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On the Outer Connected Geodetic Number of a Graph

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Abstract. For a connected graph G of order at least two, a *connected outer connected geodetic set* S of G is an outer connected geodetic set such that the subgraph induced by S is connected. The minimum cardinality of a connected outer connected geodetic set of G is the *connected outer connected geodetic number* of G and is denoted by $cg_{co}(G)$. We determine bounds for it and characterize graphs which realize these bounds. Some realization results on the connected outer connected geodetic number of a graph are studied.

Keywords. Outer connected geodetic set · Outer connected geodetic number · Connected outer connected geodetic set · Connected outer connected geodetic number

Mathematics Subject Classification (2010): 05C12

1 Introduction

By a graph $G = (V, E)$, we mean a finite simple undirected connected graph. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [1,8]. For any two vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest $x - y$ path in G. A $x - y$ path of length $d(x, y)$ is called $x - y$ *geodesic*. A vertex v of G is said to lie on a $x - y$ geodesic P if v is a vertex of P including the vertices x and y. For any vertex u of G, the eccentricity of u is defined as $e(u) = max\{d(u, v) : v \in V(G)\}$. The radius $rad(G)$ and diameter $diam(G)$ of G are defined as $rad(G) = min\{e(v) : v \in V(G)\}$ and $diam(G) = max\{e(v) : v \in V(G)\}$, respectively. The $neighborhood$ of a vertex v is the set $N(v)$ consisting of all vertices u

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which are adjacent with v. A vertex v of G is called an *extreme vertex* of G if the subgraph induced by its neighbors is complete.

The *closed interval* $I[x,y]$ consists of all vertices lying on some $x-y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x,y]$. A set S of vertices of G is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set of G is the *geodetic number* $g(G)$ of G. The geodetic number of a graph and its variants have been studied by several authors in [2–6, 9, 10]. A set S of vertices in a graph G is said to be an *outer connected geodetic set* if S is a geodetic set of G and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected geodetic set of G is the *outer connected geodetic number* of G and is denoted by $g_{oc}(G)$. The outer connected geodetic number of a graph was introduced and studied in [7]. This concept can be mainly used in fault-tolerant in communication network design [7].

The following theorems will be used in the sequel.

Theorem 1.1 [7] Each extreme vertex of a connected graph G belongs to every outer connected geodetic set of G.

Theorem 1.2 [7] For the complete graph $K_p (p \geq 2)$, $g_{oc}(K_p) = p$.

Theorem 1.3 [7] If T is a tree with k endvertices, then $g_{oc}(T) = k$.

Throughout this paper G denotes a connected graph with at least two vertices.

2 Main Results

Definition 2.1 A *connected outer connected geodetic set* S of G is an outer connected geodetic set such that the subgraph induced by S is connected. The minimum cardinality of a connected outer connected geodetic set of G is the *connected outer connected geodetic number* of G and is denoted by $cg_{co}(G)$.

Example 1 For the graph G given in Figure 2.1, it is clear that no 2-element subset of $V(G)$ is an outer connected geodetic set of G. It is easily verified that $S = \{v_2, v_4, v_6\}$ is the unique minimum outer connected geodetic set of G and so $g_{oc}(G) = 3$. Since the subgraph induced by S is not connected, S is not a connected outer connected geodetic set of G. Clearly, $S_1 = S \cup \{v_3\}$ is a minimum connected outer connected geodetic set of G so that $cg_{co}(G) = 4$. Thus the outer connected geodetic number and the connected outer connected geodetic number of a graph are different.

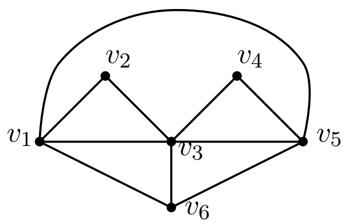

Figure 2.1: G

Theorem 2.1 Each extreme vertex of a connected graph G belongs to every connected outer connected geodetic set of G.

Proof. Since every connected outer connected geodetic set of G is also an outer connected geodetic set of G, the result follows from Theorem 1.1.

Corollary 2.1 For the complete graph $K_p (p \geq 2)$, $cg_{co}(K_p) = p$.

