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Approximate solution of a class of nonlinear singular integral equations by the Newton-Kantorovich method

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Abstract. In the paper a class of nonlinear singular integral equation with a Cauchy kernel encountered in theory of boundary value problems for analytic functions is solved by the Newton-Kantorovich method.

Keywords. Newton-Kantorovich method, nonlinear singular integral equation, Fréched derivative, Hölder condition

Mathematics Subject Classification (2010): 30E20, 30E25

1 Introduction

We consider a problem of finding the solution of a nonlinear singular integral equation (NSIE) of the form

$$f(t,u(t)) - \frac{1}{\pi}\int_{-1}^{1}\frac{u(\tau)}{\tau - t}d\tau - D = 0, \tag{1}$$

satisfying the condition

$$u(-1) = u(1) = 0, \tag{2}$$

where $f(t,u)$ is the given function. It is required to find the function $u(t)$ and constant D for which (1)-(2) holds. Problem (1)-(2) is equivalent to the following Riemann-Hilbert problem [1]:

Find a holomorphic function $W(z) = u(z) + iv(z)$, $z = x + iy$ in the upper half-plane $y > 0$ of the complex plane z, that satisfies the Hölder condition in the half-plane $y \geq 0$, is bounded at infinity, and on the straight line $y = 0$ satisfies the boundary condition

$$\begin{cases} u(x) = 0, & \text{if} \quad |x| \geq 1 \\ v(x) + f(x,u(x)) = 0, & \text{if} \quad |x| < 1. \end{cases} \tag{3}$$

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The equivalence of these problems follows directly from the representation

$$W(z) = \frac{1}{\pi i}\int_{-1}^{1}\frac{u(\xi)}{\xi - z}d\xi - Di,$$

where D is a real number, $-Di$ is the value of the function $W(z)$ at infinity.

In the paper [1] if in shown that equivalence of problems (1)-(2) and may be used when reducing problem (1)-(2) for the function $f(x, u)$ of the form

$$f(x, u) = b_0(x) + b_1(x)u + \psi(u), \tag{4}$$

where $b_i(x)$ are the functions satisfying the Hölder condition, $j = 1, 2$, the $\psi(u)$ is a function with a bounded measurable derivative $\psi'(u)$ by means of the substitution

$$U = u, \quad V = v + \psi(u)$$

to the Riemann-Hilbert problem for solution the Beltrami-type quasilinear elliptic equation $W = U + iV$ with a linear boundary conditions

$$Re[G(x)W(x)] = g(x), \tag{5}$$

where

$$G(x) = \begin{cases} 1, & if \quad |x| \geq 1, \\ b_1(x) - i, & if \quad |x| < 1, \end{cases}$$

$$g(x) = \begin{cases} 0, & if \quad |x| \geq 1, \\ -b_0(x), & if \quad |x| < 1. \end{cases}$$

We note that equations of the form (1) are encountered alow in contact theory of elasticity (see [4]).

It follows from singular integral equations that for solving problem (1)-(2) have the following relations [2]:

$$u(t) = -\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{f(\tau, u(\tau))}{\sqrt{1-\tau^2}(\tau - t)}d\tau, \tag{6}$$

$$D = \frac{1}{\pi}\int_{-1}^{1}\frac{f(\tau, u(\tau))}{\sqrt{1-\tau^2}}d\tau. \tag{7}$$

If the function $u^*(t)$ $(u^*(-1) = u^*(1) = 0)$ is the solution of NSIE (6), then the pair (u^*, D^*) is the solution of the problem (1)-(2), where

$$D^* = \frac{1}{\pi}\int_{-1}^{1}\frac{f(\tau, u^*(\tau))}{\sqrt{1-\tau^2}}d\tau.$$

We will consider NSIE in the space $\dot{H}_\alpha$ $(0 < \alpha < \frac{1}{2})$:

$$\dot{H}_\alpha = \{\varphi \in H_\alpha[-1, 1] \,|\, \varphi(-1) = \varphi(1) = 0\},$$

where $H_\alpha[-1, 1]$ is a space of functions satisfying on [-1,1] the Hölder condition with the exponent α. Note that the norm in $\dot{H}_\alpha$ is determined in the following way:

$$\|\varphi\|_\alpha = \sup_{t_1, t_2 \in [-1,1], t_1 \neq t_2}\frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\alpha}.$$

As is known [3], the operator

$$(Av)(t) = \frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{v(\tau)}{\sqrt{1-\tau^2}(\tau-t)}d\tau$$

acts from $\dot{H}_\alpha$ to $\dot{H}_\alpha$ $(0<\alpha<\frac{1}{2})$ and is bounded.

