right)$, where

$$T_{m_n} f = F^{-1} m_n(\cdot) f(\cdot),\ f \in X_1.$$

It is clear to see that, the properties (4.6) and (4.7) pass from T_{m_n} to T_m. One also has that

$$\left\|T_m\right\|_{L(X_1,X_2)} \leq C \left\|m\right\|_Y .$$

Fix $a > 0$ such that $m\left(a.\right) \in Y$. Then $I_{E_2} \circ T_{m(a.)} = T_m \circ I_{E_1}$, where $I_{\mathbb{Z}} : L_{q,\gamma}\left(\mathbb{R}^n; \mathbb{Z}\right) \to L_{q,\gamma}\left(\mathbb{R}^n; \mathbb{Z}\right)$ is the isometry

$$T\left(f\right)\left(t\right) = a^{\frac{n}{q}} f\left(at\right).$$

Thus,

$$\left\|T_m\right\|_{L(X_1,X_2)} = \left\|T_{m(a\cdot)}\right\|_{L(X_1,X_2)} \leq C \left\|m\right\|_Y ,$$

i.e.

$$\left\|T_m\right\|_{L(X_1,X_2)} \leq C M_{p,\gamma}\left(m\right).$$

The following remark collects some basic facts about the Fourier multiplier operators T_m given in Theorem 4.1 that will be used in the proof of Theorem 4.2.

Remark 4.1. Let $f \in X_1$ and let Ω be a closed subset of $\mathbb{R}^n$. Then the following are valid:

(a) Viewing f and $T_m f$ as distributions, if supp $\widehat{f} \subset \Omega$ then supp $F\left(T_m f\right) \subset \Omega$;
(b) $T_{m_1+m_2} = T_{m_1} + T_{m_2}$. If $\varphi \in S$, then $\check{\varphi} * T_m f = T_m\left(\check{\varphi} * f\right) = T_{\varphi m}\left(f\right);$
(c) If $\varphi \in S$ is 1 on supp $\widehat{f}$, then $T_{\varphi m}\left(f\right) = T_m\left(f\right);$
(d) T_m^* restricted to $L_{q',\gamma}\left(\mathbb{R}^n; E_2^*\right)$ is $T_{m^*(-.)}$.

5 Fourier multipliers on weighted Besov spaces

Let $m : \mathbb{R}^n \to L\left(E_1, E_2\right)$ be a bounded measurable function. In this section we identify conditions on m, generalizing the classical Mikhlin condition so that the multiplication operator induced by m, i.e. the operator: $u \to T_m = F^{-1} m F u$ is bounded from $B^s\left(E_1\right)$ to $B^s\left(E_2\right)$. By applying this Theorem 4.1 to the blocks of the Littlewood-Paley decomposition of Besov spaces we will now get the main result of this section. Let

$$Y_k\left(E_k\right) = B^s\left(E_k\right) = B^s_{q,r,\gamma}\left(\mathbb{R}^n; E_k\right),\ k = 1, 2.$$

Theorem 5.1. Assume that $\gamma \in A_q$ for $q \in [1, \infty]$. Let E_1, E_2 be a Banach spaces with weighted Fourier type γ and $p \in (1,\ 2]$. Then there is a constant C depending only on $C_{01}\left(p, \gamma, E_1\right)$ and $C_{02}\left(p, \gamma, E_2\right)$, so that if

$$\varphi_k m \in B^{\frac{n}{p}}_{p,1,\gamma}\left(\mathbb{R}^n; L\left(X_1, X_2\right)\right) \text{ and } M_{p,\gamma}\left(\varphi_k m\right) \leq A \text{ for each } k \in \mathbb{N}_0 \tag{5.1}$$

then m is a Fourier multiplier from Y_1 to Y_2 and $\left\|T_m\right\|_{B(Y_1,Y_2)} \leq CA$ for each $s \in \mathbb{R}$ and q, $r \in [1, \infty]$.

Proof. By properties of $\{\varphi_k\}$ we have

$$T_m f = F^{-1} m\widehat{f} = \sum_{k\in\mathbb{N}_0} F^{-1}\left[\left(\varphi_{k-1}+\varphi_k+\varphi_{k+1}\right) mF\left[\left(\check{\varphi}_k * f\right)\right]\right]$$

$$= \sum_{k\in\mathbb{N}_0} T_{(\varphi_{k-1}+\varphi_k+\varphi_{k+1})m}\left(\check{\varphi}_k * f\right), \tag{5.2}$$

where T_m is the Fourier multiplier operator on X_1 given by Theorem 4.1. From Theorem 4.1 implies that $m\varphi_k$ induces a Fourier multiplier operator $T_{m.\varphi_k}$ with

$$\left\|T_{m.\varphi_k}\right\|_{L(X_1,X_2)} \leq CM_{p,\gamma}\left(\varphi_k m\right) \leq CA$$

for some constant C depending only on $C_{0,1}\left(p,\gamma,E_1\right)$ and $C_{0,2}\left(p,\gamma,E_2\right)$. Let

$$\psi_k = \varphi_{k-1}+\varphi_k+\varphi_{k+1}.$$

Note that $\psi_k(s)\equiv 1$ when $s\in \text{supp}\ \varphi_k$. Then $m\psi_k$ induces the Fourier multiplier operator $T_{m.\psi_k}$ with

$$T_{m\psi_k} = T_{m\varphi_{k-1}} + T_{m\varphi_k} + T_{m\varphi_{k+1}} \in B\left(X_1,X_2\right)$$