Theorem 2.2 Let G be any connected graph with cut-vertices and let S be a connected outer connected geodetic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.

Proof. Let v be a cut-vertex of G and S be a connected outer connected geodetic set of G. Suppose that there exists a component, say G_1 of $G - v$ such that G_1 contains no vertex of S. Let u be a vertex of G_1. Since by Theorem 2.1, S contains all the extreme vertices of G, u is not an extreme vertex of G. Since S is a connected outer connected geodetic set of G, there exists a pair of vertices $x, y \in S$ such that u is an internal vertex of some $x - y$ geodesic $P : x = u_0, u_1, ..., u, ...u_n = y$ in G. Since v is a cut-vertex of G, the $x - u$ subpath of P and $u - y$ subpath of P both contain v, and it follows that P is not a path, which is a contradiction.

Theorem 2.3 Every cut-vertex of a connected graph G belongs to every connected outer connected geodetic set of G.

Proof. Let S be a connected outer connected geodetic set of G and let v be a cut-vertex of G. Let $G_1, G_2, \dots, G_r (r \geq 2)$ be the component of $G - v$. By Theorem 2.2, S contains at least one vertex from each $G_i (1 \leq i \leq r)$. Since the subgraph induced by S is connected and v is a cut-vertex of G, it follows that $v \in S$.

The next corollaries follows from Theorems 2.1 and 2.3

Corollary 2.2 For the star $K_{1,p-1} (p \geq 1)$, $cg_{co}(K_{1,p-1}) = p$.

Corollary 2.3 For a connected graph G with k extreme vertices and l cut-vertices, $max\{2, k + l\} \leq cg_{co}(G) \leq p$.

Corollary 2.4 For any non-trivial tree T of order p, $cg_{co}(G) = p$.

For any real x, $\lfloor X \rfloor$ denotes the largest integer less than or equal to X.

Theorem 2.4 For any cycle $C_p (p \geq 3)$, $cg_{co}(C_p) = \begin{cases} \frac{p}{2} + 1 \text{ if } p \text{ is even} \\ \lfloor \frac{p}{2} \rfloor + 2 \text{ if } p \text{ is odd.} \end{cases}$

Proof. We prove this theorem by considering two cases.
Case 1. Suppose that p is even. Let $p = 2n$. Let $C_{2n} : v_1, v_2, v_3, \dots, v_{2n}, v_1$ be a cycle of order $2n$. Let $S = \{v_1, v_2, v_3, \dots, v_{n+1}\}$. It is clear that S is an outer connected geodetic set of C_p and the subgraph induced by S is connected. Thus S is a connected outer connected geodetic set of G, $cg_{co}(G) \leq n + 1$. It is easily verified that G has no connected outer connected geodetic set of G with cardinality at most n. Hence $cg_{co}(C_p) = n + 1$.
Case 2. Suppose that p is odd. Let $p = 2n + 1$. Let $C_{2n+1} : v_1, v_2, v_3, \dots, v_{2n+1}, v_1$ be a cycle of order $2n + 1$. Let $S = \{v_1, v_2, v_3, ..., v_{n+1}, v_{n+2}\}$. Then, similar to Case 1, it is easily verified that S is a minimum connected outer connected geodetic set of C_p and $cg_{co}(C_p) = n + 2$.

Theorem 2.5 For a connected graph G of order $p \geq 2$, $2 \leq g_{oc}(G) \leq cg_{co}(G) \leq p$.

Proof. Any outer connected geodetic set of G needs at least two vertices and so $g_{oc}(G) \geq 2$. Since every connected outer connected geodetic set of G is an outer connected geodetic set of G, it follows that $g_{oc}(G) \leq cg_{co}(G)$. Also, $V(G)$ is a connected outer connected geodetic set of G, it is clear that $cg_{co}(G) \leq p$. Hence $2 \leq g_{oc}(G) \leq cg_{co}(G) \leq p$.

Corollary 2.5 Let G be a connected graph G of order $p(p \geq 2)$. If $cg_{co}(G) = 2$ then $g_{oc}(G) = 2$.