We introduce the following operator

$$(Pu)(t) = u(t) + \frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{f(\tau,u(\tau))}{\sqrt{1-\tau^2}(\tau-t)}d\tau.$$

Then NSIE (6) may be written in the form

$$Pu = 0.$$

Let the function $f(t,u)$ be determined in the domain $\Omega=\{(t,u)|-1\le t\le 1, -\infty < u < +\infty\}$ and have the following properties:

1) $f(t,0)\in \dot{H}_\alpha$,

2) $\exists l>0$ such that for $\forall (t_1,u_1),(t_2,u_2)\in\Omega$ $|f(t_1,u_1)-f(t_2,u_2)|\le l(|t_1-t_2|^\alpha + |u_1-u_2|)$.

Under there conditions, it is known [3] that the operator P acts from $\dot{H}_\alpha$ to $\dot{H}_\alpha$ and is bonded. We prove the following theorem.

Theorem 1 *Let in the domain Ω there exist $f'_u(t,u)$, $f''_{u^2}(t,u)$ and the following conditions, the fulfilled:*

a) $f(t,0)$, $f'_u(t,0), f''_{u^2}(t,0)\in\dot{H}_\alpha$,

b) $\exists l_0, l_1>0$ *are such that for* $\forall (t_1,u_1),(t_2,u_2)\in\Omega$

$$|f(t_1,u_1)-f(t_2,u_2)|\le l_0(|t_1-t_2|^\alpha+|u_1-u_2|),$$

$$|f'_u(t_1,u_1)-f'_u(t_2,u_2)|\le l_1(|t_1-t_2|^\alpha+|u_1-u_2|),$$

c) $\forall u\in B(0,r)$, $\forall t_1,t_2\in[-1,1]$

$$|f''_{u^2}(t_1,u(t_1))-f''_{u^2}(t_2,u(t_2))|\le A_r|t_1-t_2|^\alpha,$$

where $B(0,r)$ is a ball of radius r in $\dot{H}_\alpha$.

Then the operator P is Fréched differentiable and the Fréched derivative has a bounded inverse at each point of the space $\dot{H}_\alpha$.

Proof. Let u_0 be an arbitrary fixed element of the space $\dot{H}_\alpha$ and h be any element of the some space.

Using the identities

$$f(\tau,u_0+h)-f(\tau,u_0)=hf'_u(\tau,u_0)+h^2\int_0^1 f''_{u^2}(\tau,u_0+\theta h)(1-\theta)d\theta,$$

we have

$$P(u_0+h)-P(u_0)=h(t)+\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{f'_u(\tau,u_0(\tau))h(\tau)}{\sqrt{1-\tau^2}(\tau-t)}d\tau+w(t,h),$$

where

$$w(t,h)=\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{h^2(\tau)\int_0^1 f''_{u^2}(\tau,u_0(\tau)+\theta h(\tau))(1-\theta)d\theta}{\sqrt{1-\tau^2}(\tau-t)}d\tau.$$

By the theorem conditions

$$\psi(\tau)=h^2(\tau)\int_0^1 f''_{u^2}(\tau,u_0(\tau)+\theta h(\tau))(1-\theta)d\theta\in\dot{H}_\alpha.$$

Then we have

$$\left\|\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{h^2(\tau)\int_0^1 f''_{u^2}(\tau,u_0(\tau)+\theta h(\tau))(1-\theta)d\theta}{\sqrt{1-\tau^2}(\tau-t)}d\tau\right\|_{\dot{H}_\alpha}$$
$$\leq C_0\left\|h^2(\tau)\int_0^1 f''_{u^2}(\tau,u_0(\tau)+\theta h(\tau))(1-\theta)d\theta\right\|_{\dot{H}_\alpha}$$
$$\leq C_0\|h\|^2_{\dot{H}_\alpha}\left\|\int_0^1 f''_{u^2}(\tau,u_0(\tau)+\theta h(\tau))(1-\theta)d\theta\right\|_{\dot{H}_\alpha}$$
$$\leq\frac{1}{2}C_0\|h\|^2_{\dot{H}_\alpha}\cdot A_{\|u_0\|+\|h\|},$$

where $C_0=\|A\|_{\dot{H}_\alpha\to\dot{H}_\alpha}$.