and

$$\left\|T_{m.\psi_k}\right\|_{B(X_1,X_2)} \leq 3CA.$$

Define T_0: $S(E_1)\to S'(E_1)$ by

$$T_0 f = F^{-1} m(\cdot) Ff(\cdot).$$

If $f\in S(E_1)$, then $\check{\varphi}_k * T_0 f = T_{m\psi_k}\left(\check{\varphi}_k * f\right)$ for each $k\in\mathbb{N}_0$ since

$$F\left[T_{m\psi_k}\left(\check{\varphi}_k * f\right)\right](.) = m(\cdot)\psi_k(\cdot)F\left[\left(\check{\varphi}_k * f\right)(\cdot)\right]$$

$$= \varphi_k(\cdot)m(\cdot)\widehat{f}(\cdot) = \varphi_k(\cdot)F\left(T_0 f\right) = F\left[\left(\check{\varphi}_k * T_0 f\right)(\cdot)\right].$$

So, by definition of the Besov norm

$$\left\|T_0 f\right\|_{Y_2} \leq 3CA\left\|T_0 f\right\|_{Y_1}.$$

Thus T_0 extends to a bounded linear operator from $\mathring{B}^s_{q,r,\gamma}\left(\mathbb{R}^n;E_1\right)$ into

$$\mathring{B}^s_{q,r,\gamma}\left(\mathbb{R}^n;E_2\right).$$

If q, $r<\infty$ then all that would remain is to verify the weak continuity condition (4.7). However, we continue with the proof in order to also cover the case $q=\infty$ or $r=\infty$. We shall show that the operator $T_m: Y_1\to Y_2$ defined by

$$T_m f = \sum_{k=1}^{\infty} f_k,\ f_k = T_{m\psi_k}\left(\check{\varphi}_k * f\right)\in X_2 \tag{5.3}$$

is indeed a (norm) continuous operator. Fix $f\in Y_1$. First, we show that the formal series (5.3) defines an element in $S'(E_2)$. Towards this, fix $\varphi\in S$. From Remark 4.1 implies that supp $f_k\subset \bar{I}_k$. Thus

$$f_k\left(\varphi\right) = \widehat{f}_k\left(\check{\varphi}\right) = \hat{f}_k\left(\psi_k\left(-\cdot\right)\check{\varphi}\right) = f_k\left(\psi_k * \varphi\right)$$

and so by using Hölder inequality with weight $\gamma \in A_q$ as in (3.7) we get

$$\sum_{k=1}^{\infty} \|f_k(\varphi)\|_{E_2} \leq \sum_{k=1}^{\infty} \|f_k\|_{X_2} \left\|\gamma^{-\frac{1}{q}}(\psi_k * \varphi)\right\|_{L_{q'}(\mathbb{C})}$$

$$\leq M \sum_{k=1}^{\infty} 2^{ks} \|\check{\varphi}_k * f\|_{X_2} \left\|2^{-ks}\psi_k * \varphi\right\|_{L_{q',\sigma}(\mathbb{C})} \leq M 2^{|s|} \|f\|_{Y_2} \|\varphi\|_{B^{-s}_{q',r',\sigma}}(\mathbb{C}),$$

where

$$\sigma(\cdot) = \gamma^{1-q}(\cdot).$$

Thus $(T_m f)(\varphi)$ for $\varphi \in S$ defines a linear map from S into E_2 which is continuous by well known inclusion

$$S(E_2) \subset Y_2 \subset S'(E_2).$$

By Remark 4.1, for each j, $k \in \mathbb{N}_0$

$$\check{\varphi}_j * T_{m\psi_k}(\check{\varphi}_k * f) = T_{m\psi_k}(\check{\varphi}_j * \check{\varphi}_k * f) = \check{\varphi}_k * T_{m\psi_k}(\check{\varphi}_j * f).$$

Since the support of φ_k intersects the support of φ_j only for $|k-j| \leq 1$, applying Remark 4.1 further gives

$$\check{\varphi}_k * T_m f = \sum_{j=k-1}^{k+1} \check{\varphi}_k * T_{m\psi_j}(\check{\varphi}_j * f) = \sum_{j=k-1}^{k+1} \check{\varphi}_j * T_{m\psi_j}(\check{\varphi}_k * f) \tag{5.4}$$

$$= \sum_{j=k-1}^{k+1} T_{m\varphi_j\psi_j}(\check{\varphi}_k * f) = T_{m\psi_k}(\check{\varphi}_k * f).$$

Hence, $\check{\varphi}_k * T_m f \in X_2$ and

$$\|\check{\varphi}_k * T_m f\|_{X_2} \leq 3CA \|\check{\varphi}_k * f\|_{X_1},$$

from which and in view of (5.2) it follows that range of T_m is contained in Y_1 and that norm of T_m as an operator from Y_1 to Y_2 is bounded by a constant depending on the items claimed. Furthermore, T_m extends T_0; indeed, if $f \in S(E_1)$ then

$$F(T_m f) = \sum_{k=1}^{\infty} F[T_{m\psi_k}(\check{\varphi}_k * f)] = \sum_{k=1}^{\infty} m\psi_k\varphi_k \widehat{f}$$

