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$(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators in product generalized local Morrey spaces

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Abstract. In this paper we prove the boundedness of the $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators T_m from product generalized local Morrey space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$ with $1/p = 1/p_1 + \ldots + 1/p_m$. In all cases the conditions for the boundedness of T_m are given in terms of Zygmund-type integral inequalities on $(\varphi_1, \ldots, \varphi_m, \varphi)$, which do not require any assumption on monotonicity of $\varphi_1, \ldots, \varphi_m, \varphi$ in r.

Keywords. Product generalized local Morrey space; $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operator

Mathematics Subject Classification (2010): 42B20, 42B35, 47G10

1 Introduction

Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderón-Zygmund operators was done by Coifman and Meyer in [5] and was later systematically studied by Grafakos and Torres in [10, 11].

The classical Morrey spaces, introduced by Morrey [25] in 1938, have been studied intensively by various authors and together with Lebesgue spaces play an important role in the theory of partial differential equations. Although such spaces allow to describe local properties of functions better than Lebesgue spaces, they have some unpleasant issues. It is well known that Morrey spaces are non separable and that the usual classes of nice functions are not dense in such spaces. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [12, 23, 26] introduced generalized Morrey spaces $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ (see, also [13, 14, 21, 27]). In [14] is defined the generalized Morrey spaces $\mathcal{M}_{p,\varphi}$ with normalized norm

$$\|f\|_{\mathcal{M}_{p,\varphi}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-1/p} \, \|f\|_{L_p(B(x,r))},$$

where the function φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

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For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \dots, f_m)$, $\overrightarrow{y} = (y_1, \dots, y_n)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

Let $\overrightarrow{f} \in L^{loc}_{p_1}(\mathbb{R}^n) \times \ldots \times L^{loc}_{p_m}(\mathbb{R}^n)$. The multi-sublinear maximal operator M_m is defined by

$$M_m(\overrightarrow{f})(x) = \sup_{r>0} \prod_{i=1}^{m} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_i(y_i)| dy_i.$$

In [11] Grafakos and Torres studied the multilinear Calderón-Zygmund operator which can be written for $x \notin \cap_{j=1}^m \mathrm{supp} f_j$ as

$$K_m(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m)\, dy_1 dy_2 \dots dy_m,$$

where $K(x, y_1, \dots, y_m)$ is the kernel function defined of the diagonal $x = y_1 = \dots y_m =$ in $(\mathbb{R}^n)^{m+1}$ satisfying

$$|K(y_0, y_1, \dots, y_m)| \le c_1 \Big(\sum_{k,l=0}^{m} |y_k - y_l| \Big)^{-mn},$$

and whenever $2|y_j - y_j'| \le \frac{1}{2} \max\limits_{0 \le k \le m} |y_j - y_k|$,

$$|K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y_j', \dots, y_m)| \le \frac{c_1 |y_j - y_j'|^{\epsilon}}{\Big(\sum\limits_{k,l=0}^{m} |y_k - y_l| \Big)^{mn+\epsilon}},$$

for some $\epsilon > 0$ and all $0 \le j \le m$. Grafakos and Torres [11] proved that the operator $K_m(\overrightarrow{f})$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ for $p_i > 1 (i = 1, \dots, m)$ and $1/p = 1/p_1 + \ldots + 1/p_m$, and bounded from $L_1(\mathbb{R}^n) \times \ldots \times L_1(\mathbb{R}^n)$ to $L_{\frac{1}{m},\infty}(\mathbb{R}^n)$.

It is well known that multi-sublinear maximal operator and multilinear Calderón-Zygmund operators play an important role in harmonic analysis (see [4,8,9,11,24]).

Let T_m be a multi-sublinear operator.