Taking into account the last inequality, we have

$$\lim_{\|h\|\to0}\frac{\|w(t,h)\|_{\dot{H}_\alpha}}{\|h\|_{\dot{H}_\alpha}}\leq\lim_{\|h\|\to0}\frac{\frac{1}{2}C_0\|h\|^2\cdot A_{\|u_0\|+\|h\|}}{\|h\|_{\dot{H}_\alpha}}=0$$

Thus, at any point $u_0\in\dot{H}_\alpha$ there exists a Fréched derivative of the operator P and it has the form

$$P'(u_0)h=h(t)+\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{f'_u(\tau,u_0(\tau))}{\sqrt{1-\tau^2}(\tau-t)}h(\tau)d\tau.\tag{8}$$

We prove that there exists an inverse operator $[P'(u_0)]^{-1}$.

Let $u_0(t)\in\dot{H}_\alpha$ be an initial approximation. For $g(t)\in\dot{H}_\alpha$ we consider the equation

$$P'(u_0)h=h(t)+\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{f'_u(\tau,u_0(\tau))h(\tau)}{\sqrt{1-\tau^2}(\tau-t)}d\tau=g(t).\tag{9}$$

Taking into account condition a) of the theorem, it is easy to calculate that the index of equation (9) equals zero.

Then for any right part of $g(t)$, equation (9) has a unique solution in H_α, that has the following form [2].

$$h(t)=-\frac{b^2(t)g(t)}{1+b^2(t)}-\frac{z(t)}{1+b^2(t)}\cdot\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{b(\tau)g(\tau)}{\sqrt{1-\tau^2}z(\tau)(\tau-t)}d\tau+g(t),\tag{10}$$

where

$$z(t)=\sqrt{1+b^2(t)}\exp\Big(-\int_{-1}^{1}\frac{\theta(\tau)}{\tau-t}d\tau\Big),\quad\theta(t)=\frac{1}{\pi}arg[1+ib(t)],$$
$$b(t)=f'_u(t,u_0(t)).$$

It can be seen from formula (10) that $h(\pm 1) = 0$, i.e. $h(t) \in \dot{H}_\alpha$.

Thus, we proved that equation (9), for any right part of from $\dot{H}_\alpha$ has a unique solution on $\dot{H}_\alpha$, that is, there exists an inverse operator $[P'(u_0)]^{-1}$ and it is of the form

$$[P'(u_0)]^{-1}g(t) = -\frac{b^2(t)g(t)}{1+b^2(t)} - \frac{z(t)}{1+b^2(t)} \cdot \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^{1} \frac{b(\tau)g(\tau)}{\sqrt{1-\tau^2}z(\tau)(\tau-t)} d\tau + g(t). \quad (11)$$

Estimate the norm

$$\|[P'(u_0)]^{-1}\|_{\dot{H}_\alpha \to \dot{H}_\alpha}.$$

We have

$$\|[P'(u_0)]^{-1}\| \le \Big\| \frac{1}{1+b^2(t)} \Big\|_{H_\alpha} \cdot \|b(t)\|^2_{H_\alpha} + \Big\| \frac{1}{1+b^2(t)} \Big\|_{H_\alpha} \cdot \|z(t)\|_{H_\alpha} C_0 \Big\| \frac{1}{z(t)} \Big\|_{H_\alpha} \cdot \|b(t)\|_{\dot{H}_\alpha} + 1.$$

Estimate each norm separably. At first we estimate $\Big\| \frac{1}{1+b^2(t)} \Big\|_{H_\alpha}$.

Obviously, $\Big\| \frac{1}{1+b^2(t)} \Big\|_{C} \le 1$.

In what follows,

$$\Big| \frac{1}{1+b^2(t_1)} + \frac{1}{1+b^2(t_2)} \Big| \le |b^2(t_2) - b^2(t_1)||b(t_2)+b(t_1)| \le 2\|b\|_C |b(t_2)-b(t_1)|$$
$$\le 2\|b\|_C l_1 \big(|t_1-t_2|^\alpha + |u_0(t_1)-u_0(t_2)|\big) \le 2\|b\|_C l_1 (1+\|u_0\|)|t_1-t_2|^\alpha$$
$$\le 2\, l_1\,(\|u_0\|+1)\, l_1\,(1+\|u_0\|)|t_1-t_2|^\alpha.$$

Thus,

$$\Big\| \frac{1}{1+b^2(t)} \Big\|_{H_\alpha} \le 2\, l_1^2\,(1+\|u_0\|)^2 + 1 \equiv M_1.$$