$$= \sum_{k=1}^{\infty} m\varphi_k \widehat{f} = F(T_0 f).$$

It remains to show only that T_m satisfies (4.7). Since $[m(-.)]^* : \mathbb{R}^n \to L(Y_2^*; Y_1^*)$ also satisfies condition (5.1), the Fourier multiplier operator $T_{m^*(-.)}$, defined by (4.6), extends to $T_{m^*(-.)} \in L(Y_2^*; Y_1^*)$. It suffices to show that T_m^* restricted to Y_2^* is $T_{m^*(-.)}$. Hence, fix $g \in Y_2^*$, $f \in B^s(E_1)$ and by using (5.4) and (2.3) we have

$$\langle T_m^* g, f\rangle_{Y_1} = \sum_{n,k \in \mathbb{N}_0} \langle \check{\varphi}_n * g, \check{\varphi}_k * T_m f\rangle_{L_{q,\gamma}(E_2)}$$

$$= \sum_{n,k \in \mathbb{N}_0} \langle \check{\varphi}_n * g, T_{m\psi_k}(\check{\varphi}_k * f)\rangle_{L_{q,\gamma}(E_2)}. \tag{5.5}$$

and

$$\langle T_{m^*(-.)}g, f\rangle_{Y_1} = \sum_{n,k\in\mathbb{N}_0} \langle \check{\varphi}_n * T_{m^*(-.)}g, \check{\varphi}_k * f\rangle_{L_{q,\gamma}(E_1)}$$

$$= \sum_{n,k\in\mathbb{N}_0} \langle T_{m^*(-.)\psi_n(.)} (\check{\varphi}_n * g), \check{\varphi}_k * f\rangle_{L_{q,\gamma}(E_1)}. \tag{5.6}$$

Fix $K_0 \subset \mathbb{N}_0$ and choose a radial $\psi \in S$ with compact support such that ψ is 1 on $\bigcup_{k=1}^{K_0+1}$ supp φ_k. If n, $k \in \{0, 1, ..., K_0\}$, then by Remark 4.1 we get

$$T_{m\psi_k} (\check{\varphi}_k * f) = T_{m\psi\psi_k} (\check{\varphi}_k * f) = T_{m\psi} (\check{\varphi}_k * f) \tag{5.7}$$

and

$$T_{m^*(-.)\psi_n(\cdot)} (\check{\varphi}_n * f) = T_{m^*(-.)\psi(\cdot)\psi_n(\cdot)} (\check{\varphi}_n * f) = T_{m^*(-.)\psi_n(.)} (\check{\varphi}_n * f). \tag{5.8}$$

since $m\psi$ and $m^*(-.)\psi_n(\cdot)$ satisfy the assumptions of Theorem 4.1. Hence, by (5.5)−(5.8) and by Remark 4.1 we have

$$\langle T_m^* g, f\rangle = \langle T_{m^*(-.)}g, f\rangle.$$

The next lemma gives a convenient way to verify the assumption of Theorem 4.8 in terms of derivatives.

By reasoning as Lemma 4.10 and Corollary 4.11 in [11] we obtain

Lemma 5.1. Let $\frac{n}{p} < l \in \mathbb{N}$ and $\sigma \in [p, \infty]$. If $m \in C^l(\mathbb{R}^n; L(E_1, E_2))$ and there exists a positive constant A so that

$$\|D^\alpha m\|_{L_\sigma(\mathbb{R}^n; L(E_1,E_2))} \leq A \tag{5.9}$$

for each $k \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l-1$. Then m satisfies condition (5.1) of Theorem 5.1.

Corollary 5.1. Let q, $r \in [1, \infty]$ and $s \in \mathbb{R}$. If $m \in C^l(\mathbb{R}^n; L(E_1, E_2))$ and there exists a positive constant A so that

$$\sup_{x\in\mathbb{R}^n} (1 + |x|)^{|\alpha|} \|D^\alpha m\|_{L_\sigma(\mathbb{R}^n; L(E_1,E_2))} \leq A \tag{5.10}$$

for each $k \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l$ and $m_k(\cdot) = m(2^{k-1}.)$. Then m is a Fourier multiplier from Y_1 to Y_2 provided one of the following conditions hold:

(a) E_1 and E_2 are arbitrary Banach spaces and $l = n + 1$;

(b) E_1 and E_2 are uniformly convex Banach spaces and $l = n$;

(c) E_1 and E_2 have Fourier type p and $l = \left[\frac{n}{p}\right] + 1$.

6 Embedding theorems in Besov-Lions type spaces

In this section embedding theorems in abstract Besov spaces in terms of interpolation of Banach spaces are derived. Note, that embedding of function spaces were studied e.g. in [1, 2, 4], [12]. Embedding in abstract function spaces in terms of interpolation were studied e.g in [21-27]. From [23] we have

Lemma 6.1. Let A be a positive operator in a Banach space E, b be a nonnegative real number and $r = (r_1, r_2, ..., r_n)$ where $r_k \in \{0, b\}$. Let $t = (t_1, t_2, ..., t_n)$, t_k are positive parameters, $0 < t_k \leq T < \infty$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $l = (l_1, l_2, ..., l_n)$, where l_k are positive and α_k are nonnegative integers such that $\varkappa = |(\alpha + r) : l| \leq 1$. For $0 < h \leq h_0 < \infty$ and, $0 \leq \mu \leq 1 - \varkappa$ the operator-function

$$\Psi_t(\xi) = \prod_{k=1}^{n} t_k^{\frac{\alpha_k + r_k}{l_k}} \xi^r (i\xi)^\alpha A^{1-\varkappa-\mu} h^{-\mu} [A + \psi(t, \xi)]^{-1}$$