For any non-trivial path $P_n(n \geq 3)$, the outer connected geodetic number is 2 and the connected outer connected geodetic number is n. This shows that the converse of Corollary 2.5 need not be true.

Remark 2.1 The bounds in Theorem 2.5 are sharp. For any non-trivial path $P_n(n \geq 3)$, $g_{oc}(P_n) = 2$ and $cg_{co}(P_n) = n$. Also, all the inequalities in Theorem 2.5 can be strict. For the graph G given in Figure 2.1, $g_{oc}(G) = 3$, $cg_{co}(G) = 4$ and $p = 6$. Thus, we have $2 < g_{oc}(G) < cg_{co}(G) < p$.

Now we proceed to characterize graphs G for which the bounds in Theorem 2.5 are attained.

Theorem 2.6 Let G be a connected graph of order $p(p \geq 2)$. Then every vertex of G is either an extreme vertex or a cut-vertex if and only if $cg_{co}(G) = p$.

Proof. Let G be a connected graph with every vertex of G either an extreme vertex or a cut-vertex. Then the result follows from Theorems 2.1 and 2.3. Conversely, let $cg_{co}(G) = p$. Suppose that there is a vertex x in G which is neither a cut-vertex nor an extreme vertex. Since x is not an extreme vertex, the subgraph induced by $N(x)$ is not complete. Then there exists two vertices u and v in $N(x)$ such that $d(u, v) \geq 2$. It is clear that x lies on a $u - v$ geodesic in G. Since x is not a cut-vertex of G, $G - x$ is connected. Clearly, $V - \{x\}$ is a connected outer connected geodetic set of G and so $cg_{co}(G) \leq |V - \{x\}| = p - 1$, which is a contradiction.

Theorem 2.7 For any connected graph G of order $p \geq 2$, $cg_{co}(G)$ = 2 if and only if $G = K_2$.

Proof. If $G = K_2$, then $cg_{co}(G) = 2$. Conversely, let $cg_{co}(G) = 2$. Let $S = \{u, v\}$ be a minimum connected outer connected geodetic set of G. Then uv is an edge. It is clear that a vertex different from u and v cannot lie on a $u - v$ geodesic and so $G = K_2$.

3 Some realization results

In view of Theorem 2.5, we have the following realization result.

Theorem 3.1 If p, a and b are integers such that $3 \leq a < b \leq p$, then there exists a connected graph G of order p with $g_{oc}(G) = a$ and $cg_{co}(G) = b$.

Proof. We prove this theorem by considering two cases.
Case 1. $3 \leq a < b = p$. Let G be any tree of order p with a end-vertices. Then by Theorem 1.3 and Corollary 2.4, $g_{oc}(G)$ = a and $cg_{co}(G) = p$.