Definition 1 *($(\mathbf{p},p)$-admissible multi-sublinear singular integral operator). Let multi-sublinear operator T_m will be called $(\mathbf{p},p)$-admissible multi-sublinear singular integral operator, if:*

1) *T_m satisfies the size condition of the form*

$$\chi_{B(x,r)}(z) \left| T_m \Big(f_1 \chi_{{}^{\complement}B(x,2r)}, \dots, f_m \chi_{{}^{\complement}B(x,2r)} \Big)(z) \right| \le C \chi_{B(x,r)}(z) \int_{\left({}^{\complement}B(x,2r)\right)^m} \frac{\big| f_1(y_1) \dots f_m(y_m) \big|}{|(z - y_1, \dots, z - y_m)|^{nm}} dy \qquad (1)$$

for $x \in \mathbb{R}^n$ and $r > 0$;

2) *T_m is bounded from product $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$.*

Definition 2 *(weak $(\mathbf{p},p)$-admissible multi-sublinear singular integral operator). Let multi-sublinear operator T_m will be called the weak $(\mathbf{p},p)$-admissible multi-sublinear singular integral operator, if:*
1) *T_m satisfies the size condition (1).*
2) *T_m is bounded from product $L_{p_1}(\mathbb{R}^n)\times\ldots\times L_{p_m}(\mathbb{R}^n)$ to the weak $WL_p(\mathbb{R}^n)$.*

In this study, we prove the boundedness of the $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators T_m from product generalized local Morrey space $LM^{\{x_0\}}_{p_1,\varphi_1}\times\ldots\times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$, if $1<p_1,\ldots,p_m<\infty$ and $1/p_1+\cdots+1/p_m=1/p$. Also we prove the boundedness of the weak $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators T_m from the space $LM^{\{x_0\}}_{p_1,\varphi_1}\times\ldots\times LM^{\{x_0\}}_{p_m,\varphi_m}$ to the weak space $WLM^{\{x_0\}}_{p,\varphi}$, if $1\le p_1,\ldots,p_m<\infty$, $1/p_1+\cdots+1/p_m=1/p$ and at least one exponent p_i equals one.

By $A\lesssim B$ we mean that $A\le CB$ with some positive constant C independent of appropriate quantities. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ and say that A and B are equivalent.

2 Preliminaries

For $x\in\mathbb{R}^n$ and $r>0$, we denote by $B(x,r)$ the open ball centered at x of radius r, by $|B(x,r)|$ the Lebesgue measure of the ball $B(x,r)$, and by ${}^{\complement}B(x,r)$ its complement. Morrey spaces $M_{p,\lambda}\equiv M_{p,\lambda}(\mathbb{R}^n)$ introduced by C. Morrey [25] in 1938, they are defined by the norm

$$\|f\|_{M_{p,\lambda}}:=\sup_{x\in\mathbb{R}^n,\ r>0} r^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,r))},$$

where $0\le\lambda<n$, $1\le p<\infty$.

We also denote by $WM_{p,\lambda}\equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f\in WL^{\mathrm{loc}}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}}=\sup_{x\in\mathbb{R}^n,\ r>0} r^{-\frac{\lambda}{p}}\|f\|_{WL_p(B(x,r))}<\infty,$$

where WL_p denotes the weak L_p-space. These spaces play an important role in the study of local properties of the solutions of partial differential equations, together with weighted Lebesgue spaces, see [7], [22].

We define the generalized local Morrey spaces as follows.

Definition 3 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and $1\le p<\infty$. We denote by $M_{p,\varphi}$ the generalized Morrey space, the space of all functions $f\in L^{\mathrm{loc}}_p(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}}=\sup_{x\in\mathbb{R}^n,r>0}\varphi(x,r)^{-1}\,|B(x,r)|^{-\frac{1}{p}}\,\|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi}\equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f\in WL^{\mathrm{loc}}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}}=\sup_{x\in\mathbb{R}^n,r>0}\varphi(x,r)^{-1}\,|B(x,r)|^{-\frac{1}{p}}\,\|f\|_{WL_p(B(x,r))}<\infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r)=r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda}=M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM_{p,\lambda}=WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 4 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and $1\le p<\infty$. We denote by $LM_{p,\varphi}\equiv LM_{p,\varphi}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f\in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}}=\sup_{r>0}\varphi(0,r)^{-1}\,|B(0,r)|^{-\frac{1}{p}}\,\|f\|_{L_p(B(0,r))}.$$

Also by $WLM_{p,\varphi}\equiv WLM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f\in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}}=\sup_{r>0}\varphi(0,r)^{-1}\,|B(0,r)|^{-\frac{1}{p}}\,\|f\|_{WL_p(B(0,r))}<\infty.$$

Definition 5 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and $1\le p<\infty$. For any fixed $x_0\in\mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}}\equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f\in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}}=\|f(x_0+\cdot)\|_{LM_{p,\varphi}}.$$