is bounded operator in E uniformly with respect to $\xi \in \mathbb{R}^n$, $h > 0$ and t, i.e there is a constant C_μ such that

$$\|\Psi_{t,h,\mu}(\xi)\|_{L(E)} \leq C_\mu$$

for all $\xi \in \mathbb{R}^n$ and $h > 0$, where,

$$\psi = \psi(t, \xi) = \sum_{k=1}^{n} t_k |\xi_k|^{l_k} + h^{-1}.$$

Let

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \ l = (l_1, l_2, ..., l_n), \ \varkappa = \sum_{k=1}^{n} \frac{\alpha_k}{l_k},$$

$$Y = B^s_{p,\theta,\gamma}(\mathbb{R}^n; E), \ B^{l,s}(\mathbb{R}^n) = B^{l,s}_{p,\theta,\gamma}(\mathbb{R}^n; E(A), E).$$

Let $l = (l_1, l_2, ..., l_n)$, where l_k are positive integers. Let

$$\nu(l) = \max_{k,,j \in \{1,2,...,n\}} \left[\frac{1}{l_k} - \frac{1}{l_j}\right], \ \eta = \eta(t) = \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}}.$$

Theorem 6.1. Suppose that the following conditions hold:

(1) $\gamma \in A_p$ for $p \in [1, \infty]$, E is a Banach spaces with weighted Fourier type γ and $\sigma \in (1, 2]$;

(2) $t = (t_1, t_2, ..., t_n)$, $0 < t_k \leq T < \infty$, $1 < p \leq q < \infty$, $\theta \in [1, \infty]$;

(3) l_k are positive and α_k are nonnegative integers such that $0 < \varkappa + \nu(l) \leq 1$, and let $0 \leq \mu \leq 1 - \varkappa - \nu(l)$;

(4) A is a φ-positive operator in E.

Then an embedding

$$D^\alpha B^{l,s}_{p,\theta,\gamma}(\mathbb{R}^n; E(A), E) \subset B^s_{p,\theta,\gamma}\left(\mathbb{R}^n; E\left(A^{1-\varkappa-\mu}\right)\right)$$

is continuous and there exists a constant $C_\mu > 0$, depending only on μ, such that the following uniform estimate holds

$$\eta(t) \|D^\alpha u\|_{B^s_{p,\theta,\gamma}(\mathbb{R}^n; E(A^{1-\varkappa-\mu}))} \leq C_\mu \left[h^\mu \|u\|_{B^{l,s}(\mathbb{R}^n)} + h^{-(1-\mu)} \|u\|_Y\right] \tag{6.1}$$

for all $u \in B^{l,s}(\mathbb{R}^n)$ and $0 < h \leq h_0 < \infty$.

Proof. We have

$$\|D^\alpha u\|_{B^s_{p,\theta,\gamma}(\mathbb{R}^n; E(A^{1-\varkappa-\mu}))} = \left\|A^{1-\varkappa-\mu} D^\alpha u\right\|_Y \tag{6.2}$$

for all u such that

$$\|D^\alpha u\|_{B^s_{p\ddot{o}\theta,\gamma}(\mathbb{R}^n; E(A^{1-\varkappa-\mu}))} < \infty.$$

On the other hand by using the relation (6.2) we have

$$A^{1-\alpha-\mu} D^\alpha u = F^{-1} F A^{1-\varkappa-\mu} D^\alpha u = F^{-1} A^{1-\varkappa-\mu} F D^\alpha u$$

$$= F^{-1}A^{1-\varkappa-\mu}\left(i\xi\right)^{\alpha}Fu = F^{-1}\left(i\xi\right)^{\alpha}A^{1-\varkappa-\mu}Fu. \tag{6.3}$$

Hence denoting Fu by $\widehat{u}$, we get from the relations (6.2) and (6.3)

$$\left\|D^{\alpha}u\right\|_{B^{s}_{p,\theta,\gamma}(\mathbb{R}^n;E(A^{1-\varkappa-\mu}))} \backsim \left\|F^{-1}\left(i\xi\right)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}\right\|_{Y}.$$

Similarly, from definition of for all $u\in Y$ we have

$$\left\|u\right\|_{B^{l,s}(\mathbb{R}^n)} = \left\|u\right\|_{B^{s}_{p,\theta,\gamma}(\mathbb{R}^n;E(A))} + \sum_{k=1}^{n}\left\|t_k D_k^{l_k}u\right\|_{Y}$$

$$= \left\|F^{-1}\widehat{u}\right\|_{B^{s}_{p,\theta,\gamma}(\mathbb{R}^n;E(A))} + \sum_{k=1}^{n}\left\|t_k F^{-1}\left[\left(i\xi_k\right)^{l_k}\widehat{u}\right]\right\|_{Y}$$

$$\backsim \left\|F^{-1}A\widehat{u}\right\|_{Y} + \sum_{k=1}^{n}\left\|t_k F^{-1}\left[\left(i\xi_k\right)^{l_k}\widehat{u}\right]\right\|_{Y}.$$

Thus proving the inequality (6.1) for some constants C_{μ} is equivalent to proving

$$\eta\left\|F^{-1}\left(i\xi\right)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}\right\|_{B^{l,s}(\mathbb{R}^n)}$$

$$\leq C_{\mu}\left[h^{\mu}\left(\left\|F^{-1}A\widehat{u}\right\|_{Y} + \sum_{k=1}^{n}\left\|t_k F^{-1}\left[\left(i\xi_k\right)^{l_k}\widehat{u}\right]\right\|_{Y}\right) + h^{-(1-\mu)}\left\|F^{-1}\widehat{u}\right\|_{Y})\right].$$

Thus the inequality (6.1) will be followed if we prove the following inequality

$$\eta\left\|F^{-1}\left[\left(i\xi\right)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}\right]\right\|_{Y} \leq C_{\mu}\left\|F^{-1}\left[h^{\mu}(A+\psi\left(t,\xi\right)\right]\widehat{u}\right\|_{Y} \tag{6.4}$$

for a suitable $C_{\mu}>0$ and for all $u\in Y$.