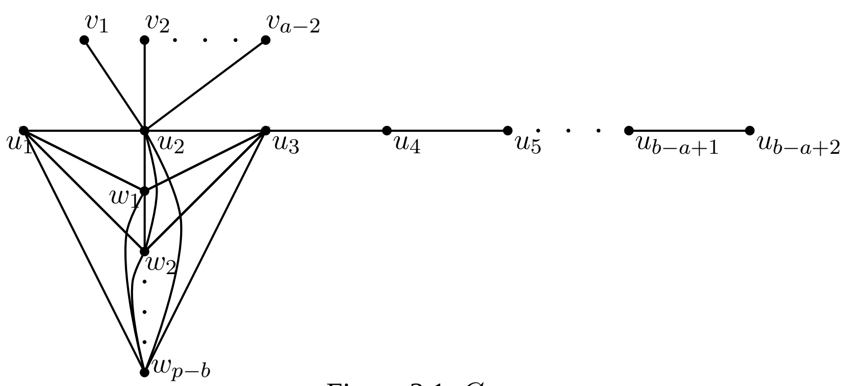

Figure 3.1: G

Case 2. $3 \leq a < b < p$. Let P_{b-a+2}: $u_1, u_2, ..., u_{b-a+2}$ be a path of order $b - a + 2$. Add $p-b+a-2$ new vertices $v_1, v_2, ..., v_{a-2}, w_1, w_2, ..., w_{p-b}$ to P_{b-a+2} and join each $v_i(1 \leq i \leq a-2)$ with the vertex u_2; and join each $w_i(1 \leq i \leq p-b)$ with the vertices u_1, u_2, u_3; and also join each $w_i(1 \leq i \leq p-b-1)$ to each $w_j(i+1 \leq j \leq p-b)$, thereby producing the graph G of order p, shown in Figure 3.1. Let $S = \{v_1, v_2, ..., v_{a-2}, u_1, u_{b-a+2}\}$ be the set of all extreme vertices of G. By Theorems 1.1 and 2.1, every outer connected geodetic set and every connected outer connected geodetic set of G contain S. It is clear that S is the unique minimum outer connected geodetic set of G and so $g_{oc}(G)$ = a. Since the subgraph induced by S is not connected, S is not a connected outer connected geodetic set of G. Let $S_1 = S \cup \{u_2, u_3, u_4, ..., u_{b-a+1}\}$ be the set of all extreme vertices and cut-vertices of G. By Theorems 2.1 and 2.3, every connected outer connected geodetic set of G contain S_1, and the subgraph induced by S_1 is connected. It is clear that S_1 is the unique minimum connected outer connected geodetic set of G and so $cg_{co}(G) = b$.

For any connected graph G, $rad(G) \leq diam(G) \leq 2rad(G)$. Ostrand[11] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively, of some connected graph. Now, Ostrands theorem can be extended so that the connected outer connected geodetic number can also be prescribed.

Theorem 3.2 For any three integers r, d and $k \geq d+1$ with $r \leq d \leq 2r$ there exists a connected graph G with $rad(G) = r$, $diam(G) = d$ and $cg_{co}(G) = k$.

Proof. We prove this theorem by considering three cases.
Case 1. If $r = 1$, then $d = 1$ or 2. If $d = 1$, let $G = K_k$. Then by Corollary 2.1, $cg_{co}(G) = k$. If $d = 2$, let $G = K_{1,k-1}$. Then by Corollary 2.2, $cg_{co}(G) = k$.

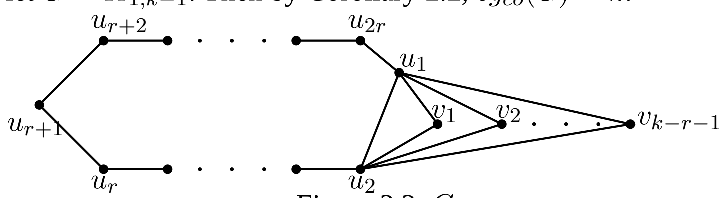

Figure 3.2: G

Case 2. $r \geq 2$ and $r = d$. First, let $k \geq r+1$. Let $C_{2r} : u_1, u_2, ..., u_{2r}, u_1$ be a cycle of order $2r$. Let G be the graph obtained from C_{2r} by adding '$k - r + 1$' new vertices $v_1, v_2, ..., v_{k-r-1}$ and joining each $v_i(1 \leq i \leq k-r-1)$ with the vertices u_1 and u_2 of C_{2r}. The graph G is shown in Figure 3.2. It is easily verified that the eccentricity of each vertex of G is r so that $rad(G)$ = $diam(G)$ = r. Let $S = \{v_1, v_2, ..., v_{k-r-1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, every connected outer connected geodetic set of G contains S. It is clear that S is not a connected outer connected geodetic set of G. It follows from Theorems 2.1 and 2.4 that $S \cup \{u_1, u_2, ..., u_{r+1}\}$ is a minimum connected outer connected geodetic set of G and so $cg_{co}(G) = k$.