Also by $WLM_{p,\varphi}^{\{x_0\}}\equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f\in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}}=\|f(x_0+\cdot)\|_{WLM_{p,\varphi}}<\infty.$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ and weak local Morrey space $WLM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}$:

$$LM_{p,\lambda}^{\{x_0\}}=LM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \qquad WLM_{p,\lambda}^{\{x_0\}}=WLM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Wiener [28,29] looked for a way to describe the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted L_q spaces. Beurling [3] extended this idea and defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q+1/q'=1$. To be precise, A_q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L_q spaces; the space $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [6] observed that the space B_q can be described by

$$\|f\|_{B_q}=\sup_{k\ge 0}2^{-\frac{kn}{q}}\,\|f\chi_k\|_{L_q(\mathbb{R}^n)}, \tag{2}$$

where χ_0 is the characteristic function of the unit ball $\{x\in\mathbb{R}^n:|x|\le 1\}$, χ_k is the characteristic function of the annulus $\{x\in\mathbb{R}^n:2^{k-1}<|x|\le 2^k\}$, $k=1,2,\ldots$. By duality, the space $A_q(\mathbb{R}^n)$, called Beurling algebra now, can be described by

$$\|f\|_{A_q}=\sum_{k=0}^{\infty}2^{-\frac{kn}{q'}}\,\|f\chi_k\|_{L_q(\mathbb{R}^n)}. \tag{3}$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be the homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k \in \mathbb{Z}$ in (2) and (3) instead of $k \geq 0$ there.

If $\lambda < 0$ or $\lambda > n$, then $LM_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Note that $LM_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $LM_{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$.

$$\dot{B}_{p,\mu} = LM_{p,\varphi}\Big|_{\varphi(0,r)=r^{\mu n}}, \quad W\dot{B}_{p,\mu} = WLM_{p,\varphi}\Big|_{\varphi(0,r)=r^{\mu n}}.$$

Alvarez, Guzman-Partida and Lakey [2] in order to study the relationship between central BMO spaces and Morrey spaces, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv LM_{p,n+np\mu}(\mathbb{R}^n)$, $\mu \in [-\frac{1}{p}, 0]$. If $\mu < -\frac{1}{p}$ or $\mu > 0$, then $\dot{B}_{p,\mu}(\mathbb{R}^n) = \Theta$. Note that $\dot{B}_{p,-\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\dot{B}_{p,0}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$. Also define the weak central Morrey spaces $W\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv WLM_{p,n+np\mu}(\mathbb{R}^n)$.

The following statements, containing results obtained in [23], [26] was proved in [12,14] (see also [1,13,20]).

Theorem A. [12,13] *Let $x_0 \in \mathbb{R}^n$, $1 \leq p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \varphi_1(x_0, t)\, \frac{dt}{t} \leq C\, \varphi_2(x_0, r), \tag{4}$$

where C does not depend on x_0 and r. Then the Calderón-Zygmund operator $K \equiv K_1$ is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{1,\varphi_2}^{\{x_0\}}$.

The following statements, containing results in Theorem A was proved in [1], see also [15,20].

Theorem B. *Let $x_0 \in \mathbb{R}^n$, $1 \leq p < \infty$ and (φ_1, φ) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}}}\, \frac{dt}{t} \leq C\, \varphi(x_0, r), \tag{5}$$

where C does not depend on x_0 and r. Let the operator K is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{1,\varphi}^{\{x_0\}}$.

3 $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators in the product spaces $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$

In this section, we prove the boundedness of the $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators T_m from product generalized local Morrey space $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to $LM_{p,\varphi}^{\{x_0\}}$, if $1 < p_1, \ldots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$. Also we prove the boundedness of the weak $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators T_m from the space $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to the weak space $WLM_{p,\varphi}^{\{x_0\}}$, if $1 \leq p_1, \ldots, p_m < \infty$, $1/p_1 + \cdots + 1/p_m = 1/p$ and at least one exponent p_i equals one.

We will use the following statements on the boundedness of the weighted Hardy operators

$$H_w g(r) := \int_r^\infty g(t) w(t) dt, \ 0 < t < \infty$$

where w is a fixed function non-negative and measurable on $(0, \infty)$.
The following theorem was proved in [16,17].