Let us express the left hand side of (6.3) as follows

$$\eta\left\|F^{-1}\left[\left(i\xi\right)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}\right]\right\|_{Y} \tag{6.5}$$

$$= \eta\left\|F^{-1}\left(i\xi\right)^{\alpha}A^{1-\varkappa-\mu}\left[h^{\mu}(A+\psi\right]^{-1}\left[h^{\mu}\left(A+\psi\right)\right]\right\|_{Y}.$$

(Since A is a positive operator in E and $-\psi\left(t,\xi\right)\in S\left(\varphi\right)$ so it is possible). It is clear that the inequality (6.4) will be followed immediately from (6.5) if we can prove that the operator-function

$$\Psi_t = \Psi_{t,h,\mu} = \eta\left(t\right)\left(i\xi\right)^{\alpha}A^{1-\varkappa-\mu}\left[h^{\mu}\left(A+\psi\right)\right]^{-1}$$

is a multiplier in $M^{s,\gamma}_{p,\theta,\gamma}(E)$, which is uniformly with respect to h and t. In order to prove that $\Psi_t\in M^{s,\gamma}_{p,\theta,\gamma}(E)$ it suffices to show that there exists a constant $M_{\mu}>0$ with

$$\left|\xi\right|^{k}\left\|D^{\beta}\Psi_t\left(\xi\right)\right\|_{L(E)} \leq C,\ k=0,1,...,\left|\beta\right| \tag{6.6}$$

for all

$$\beta = \left(\beta_1,\beta_2,...,\beta_n\right),\ \beta_k\in\left\{0,1\right\},\ \xi_k\neq 0.$$

To see this, we apply Lemma 6.1 and get a constant $M_{\mu}>0$ depending only on μ such that

$$\left\|\Psi_t(\xi)\right\|_{L(E)} \leq M_{\mu}$$

for all $\xi \in \mathbb{R}^n$. This shows that the inequality (7.6) is satisfied for $\beta = (0, ..., 0)$. We next consider (6.6) for $\beta = (\beta_1, ...\beta_n)$ where $\beta_k = 1$ and $\beta_k = 0$ for $j \neq k$. By using the condition $\varkappa + \nu(l) \leq 1$ and well known inequality

$$y_1^{\alpha_1} y_2^{\alpha_2} ... y_n^{\alpha_n} \leq C \left[1 + \sum_{k=1}^{n} y_k^{l_k} \right], \text{ for } y_k \geq 0,$$

we have

$$|\xi| \, |\xi_k| \, \|D_k \Psi_t(\xi)\|_{L(E)} \leq M_\mu, \; k = 1, 2...n.$$

Repeating the above process we obtain the estimate (7.6). Thus the operator-function $\Psi_{t,h,\mu}(\xi)$ is a uniform collection of multiplier with respect to h and t i.e

$$\Psi_{t,h,\mu} \in \Phi_h \subset M_{p,\theta,\gamma}^{s,\gamma}(E).$$

This completes the proof of the Theorem 6.1. It is possible to state Theorem 6.1 in a more general setting. For this, we use the conception of extension operator.

Let

$$Y = B_{p,\theta,\gamma}^{s}(\Omega; E), \; B^{l,s}(\Omega) = B_{p,\theta,\gamma}^{l,s}(\Omega; E(A), E).$$

Condition 6.1. Let $\gamma \in A_\nu$ for $\nu \in [1, \infty]$. Assume that E is a Banach spaces with weighted Fourier type γ and $\sigma \in [1, 2]$. Suppose A is a φ-positive operator in Banach spaces E. Let a region $\Omega \subset \mathbb{R}^n$ be such that there exists a bounded linear extension operator B from $B^{l,s}(\Omega)$ to $B^{l,s}(\mathbb{R}^n)$ for $p, \theta \in [1, \infty]$.

Remark 6.1. If $\Omega \subset \mathbb{R}^n$ is a region satisfying a strong l-horn condition (see [4], § 18) $E = \mathbb{C}$, $A = I$, then there exists a bounded linear extension operator from $B_{p,\theta}^{s}(\Omega) = B_{p,\theta}^{s}(\Omega; \mathbb{C}, \mathbb{C})$ to

$$B_{p,\theta}^{s}(\mathbb{R}^n) = B_{p,\theta}^{s}(\mathbb{R}^n; \mathbb{C}, \mathbb{C}).$$

Theorem 6.2. Suppose that the all conditions of the Theorem 6.1 and the Condition 6.1 are hold. Then the embedding

$$D^\alpha B^{l,s}(\Omega) \subset B_{q,\theta,\gamma}^{s}\left(\Omega; E\left(A^{1-\varkappa-\mu}\right)\right)$$

is continuous and there exists a constant C_μ depending only on μ such that

$$\eta \|D^\alpha u\|_{B_{q,\theta,\gamma}^{s}(\Omega; E(A^{1-\varkappa-\mu}))} \leq C_\mu \left[h^\mu \|u\|_{B^{l,s}(\Omega)} + h^{-(1-\mu)} \|u\|_Y \right] \tag{6.7}$$

for all $u \in B^{l,s}(\Omega)$ and $0 < h \leq h_0 < \infty$.