In a similar way we find

$$\|b(t)\|_{\dot{H}_\alpha} \le l_1(1+\|u_0\|) \equiv M_2.$$

Hence,

$$\Big\| \frac{1}{1+b^2(t)} \Big\|_{H_\alpha} \|b(t)\|^2 \le l_1(1+\|u_0\|)[2l_1^2(1+\|u_0\|)^2+1] \equiv M_3.$$

Estimate the norm $z(t)$:

$$z(t) = \sqrt{1+b^2(t)} \exp\Big(-\int_{-1}^{1} \frac{\theta(\tau)}{\tau - t} d\tau \Big).$$

Applying the Lagrange theorem, we get

$$|\sqrt{1+b^2(t_1)} - \sqrt{1+b^2(t_2)}| \le l_1^2\big(\|u_0\|_C + 1\big)\big(|t_1-t_2|^\alpha + |u_0(t_1)-u_0(t_2)|\big). \quad (12)$$

On the other hand, we have

$$\|\sqrt{1+b^2(t_1)}\|_C = \sqrt{1+\|b(t)\|_C^2} \le \sqrt{1+l_1(\|u_0\|_C+1)}. \tag{13}$$

It follows from (12) and (13) that

$$\|\sqrt{1+b^2(t)}\|_{H_\alpha} \le l_1^2(1+\|u_0\|)^2 + \sqrt{1+l_1(\|u_0\|_C+1)} \equiv M_4. \tag{14}$$

Introduce a singular operator

$$(Bv)(t) = -\int_{-1}^{1} \frac{v(\tau)}{\tau - t} d\tau.$$

It is known [3] that the operator B acts from $\dot{H}_\alpha$ to H_α and is bounded. Taking this into account, we have

$$\Big\| -\int_{-1}^{1} \frac{\theta(\tau)}{\tau-t} d\tau \Big\|_{H_\alpha} \le C_1 \|\theta(t)\|_{\dot{H}_\alpha}, \quad \text{where} \quad C_1 = \|B\|_{\dot{H}_\alpha \to H_\alpha}.$$

Estimate $\|\theta(t)\|_{\dot{H}_\alpha}$.

$$\begin{aligned} |\theta(t_1) - \theta(t_2)| &\le \frac{1}{\pi}\big|f'_u(t_1,u_0(t_1)) - f'_u(t_2,u_0(t_2))\big| \\ &\le \frac{1}{\pi} l_1(1+\|u_0\|)|t_1-t_2|^\alpha \Rightarrow \|\theta(t)\|_{\dot{H}_\alpha} \le \frac{1}{\pi} l_1(1+\|u_0\|) \equiv M_5. \end{aligned}$$

Estimate the last one, we get

$$\Big\| \exp\Big(-\int_{-1}^{1} \frac{\theta(\tau)}{\tau-t} d\tau\Big) \Big\|_{H_\alpha} \le e^{\frac{c_1 l_1}{\pi}(1+\|u_0\|)} \Big(1 + \frac{c_1 l_1}{\pi}(1+\|u_0\|)\Big). \tag{15}$$

From estimations (14) and (15), for $\|z(t)\|$ we find

$$\|z(t)\| \le M_3 \Big(1 + \frac{c_1 l_1}{\pi}(1+\|u_0\|)\Big) \cdot \exp\Big(\frac{c_1 l_1}{\pi}(1+\|u_0\|)\Big) \equiv M_6. \tag{16}$$

By the similar calculations we can prove the validity of the following estimation

$$\begin{aligned} \Big\| \frac{1}{z(t)} \Big\|_{H_\alpha} &\le \Big[1 + l_1^2(1+\|u_0\|)^2\Big] \Big[1 + \frac{C_1 l_1}{\pi}(1+\|u_0\|)\Big] \\ &\quad \times \exp\Big(\frac{C_1 l_1}{\pi}(1+\|u_0\|)\Big) \equiv M_7. \end{aligned} \tag{17}$$

We put together are the obtained estimations and have

$$\|[P'(u_0)]^{-1}\|_{\dot{H}_\alpha \to \dot{H}_\alpha} \le 1 + M_3 + C_0 M_1 M_2 M_6 M_7 \equiv M_0. \tag{18}$$

Now we estimate $\|[P'(u_0)]^{-1} P(u_0)\|_{\dot{H}_\alpha}$.

$$\|[P'(u_0)]^{-1} P(u_0)\| \le \|[P'(u_0)]^{-1}\| \cdot \|P(u_0)\| \le M_0 \|P(u_0)\|,$$

$$\|P(u_0)\| \le \|u_0\| + C_0 \|f(t,u_0(t))\| \le \|u_0\| + C_0 l_0 (1+\|u_0\|).$$