Theorem 1 *[16, 17] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{6}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C=B$ is the best constant for (6).

Remark 1 In (6) it is assumed that $0\cdot\infty=0$.

The following Guliyev local estimates are valid (see [14,18]).

Lemma 1 *Let $x_0\in\mathbb{R}^n$, $m\ge 2$, $1\le p_1,\dots,p_m<\infty$ with $1/p=1/p_1+\ldots+1/p_m$.*
If T_m be a $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators, then for $p_1,\dots,p_m>1$ the inequality

$$\|T_m(\overrightarrow{f})\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))}\, t^{-\frac{n}{p_i}}\, \frac{dt}{t} \tag{7}$$

holds for any $B(x_0,r)$ and for all $\overrightarrow{f}\in L_{p_1}^{loc}(\mathbb{R}^n)\times\ldots\times L_{p_m}^{loc}(\mathbb{R}^n)$.
If T_m be a weak $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators and at least one exponent p_i equals one, then the inequality

$$\|T_m(\overrightarrow{f})\|_{WL_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))}\, t^{-\frac{n}{p_i}}\, \frac{dt}{t} \tag{8}$$

holds for any $B(x_0,r)$ and for all $\overrightarrow{f}\in L_{p_1}^{loc}(\mathbb{R}^n)\times\ldots\times L_{p_m}^{loc}(\mathbb{R}^n)$.

Proof. Let $p_1,\dots,p_m\in(1,\infty)$, $1/p=\sum_{k=1}^m 1/p_k$. For arbitrary $x\in\mathbb{R}^n$, set $B=B(x,r)$ for the ball centered at x with a radius r, $2B=B(x,2r)$. We represent $\overrightarrow{f}=(f_1,\dots,f_m)$ as

$$f_j=f_j^0+f_j^\infty,\qquad f_j^0=f_j\chi_{2B},\qquad f_j^\infty=f_j\chi_{\complement_{(2B)}},\qquad j=1,\dots,m. \tag{9}$$

Then we write

$$\prod_{i=1}^m f_i(y_i)=\prod_{i=1}^m\left(f_i^0(y_i)+f_i^\infty(y_i)\right)=\sum_{\beta_1,\dots,\beta_m\in\{0,\infty\}} f_1^{\beta_1}(y_1)\dots f_m^{\beta_m}(y_m)$$
$$=\prod_{i=1}^m f_i^0(y_i)+\sum_{\beta_1,\dots,\beta_m}{}' f_1^{\beta_1}(y_1)\dots f_m^{\beta_m}(y_m),$$

where each term in $\sum'$ contains at least one $\beta_i\ne 0$. Since T_m is an m-linear operator, then we split $T_m(\overrightarrow{f})$ as follows:

$$\left|T_m(\overrightarrow{f})(y)\right| \le \left|T_m(f_1^0,\dots,f_m^0)(y)\right| + \Big|\sum_{\beta_1,\dots,\beta_m}{}' T_m(f_1^{\beta_1},\dots,f_m^{\beta_m})(y)\Big|,$$

where $\beta_1, \dots, \beta_m \in \{0, \infty\}$ and each term in $\sum'$ contains at least one $\beta_i \neq 0$. Then,

$$\left\| T_m(\vec{f}) \right\|_{L_p(B(x,r))} \leq \| T_m(f_1^0, \dots, f_m^0) \|_{L_p(B(x,r))}$$
$$+ \Big\| \sum_{\beta_1,\dots,\beta_m}' T_m(f_1^{\beta_1}, \dots, f_m^{\beta_m}) \Big\|_{L_p(B(x,r))} \leq J^0 + \sum' J^{\beta_1,\dots,\beta_m}.$$

Thus,

$$J^0 = \| T_m(\vec{f^0}) \|_{L_p(B(x,r))} \leq \| T_m(\vec{f^0}) \|_{L_p(\mathbb{R}^n)}$$
$$\lesssim \prod_{i=1}^m \| f_i^0 \|_{L_{p_i}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \| f_i \|_{L_{p_i}(B(x,2r))}.$$