Proof. It suffices to prove the estimate (7.7). Let P be a bounded linear extension operator from $B_{q,\theta,\gamma}^{s}(\Omega; E)$ to $B_{q,\theta,\gamma}^{s}(\mathbb{R}^n; E)$ and also from to $B^{l,s}(\Omega)$ to $B^{l,s}(\mathbb{R}^n)$. Let P_Ω a restriction operator from $\mathbb{R}^n$ to Ω. Then for any $u \in Y$ we have

$$\|D^\alpha u\|_{B_{q,\theta,\gamma}^{s}(\Omega; E(A^{1-\varkappa-\mu}))}$$

$$= \|D^\alpha P_\Omega P u\|_{B_{q,\theta,\gamma}^{s}(\Omega; E(A^{1-\varkappa-\mu}))} \leq C \|D^\alpha P u\|_{B_{q,\theta,\gamma}^{s}(\mathbb{R}^n; E(A^{1-\varkappa-\mu}))}$$

$$\leq C_\mu \left[h^\mu \|Pu\|_{B^{l,s}(\mathbb{R}^n)} + h^{-(1-\mu)} \|Pu\|_{B_{p,\theta,\gamma}^{s}(\mathbb{R}^n; E)} \right]$$

$$\leq C\mu \left[h^\mu \|u\|_{B^{l,s}(\Omega)} + h^{-(1-\mu)} \|u\|_Y \right].$$

Result 6.1. Let the all conditions of Theorem 6.2 hold. Then for all $u \in Y_0$ we have the following multiplicative estimate

$$\|D^{\alpha}u\|_{B^{s}_{q,\theta,\gamma}(\Omega;E(A^{1-\varkappa-\mu}))} \leq C_{\mu}\, \|u\|^{1-\mu}_{B^{l,s}_{p}(\Omega)}\, \|u\|^{\mu}_{Y}\,. \tag{6.8}$$

Indeed setting $h = \|u\|_{Y} \,. \, \|u\|^{-1}_{B^{l,s}(\Omega)}$ in (6.7) we obtain (6.8).

Result 6.2. If $l_1 = l_2 = \ldots = l_n = m$, then we obtain the continuity of embedding operators in the isotropic class

$$B^{m,s}_{p,\theta,\gamma}\left(\Omega; E\left(A\right)E\right).$$

For $E = \mathbb{C}$, $A = I$ we obtain the embedding of weighted Besov type spaces

$$D^{\alpha}B^{l,s}_{p,\theta,\gamma}\left(\Omega\right) \subset B^{s}_{q,\theta,\gamma}\left(\Omega\right).$$

7 $\mathbb{B}$-separable abstract differential equation on $\mathbb{R}^n$

Let us consider the equation (1.1).

Condition 7.1. Let

$$\text{(a) } K(\xi) = \sum_{|\alpha:l|=1} a_{\alpha}\left(i\xi_1\right)^{\alpha_1}\left(i\xi_2\right)^{\alpha_2}...\left(i\xi_n\right)^{\alpha_n} \in S(\varphi),\ \varphi < \frac{\pi}{2};$$

(b) There exists a positive constat M_0 so that

$$|K(\xi)| \geq M_0 \sum_{k=1}^{n} |\xi_k|^{l_k} \text{ for all } \xi \in \mathbb{R}^n,\ \xi \neq 0.$$

Consider the following degenerate abstract differential equation

$$Lu = \sum_{|\alpha:l|=1} a_{\alpha}D^{[\alpha]}u + Au + \sum_{|\alpha:l|<1} A_{\alpha}D^{[\alpha]}u = f, \tag{7.1}$$

where A, $A_{\alpha}(x)$ are linear operators in a Banach space E, a_k are complex-valued functions and

$$D^{[i]}_{x_k} = \left(\gamma\left(x_k\right)\frac{\partial}{\partial x_k}\right)^{i},\ D^{[\alpha]} = D^{[\alpha_1]}_{1}D^{[\alpha_2]}_{2}...D^{[\alpha_n]}_{n}.$$

Here, $B^{[l],s}_{q,\theta,\gamma}\left(\Omega; E_0, E\right)$ denote a E-valued Sobolev-Besov weighted space of functions $u \in B^{s}_{q,\theta}\left(\mathbb{R}^n; E\right)$ that have weak derivatives $D^{[l_k]}_{k}u \in B^{s}_{q,\theta}\left(\mathbb{R}^n; E\right)$ with the norm

$$\|u\|_{B^{[l,],s}_{q,\theta}(\mathbb{R}^n;E_0,E)} = \|u\|_{B^{s}_{q,\theta}(\mathbb{R}^n;E_0)} + \sum_{k=1}^{n}\left\|D^{[l_k]}_{k}u\right\|_{B^{s}_{q,\theta}(\mathbb{R}^n;E)} < \infty.$$