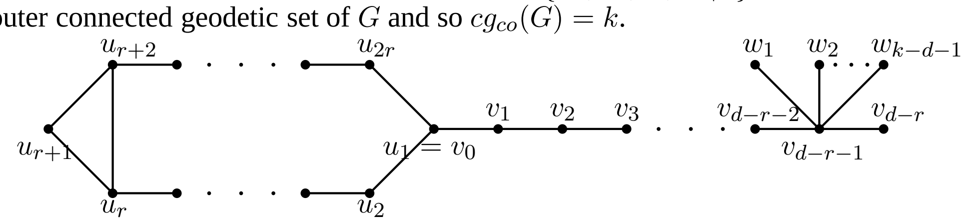

Figure 3.3: G

Case 3. $r \geq 2$ and $r < d \leq 2r$. Let $C_{2r} : u_1, u_2, ..., u_{2r}, u_1$ be a cycle of order $2r$ and let P_{d-r+1}: $v_0, v_1, ..., v_{d-r}$ be a path of order $d - r + 1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying the vertex v_0 of P_{d-r+1} and the vertex u_1 of C_{2r} and joining the vertex u_{r+2} to the vertex u_r. Let G be the graph obtained from H by adding $k - d - 1$ new vertices $w_1, w_2, ..., w_{k-d-1}$ and joining each vertex $w_i(1 \leq i \leq k-d-1)$ to the vertex v_{d-r-1}. The graph G is shown in Figure 3.3. It is easy to verify that $r \leq e(x) \leq d$ for any vertex x in G and $e(u_1) = r$ and $e(v_{d-r}) = d = e(u_{r+1})$. Then $rad(G) = r$ and $diam(G) = d$. Let $S = \{u_1, v_1, v_2, ..., v_{d-r-1}, v_{d-r}, u_{r+1}, w_1, w_2, ..., w_{k-d-1}\}$ be the set of all cut-vertices and extreme vertices of G. By Theorems 2.1 and 2.3, every connected outer connected geodetic set of G contain S. It is clear that S is an outer connected geodetic set of G and the subgraph induced by S is not connected, S is not a connected outer connected geodetic set of G. It is easily verify that $S \cup \{u_2, u_3, ..., u_r\}$ is a minimum connected outer connected geodetic set of G and so $cg_{co}(G) = k$.

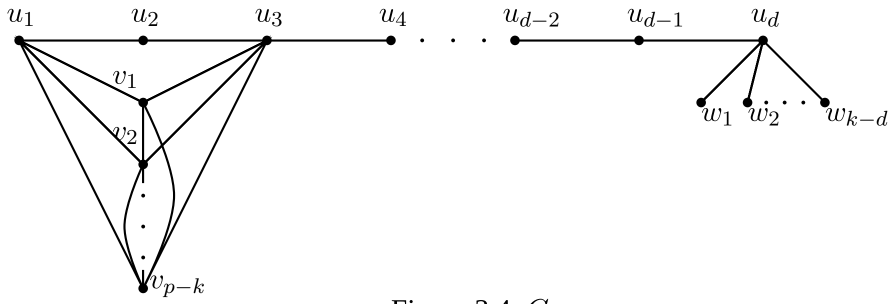

Figure 3.4: G

Theorem 3.3 If p, d and k are integers such that $3 \leq d \leq k-1$ and $p \geq k+1$, then there exists a connected graph G of order p, diameter d and $cg_{co}(G) = k$.

Proof. Let $P_d : u_1, u_2, \ldots, u_d$ be a path of order d. Add $p-d$ new vertices $v_1, v_2, \ldots, v_{p-k}$, $w_1, w_2, \ldots, w_{k-d}$ to P_d and join each $v_i (1 \leq i \leq p-k)$ with the vertices u_1 and u_3; and join each $w_j (1 \leq j \leq k-d)$ with the vertex u_d and also join each $v_i (1 \leq i \leq p-k-1)$ with $v_j (i+1 \leq j \leq p-k)$, thereby producing the graph G of order p with diameter d is shown in Figure 3.4. Let $S = \{w_1, w_2, \ldots, w_{k-d}, u_3, u_4, \ldots, u_d\}$ be the set of all extreme vertices and cut-vertices of G. By Theorems 2.1 and 2.3 every connected outer connected geodetic set of G contain S. It is clear that S is not a connected outer connected geodetic set of G. Also, for any vertex $x \in V - S$, $S \cup \{x\}$ is not a connected outer connected geodetic set of G. It is easily verified that $S_1 = S \cup \{u_1, u_2\}$ is a connected outer connected geodetic set of G so that $cg_{co}(G) = k$.

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