On the other hand, we have

$$\prod_{i=1}^m \| f_i \|_{L_{p_i}(2B)} \approx |B|^m \prod_{i=1}^m \| f_i \|_{L_{p_i}(2B)} \int_{2r}^\infty \frac{dt}{t^{nm+1}}$$
$$\leq |B|^m \int_{2r}^\infty \prod_{i=1}^m \| f_i \|_{L_{p_i}(B(x,t))} \frac{dt}{t^{nm+1}}$$
$$\lesssim \prod_{i=1}^m |B|^{\frac{1}{p_i}} \int_{2r}^\infty \prod_{i=1}^m \| f_i \|_{L_{p_i(B(x,t))}} |B(x,t)|^{-\frac{1}{p_i'}} \frac{dt}{t}$$
$$\approx |B|^{\frac{1}{p}} \int_{2r}^\infty \prod_{i=1}^m \| f_i \|_{L_{p_i}(B(x,t))} |B(x,t)|^{-\frac{1}{p_i}} \frac{dt}{t}. \tag{10}$$

Thus

$$J^0 \lesssim |B(x,r)|^{\frac{1}{p}} \int_{2r}^\infty \prod_{i=1}^m \| f_i \|_{L_{p_i}(B(x,t))} |B(x,t)|^{-\frac{1}{p_i}} \frac{dt}{t}. \tag{11}$$

For the other terms, let us first deal with the case when $\beta_1 = \dots = \beta_m = \infty$.

When $|x - y_i| \leq r$, $|z - y_i| \geq 2r$, we have $\frac{1}{2}|z - y_i| \leq |x - y_i| \leq \frac{3}{2}|z - y_i|$, and therefore,

$$|T_m(f_1^\infty, \dots, f_m^\infty)(z)| \lesssim \int_{\left(^{\complement}B(x,2r)\right)^m} \frac{|f_1(y_1) \cdots f_m(y_m)|}{(|x - y_1| + \ldots + |x - y_m|)^{mn}} d\vec{y}$$
$$\lesssim \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x - y_i|^n} dy_i$$

and

$$\| T_m(f_1^\infty, \dots, f_m^\infty) \|_{L_p(B(x,r))} \leq \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x - y_i|^n} dy_i \, \| \chi_{B(x,r)} \|_{L_p(\mathbb{R}^n)}$$
$$\lesssim |B(x,r)|^{\frac{1}{p}} \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x - y_i|^n} dy_i.$$

By Fubini's theorem we have

$$\int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^{m} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \approx \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^{m} |f_i(y_i)| dy_i \int_{|x-y_i|}^{\infty} \frac{dt}{t^{n+1}}$$
$$\approx \int_{2r}^{\infty} \prod_{i=1}^{m} \int_{2r\le|x-y_i|<t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty} \prod_{i=1}^{m} \int_{B(x,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality, we get

$$\int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^{m} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \lesssim \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} |B(x,t)|^{-\frac{1}{p_i'}} \frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} |B(x,t)|^{-\frac{1}{p_i}} \frac{dt}{t}. \tag{12}$$

Moreover, for all $p_i \in [1,\infty)$, $i = 1,\ldots,m$ the inequality

$$\left\|T_m(f_1^{\infty},\ldots,f_m^{\infty})\right\|_{L_p(B(x,r))}$$
$$\lesssim |B(x,r)|^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} |B(x,t)|^{-\frac{1}{p_i}} \frac{dt}{t} \tag{13}$$

is valid.

We now consider the cases when exactly l of the β_l's are ∞ for some $1 \le l < m$. We only give the arguments for one of these cases. The rest are similar and can easily be obtained from the arguments below by permuting the indices. To this end we may assume that $\beta_1 = \ldots = \beta_l = \infty$ and $\beta_{l+1} = \ldots = \beta_m = 0$. Recall the fact that $|x-y_i| \approx |z-y_i|$ for $z \in B(x,r)$, $y_i \in {}^{\complement}B(x,2r)$ and $1 \le i \le l$. We have

$$\left|T_m(f_1^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_m^0)(z)\right|$$
$$\lesssim \int_{\left(^{\complement}B(x,2r)\right)^l} \int_{\left(B(x,2r)\right)^{m-l}} \frac{|f_1(y_1)\cdots f_m(y_m)|\, d\vec{y}}{(|x-y_1|+\ldots+|x-y_m|)^{mn}}$$
$$\lesssim \int_{\left(^{\complement}B(x,2r)\right)^l} \frac{|f_1(y_1)\cdots f_l(y_l)|\, dy_1\ldots dy_l}{(|x-y_1|+\ldots+|x-y_l|)^{mn}}$$
$$\times \int_{\left(B(x,2r)\right)^{m-l}} |f_{l+1}(y_{l+1})\cdots f_m(y_m)|\, dy_{l+1}\ldots dy_m$$
$$\lesssim \int_{\left(^{\complement}B(x,2r)\right)^l} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i.$$