Remark 7.1. Under the substitution

$$\tau_k = \int_{0}^{x_k} \gamma^{-1}\left(y\right)dy \tag{7.2}$$

the spaces $B^{s}_{p,\theta,\gamma}(\mathbb{R}^n;E)$, $B^{[l],s}_{p,\theta,\gamma}(\mathbb{R}^n;E(A),E)$ are mapped isomorphically onto the weighted spaces $B^{s}_{p,\theta,\tilde{\gamma}}(\mathbb{R}^n;E)$, $B^{l,s}_{p,\theta,\tilde{\gamma}}(\mathbb{R}^n;E(A),E)$, respectively, where

$$\gamma=\prod_{k=1}^{n}\gamma(x_k),\ \tilde{\gamma}=\tilde{\gamma}(\tau)=\prod_{k=1}^{n}\gamma(x_k(\tau_k)).$$

Moreover, under the substitution (7.2) the degenerate problem (7.1) is mapped to the nondegenerate problem (1.1) considered in the weighted space $B^{s}_{p,\theta,\tilde{\gamma}}(\mathbb{R}^n;E)$.

Let

$$Y=B^{s}_{q,\theta,\gamma}(\mathbb{R}^n;E),\ Y_0=B^{l,s}_{q,\theta,\gamma}(\mathbb{R}^n;E(A),E).$$

Theorem 7.1. Suppose that the following conditions hold:

(1) Condition 7.1 is hold, $s>0$, $1\leq q$, $\theta\leq\infty$ and $0<\mu<1-|\alpha:l|$;

(2) $\gamma\in A_q$ for $q\in[1,\infty]$. E is a Banach spaces with weighted Fourier type γ and $p\in[1,2]$;

(4) A is a φ-positive operator in E and

$$A_\alpha(x)A^{-(1-|\alpha:l|-\mu)}\in L_\infty(\mathbb{R}^n;L(E)).$$

Then for all $f\in Y$ and for sufficiently large $|\lambda|$, $\lambda\in S(\varphi)$ equation (1.1) has a unique solution $u(x)\in Y_0$ and

$$\sum_{|\alpha:l|=1}\|D^\alpha u\|_Y+\|Au\|_Y\leq C\|f\|_Y. \tag{7.3}$$

Proof. Firstly, we will consider the leading part of (1.1) i.e. consider the differential-operator equation

$$(L_0+\lambda)u=\sum_{|\alpha:l|=1}D^\alpha u+Au+\lambda u=f. \tag{7.4}$$

Then we apply the Fourier transform to equation (7.4) with respect to $x=(x_1,...,x_n)$ and obtain

$$K(\xi)\,\widehat{u}(\xi)+(A+\lambda)\,\widehat{u}(\xi)=\widehat{f}(\xi). \tag{7.5}$$

Since $K(\xi)\in S(\varphi)$ for all $\xi\in\mathbb{R}^n$ therefore, $\omega=\omega(\lambda,\xi)=\lambda+K(\xi)\in S(\varphi)$ for all $\xi\in\mathbb{R}^n$, i.e. operator $A+\omega$ is invertible in E. Hence (7.5) implies that the solution of equation (7.4) can be represented in the form

$$u(x)=F^{-1}(A+\omega)^{-1}\widehat{f}. \tag{7.6}$$

It is clear to see that the operator- function $\varphi_\lambda(\xi)=[A+\omega]^{-1}$ is a multiplier in Y uniformly with respect to λ. Actually, by definition of the positive operator, for all $\xi\in\mathbb{R}^n$ and $\lambda\geq 0$ we get

$$\|\varphi_\lambda(\xi)\|_{L(E)}=\left\|(A+\omega)^{-1}\right\|\leq M(1+|\omega|)^{-1}\leq M_0.$$

Moreover, since $D_k\varphi_\lambda(\xi)=\alpha_k a_\alpha\xi^\alpha(A+\omega)^{-2}\xi_k^{-1}$, then by using the resolvent properties of positive operator A we have

$$\|\xi_k D_k\varphi_\lambda\|_{L(E)}\leq|\alpha_k a_\alpha|\,\xi^\alpha\left\|(A+\omega I)^{-2}\right\|\leq M. \tag{7.7}$$

Using the estimate (7.7) we show the uniform estimate

$$|\xi|^{\beta}\left\|D_{\xi}^{\beta}\varphi_{\lambda}(\xi)\right\|_{L(E)}\leq C \tag{7.8}$$

for

$$\beta=\beta_1,...,\beta_n),\ \beta_i\in\{0,1\},\ \xi=(\xi_1,...,\xi_n),\ \xi_i\neq 0.$$

In a similar way we can prove that the operator-functions $\varphi_{\alpha\lambda}(\xi)=\xi^{\alpha}\varphi_{\lambda,t},\ k=1,2,..,n$ and $\varphi_{0\lambda}=A\varphi_{\lambda}$ satisfiy the estimates

$$(1+|\xi|)^{|\beta|}\left\|D_{\xi}^{\beta}\varphi_{\alpha,\lambda}(\xi)\right\|_{L(E)}\leq C,\ (1+|\xi|)^{|\beta|}\left\|D_{\xi}^{\beta}\varphi_{0,\lambda}(\xi)\right\|_{L(E)}\leq C. \tag{7.9}$$