Thus,

$$\|[P'(u_0)]^{-1} P(u_0)\| \le M_0(\|u_0\| + C_0 l_0(1+\|u_0\|)).$$

Prove that the operator $P'(u)$ satisfies the Lipschits condition in the ball $u_0(t)$ centered at R.

$$P'(u_1)h - p'(u_2)h = \frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{f'_u(\tau,u_1(\tau)) - f'_u(\tau,u_2(\tau))}{\sqrt{1-\tau^2}(\tau-t)}h(\tau)d\tau$$
$$= \frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{[u_1(\tau)-u_2(\tau)]a(\tau)}{\sqrt{1-\tau^2}(\tau-t)}h(\tau)d(\tau),$$

where

$$a(\tau) = \int_0^1 f''_{u_1}(\tau, u_1(\tau) + \theta(u_2(\tau)-u_1(\tau)))d\theta,\ a(\tau)\in \dot{H}_\alpha.$$

Hence it follows that

$$\|P'(u_1) - P'(u_2)\| = \sup_{\|h\|=1}\left\|\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{[u_1(\tau)-u_2(\tau)]a(\tau)}{\sqrt{1-\tau^2}(\tau-t)}h(\tau)d(\tau)\right\|$$
$$\le C_0\|u_1-u_2\|_{\dot{H}_\alpha}\|a(\tau)\|_{\dot{H}_\alpha} \le C_0 A_{\|u_0\|+3R}\|u_1-u_2\|_{\dot{H}_\alpha}. \quad (19)$$

Thus, we proved that all the conditions of applicability of the Newton-Kantorovich method the equation (6) are fulfilled, i.e. thus the proof of Theorem 1 is complete. .

Theorem 2 *Let $B(u_0,R)$ be a ball centered at u_0 of radius R in $\dot{H}_\alpha$ $(0<\alpha<\frac{1}{2})$, and*

$$h = M_0^2(\|u_0\| + C_0 l_0(1+\|u_0\|))C_0 A_{\|u_0\|+3R}.$$

Then, if $h<\frac{1}{4}$ and the function $f(t,u)$ satisfies the conditions of Theorem 1, then in the ball

$$\|u-u_0\| \le M_0(\|u_0\| + C_0 l_0(1+\|u_0\|))\frac{1-\sqrt{1-4h}}{2h}$$

equations (6) has a unique solution $u^*(t)$ and the sequence $\{u_k(t)\}$ in determined by the formula

$$u_{n+1}(t) = u_n(t) - [P'(u_0)]^{-1}(P(u_n)),$$

converges to this solution. Moreover we have the following estimation of the convergence:

$$\|u_n - u^*\|_{\dot{H}_\alpha} \le \frac{q^n}{1-q}\|[P'(u_0)]^{-1}P(u_0)\|_{\dot{H}_\alpha},$$
$$q = \frac{1-\sqrt{1-4h}}{2} < \frac{1}{2}.$$

It remains to prove that the sequence of constants

$$D_n = \frac{1}{\pi}\int_{-1}^{1}\frac{f(\tau,u_n(\tau))}{\sqrt{1-\tau^2}}d\tau$$

converges to D^*. Estimate $|D_n - D^*|$. After simple calculations we have

$$
\begin{aligned}
|D_n - D^*| &= \Big|\frac{1}{\pi}\int_{-1}^{1}\frac{f(\tau,u_n(\tau))}{\sqrt{1-\tau^2}}d\tau - \frac{1}{\pi}\int_{-1}^{1}\frac{f(\tau,u^*(\tau))}{\sqrt{1-\tau^2}}d\tau\Big| \\
&\le \frac{1}{\pi}\int_{-1}^{1}\frac{f(\tau,u_n(\tau)) - f(\tau,u^*(\tau))}{\sqrt{1-\tau^2}}d\tau \le \frac{l_0}{\pi}\int_{-1}^{1}\frac{u_n(\tau)-u^*(u)}{\sqrt{1-\tau^2}}d\tau \\
&\le \frac{l_0 q^n}{\pi(1-q)}\|[P'(u_0)]^{-1}P(u_0)\|_{\dot{H}_\alpha}\int\limits_{-1}^{1}\frac{d\tau}{\sqrt{1-\tau^2}} \\
&= \frac{l_0 q^n}{\pi(1-q)}\|[P'(u_0)]^{-1}P(u_0)\|_{\dot{H}_\alpha} \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}
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