By Fubini's theorem we have

$$\int_{\left(\complement_{B(x,2r)}\right)^l} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i$$
$$\approx \int_{\left(\complement_{B(x,2r)}\right)^l} \prod_{i=1}^{l} |f_i(y_i)| dy_i \int_{|x-y_i|}^{\infty} \frac{dt}{t^{n+1}} \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i$$
$$\approx \int_{2r}^{\infty} \prod_{i=1}^{l} \int_{2r\le|x-y_i|<t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}} \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i$$
$$\lesssim \int_{2r}^{\infty} \prod_{i=1}^{l} \int_{B(x,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}} \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i.$$

Applying Hölder's inequality, we get

$$\int_{\complement_{(2B)}} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i$$
$$\lesssim \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} |B(x,t)|^{\frac{1}{p_i'}} \frac{dt}{t^{n+1}}$$
$$\le \int_{2r}^{\infty} \prod_{i=1}^{m} \|f\|_{L_{p_i}(B(x,t))} |B(x,t)|^{-\frac{1}{p_i}} \frac{dt}{t}. \tag{14}$$

From (14) we get

$$\|T_m(f_1^{\infty},\dots,f_l^{\infty},f_{l+1}^0,\dots,f_m^0)\|_{L_p(B(x,r))}$$
$$\lesssim |B(x,r)|^{\frac{1}{p}} \prod_{i=1}^{l} \int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i$$
$$\lesssim |B(x,r)|^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f\|_{L_{p_i}(B(x,t))} |B(x,t)|^{-\frac{1}{p_i}} \frac{dt}{t}.$$

We now proof the second part. For any ball $B = B(x,r) \subset \mathbb{R}^n$, decompose $f_i = f_i^0 + f_i^{\infty}$, where $f_i^0 = f_i \chi_{2B}$, $2B = B(x,2r)$, $i = 1,\dots,m$. Then for any given $\lambda > 0$, we can write

$$\left(\left\{y \in B(x,r) : \left|T_m(\overrightarrow{f})(y)\right| > \lambda\right\}\right)^{\frac{1}{p}}$$
$$\le \left(\left\{y \in B(x,r) : \left|T_m(f_1^0,\dots,f_m^0)(y)\right| > \lambda/2^m\right\}\right)^{\frac{1}{p}}$$
$$+ \sum{}' \left(\left\{y \in B(x,r) : \left|T_m(f_1^{\beta_1},\dots,f_m^{\beta_m})(y)\right| > \lambda/2^m\right\}\right)^{\frac{1}{p}} = J_*^0 + \sum{}' J_*^{\beta_1,\dots,\beta_m},$$

where each term in $\sum{}'$ contains at least one $\beta_i \ne 0$. We have

$$J_*^0 = \|T_m(\overrightarrow{f^0})\|_{WL_p(B(x,r))} \le \|T_m(\overrightarrow{f^0})\|_{WL_p(\mathbb{R}^n)}$$
$$\lesssim \prod_{i=1}^{m} \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} = \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,2r))}.$$

We have the following estimate:

$$\|T_m(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{L_p(B(x,r))} \lesssim |B(x,r)|^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f\|_{L_{p_i}(B(x,t))} \, |B(x,t)|^{-\frac{1}{p_i}} \, \frac{dt}{t}.$$

Then

$$\begin{aligned}\sum{}' J_*^{\beta_1,\dots,\beta_m} &= \sum{}' \|T_m(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{WL_p(B(x,r))} \\ &\le \sum{}' \|T_m(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{L_p(B(x,r))} \\ &\lesssim |B(x,r)|^{1/q} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f\|_{L_{p_i}(B(x,t))} \, |B(x,t)|^{-\frac{1}{p_i}} \, \frac{dt}{t}.\end{aligned}$$

Now we give the boundedness of multi-sublinear singular integral operators in product generalized local Morrey spaces.