Then in view of (7.8) and (7.9) we obtain that operator-functions $\varphi_{\lambda},\varphi_{\alpha\lambda},\varphi_{0,\lambda}$ are multipliers in Y. By (7.9) and in view of

$$\|D^{\alpha}u\|_{Y}=\left\|F^{-1}\xi^{\alpha}\widehat{u}\right\|_{Y}=\left\|F^{-1}\xi^{\alpha}\left(A+\omega\right)^{-1}\widehat{f}\right\|_{Y},$$

$$\|Au\|_{Y}=\left\|F^{-1}A\widehat{u}\right\|_{Y}=\left\|F^{-1}\left[A\left(A+\omega\right)^{-1}\right]\widehat{f}\right\|_{Y}.$$

we obtain that there exists a unique solution of equation (7.4) for all $f\in Y$ and the uniform estimate holds

$$\sum_{|\alpha:l|=1}\|D^{\alpha}u\|_{\ Y}+\|Au\|_{\ Y}\leq C\left\|f\right\|_{\ Y}. \tag{7.10}$$

We consider the differential operator G_0 generated by problem (7.4), that is

$$D\left(G_0\right)=Y_0,\ G_0u=\sum_{|\alpha:l|=1}D^{\alpha}u+Au.$$

The estimate (7.10) implies that the operator $G_0+\lambda$ has a bounded inverse from Y into Y_0 for all $\lambda\geq 0$. Let G denote the differential operator in Y generated by problem (1.1). Namely,

$$D\left(G\right)=Y_0,\ Gu=G_0u+L_1u,\quad L_1u=\sum_{|\alpha:l|<1}A_{\alpha}\left(x\right)D^{\alpha}u. \tag{7.11}$$

In view of (4) condition, by virtue of Theorem 6.1, for all $u\in Y$ we have

$$\|L_1u\|_{Y}\leq\sum_{|\alpha:l|<1}\|A_{\alpha}(x)D^{\alpha}u\|_{Y}\leq\sum_{|\alpha:l|<1}\left\|A^{1-|\alpha:l|-\mu}D^{\alpha}u\right\|_{Y} \tag{7.12}$$

$$\leq C\left[h^{\mu}\left(\sum_{|\alpha:l|=1}\|D^{\alpha}u\|_{Y}+\|Au\|_{Y}\right)+h^{-(1-\mu)}\left\|u\right\|_{Y}\right].$$

Then from estimates (7.10) and (7.12) for $u\in Y_0$ we obtain

$$\|L_1u\|_{Y}\leq C\left[h^{\mu}\left\|(G_0+\lambda)u\right\|_{Y}+h^{-(1-\mu)}\left\|u\right\|_{Y}\right]. \tag{7.13}$$

Since $\|u\|_{Y}=\frac{1}{\lambda}\left\|\left(G_0+\lambda\right)u-G_0u\right\|_{Y}$ for all $u\in Y_0$ we get

$$\|u\|_{Y}\leq\frac{1}{|\lambda|}\left[\left\|\left(G_0+\lambda\right)u\right\|_{Y}+\|G_{0t}u\|_{Y}\right], \tag{7.14}$$

$$\|G_0 u\|_Y \le C \left[\sum_{|\alpha|=l} \|D^\alpha u\|_{B^s_{p,\theta,\gamma}} + \|Au\|_{B^s_{p,\theta,\gamma}} \right].$$

From estimates (7.12) – (7.14) for all $u \in Y_0$ we obtain

$$\|L_1 u\|_Y \le C h^\mu \|(G_0 + \lambda) u\|_Y + C_1 |\lambda|^{-1} h^{-(1-\mu)} \|(G_0 + \lambda) u\|_Y . \tag{7.15}$$

Then by choosing h and λ such that $Ch^\mu < 1$, $C_1 |\lambda|^{-1} h^{-(1-\mu)} < 1$ from (7.15) we obtain the uniform estimate

$$\left\| L_1 (G_0 + \lambda)^{-1} \right\|_{L(E)} < 1. \tag{7.16}$$

Using the relation (7.11), estimates (7.10) and (7.16) and the perturbation theory of linear operators we obtain that the differential operator $G + \lambda$ is invertible from Y into Y_0. Hence, inequality (7.3) is valid and this complete the proof.

Result 7.1. The Theorem 8.1 implies that G has a resolvent operator $(G + \lambda)^{-1}$ for $|\arg \lambda| \le \varphi$ and the following uniform estimate holds

$$\sum_{|\alpha:l|\le 1} |\lambda|^{1-|\alpha:l|} \left\| D^\alpha (G+\lambda)^{-1} \right\|_{L(Y)} + \left\| A (G + \lambda)^{-1} \right\|_{L(Y)} \le C.$$

Let

$$Y = B^s_{q,\theta} (\mathbb{R}^n; E), \; Y_0 = B^{[l],s}_{q,\theta,\gamma} (\mathbb{R}^n; E(A), E).$$

Let Q denote the operator in $B^s_{q,\theta} (\mathbb{R}^n, E)$ generated by problem (7.1). Theorem 7.1 and Remark 7.1 imply

Result 7.2. Let all conditions of Theorem 7.1 hold. Then for all $f \in Y$, $\lambda \in S(\varphi)$

and for sufficiently large $|\lambda|$, the equation (7.1) has a unique solution $u \in Y_0$ and the coercive uniform estimate holds

$$\sum_{|\alpha:l|\le 1} |\lambda|^{1-|\alpha:l|} \left\| D^{[\alpha]} (Q+\lambda)^{-1} \right\|_{L(Y)} + \left\| A (Q + \lambda)^{-1} \right\|_{L(Y)} \le C.$$

Remark 7.2. The Result 7.2 implies that G is a positive operator in Y. Then by virtue of [28, §1.14.5] the operator G is a generator of an analytic semigroup in Y for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$.

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