Theorem 2 *Let $x_0 \in \mathbb{R}^n$, $m \ge 2$, $1 \le p_1,\dots,p_m < \infty$ with $1/p = 1/p_1 + \dots + 1/p_m$ and $(\varphi_1,\dots,\varphi_m,\varphi)$ satisfies the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x_0,s) s^{\frac{n}{p_i}}}{t^{\frac{n}{p}+1}} dt \lesssim \varphi(x_0,r), \tag{15}$$

where the implicit constant does not depend on r.

If T_m be a $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators, then the operator T_m is bounded from product space $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to $LM_{p,\varphi}^{\{x_0\}}$ for $p_i > 1$, $i = 1,\dots,m$.

If T_m be a weak $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators, then the operator T_m is bounded from product space $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to $WLM_{p,\varphi}^{\{x_0\}}$ for $p_i \ge 1$, $i = 1,\dots,m$, $\min\{p_1,\dots,p_m\} = 1$.

Proof. Let $1 < p_1,\dots,p_m < \infty$ and $\overrightarrow{f} \in LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$. By Theorem 1 and Lemma 1 with $v_2(r) = \varphi(x_0,r)^{-1}$, $v_1(r) = \prod_{i=1}^m \varphi_i(x_0,r)^{-1} r^{-\frac{n}{p_i}}$, we have

$$\begin{aligned}\|T_m(\overrightarrow{f})\|_{LM_{p,\varphi}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi(x_0,r)^{-1} \int_r^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))} \, t^{-\frac{n}{p_i}} \, \frac{dt}{t} \\ &\lesssim \sup_{r>0} \prod_{i=1}^m \varphi_i(x_0,r)^{-1} \, r^{-\frac{n}{p_i}} \, \|f_i\|_{L_{p_i}(B(x_0,r))} = \prod_{i=1}^m \|f_i\|_{LM_{p_i,\varphi_i}^{\{x_0\}}}.\end{aligned}$$

When $p_i \ge 1$, $i = 1,\dots,m$, $\min\{p_1,\dots,p_m\} = 1$, the proof is similar and we omit the details here.

From Theorem 2 we get the following corollary about boundedness of multi-sublinear maximal operator and multilinear singular integral operators on product local generalized Morrey space.

Corollary 1 *Let $x_0 \in \mathbb{R}^n$, $m \ge 2$, $1 \le p_1,\dots,p_m < \infty$ with $1/p = 1/p_1 + \dots + 1/p_m$ and $(\varphi_1,\dots,\varphi_m,\varphi)$ satisfies the condition (15). Then the operators M_m and K_m are bounded from product space $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to $LM_{p,\varphi}^{\{x_0\}}$ for $p_i > 1$, $i = 1,\dots,m$, and from product space $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to $WLM_{p,\varphi}^{\{x_0\}}$ for $p_i \ge 1$, $i = 1,\dots,m$, $\min\{p_1,\dots,p_m\} = 1$.*

From Theorem 2 we get the following corollary about boundedness of $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators on product generalized Morrey space.

Corollary 2 *Let $m \geq 2$, $1 \leq p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x,s) s^{\frac{n}{p_i}}}{t^{\frac{n}{p}}} \frac{dt}{t} \lesssim \varphi(x,r), \tag{16}$$

where the implicit constant does not depend on x and r.

If T_m be a $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators, then the operator T_m is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $M_{p,\varphi}$ for $p_i > 1$, $i = 1, \ldots, m$.

If T_m be a weak $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators, then the operator T_m is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $WM_{p,\varphi}$ for $p_i \geq 1$, $i = 1, \ldots, m$, $\min\{p_1, \ldots, p_m\} = 1$.

From Corollary 2 we get the following corollary proven in [18] (see also [19]) about boundedness of multi-sublinear maximal operator and multilinear singular integral operators on product generalized Morrey space.

Corollary 3 *Let $m \geq 2$, $1 \leq p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition* (16). *Then the operators M_m and K_m are bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $M_{p,\varphi}$ for $p_i > 1$, $i = 1, \ldots, m$, and from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $WM_{p,\varphi}$ for $p_i \geq 1$, $i = 1, \ldots, m$,* $\min\{p_1, \ldots, p_m\} = 1$.

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