Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **41** (1), 1-18 (2021).
Noname

# Determination of a time-dependent coefficient in a non-linear hyperbolic equation with non-classical boundary condition

**Ibrahim Tekin** ⋆

Received: 27.06.2019 / Revised: 26.10.2020 / Accepted: 27.01.2021

**Abstract.** *The non-linear hyperbolic equation is used to model many non-linear phenomena. In this paper, we consider an initial boundary value problem for non-linear hyperbolic equation. We determine a time-dependent coefficient multiplying non-linear term by using an additional condition, and prove the existence and uniqueness theorem for small times. We also propose a numerical scheme to solve the inverse problem for non-linear hyperbolic equation, and give test examples for sine, quadratic and cubic non-linearity.*

**Keywords.** non-linear hyperbolic equation, inverse coefficient problem, Fourier method, finite difference method, existence and uniqueness.

**Mathematics Subject Classification (2010):** 35R30, 35L70, 65M06

## 1 Introduction

Inverse problems for non-linear hyperbolic equations provide an important value for physical applications, but there are limited results in this area, ([3], [4], [5], [6], [14], [20], [24], [29]). The inverse problems associated with the recovery of the coefficient for non-linear hyperbolic equations are also scarce ([12], [30], [32]) and need more consideration for further studies.

Consider the inverse problem of finding a pair of functions $\{a(t), w(x,t)\}$ for a non-linear hyperbolic equation

$$w_{tt}(x,t) = w_{xx}(x,t) + F(x,t;a,w),\ (x,t) \in \overline{D}_T, \tag{1.1}$$

with the initial conditions

$$w(x,0) = \varphi(x),\ w_t(x,0) = \psi(x),\ 0 \le x \le 1, \tag{1.2}$$

and non-classical boundary conditions

$$w(0,t) = 0,\ 0 \le t \le T, \tag{1.3}$$

$$w_x(0,t) = w_x(1,t),\ 0 \le t \le T, \tag{1.4}$$

⋆ Corresponding author

I. Tekin
Department of Mathematics, Bursa Technical University, Yıldırım-Bursa, 16310,Turkey
E-mail: ibrahim.tekin@btu.edu.tr

and over determination condition

$$w(1,t) = h(t),\ 0 \le t \le T, \tag{1.5}$$

where $D_T = \{(x,t) : 0 < x < 1,\ 0 < t < T\}$ for some fixed $T > 0$, $F(x,t;a,w) = a(t)g(w(x,t)) + f(x,t)$ $w(x,t)$ represents the wave displacement at position $x$ and time $t$, $g(w(x,t))$ is the non-linear force and the functions $\varphi(x)$, and $\psi(x)$ are wave modes or kinks and velocity, respectively.

Because of the presence of the non-linearity $g(w)$, the problem (1.1) - (1.4) for the unknown function $w(x,t)$ is over-specified for arbitrary functions $f$, $g$, $\varphi$, and $\psi$. Thus there may exist no solution $w(x,t)$. In the case of $g(w) = w$, the equation (1.1) is linear. The inverse coefficient/source problems for the linear hyperbolic equation with different boundary conditions were satisfactorily studied in various literature, see [11], [16], [21], [25] and more recently, [8], [9], [15], [23].

The homogeneous equation (1.1) is called sine-Gordon equation, Klein-Gordon equation with quadratic non-linearity and Klein-Gordon equation with cubic non-linearity given by

$$w_{tt}(x,t) = w_{xx}(x,t) + \sin(w(x,t)),$$
$$w_{tt}(x,t) = w_{xx}(x,t) + w^2(x,t),$$
$$w_{tt}(x,t) = w_{xx}(x,t) + w^3(x,t),$$

respectively.

The sine-Gordon equation appeared in many scientific fields such as the propagation of fluxons in Josephson junctions between two superconductors, the motion of rigid pendula attached to a stretched wire, solid state physics, non-linear optics, and dislocations in metals where $\sin w$ is due to periodic structure of rows of atoms, stability of fluid motions, and in soliton theory of DNA molecule, see [22], [33], [34], [36], [37].

The non-linear Klein-Gordon equation appears in many types of non-linearities (quadratic, cubic, etc.). The non-linear Klein-Gordon equation arises in relativistic quantum mechanics, field theory, and non-linear optics. This equation also used to model many different phenomena, such as propagation of dislocations in crystals and behaviour of elementary particles, [1], [2], [7], [35].

For the some numerical aspects of initial and initial-boundary value problems (IBVPs) for the linear and non-linear hyperbolic equations are considered in [8], [9], [15], and [27], [28], [30], respectively. It is important to note that authors compared the properties of four explicit finite difference schemes used to integrate the non-linear Klein-Gordon equation in the papers [13], [31] .

In this paper, we have an initial boundary value problem for non-linear hyperbolic equation with non-classical boundary condition. Giving an over determination condition, a time-dependent coefficient multiplying non-linear term is determined and the existence and uniqueness theorem for small times is proved. The finite different method is also proposed for solving the inverse problem. Also to achieve stable numerical solutions we will use the mollification method which is one of the regularization procedure that is appropriate to stabilize a variety of ill-posed problems by restoring continuity with respect to the data. The method can also be interpreted as a simple and efficient data smoothing algorithm and, as such, it can be used to fit noisy data (a well-posed problem) and to stabilize unstable numerical marching schemes for well-posed parabolic and hyperbolic problems, [18].

The article is organized as following: In Section 2, we first present equivalent inverse problem, and its auxiliary Sturm-Liouville spectral problem and its eigenvalues and eigenfunctions. Then we introduce two Banach spaces to investigate the inverse problem. In Section 3, the numerical method for solving the direct problem of hyperbolic equation with non-linear source based on the finite difference method is presented. In Section 4, the inverse problem under investigation is formulated. The existence and uniqueness of a classical solution to the inverse IBVP is proved for small times. In Section 5, we solve the inverse problem numerically by applying finite difference method. We present three numerical examples intended to illustrate the behaviour of the proposed method and the tests are performed by using MATLAB. First one of these examples is sine-Gordon case of (1.1) that is not satisfies the conditions of existence and uniqueness theorem. Second one is the Klein-Gordon equation with cubic non-linearity which coefficient is continuous but non-differentiable. Third example is the Klein-Gordon equation with quadratic non-linearity which satisfies the conditions of theoretical results. The possibly ill-posedness of the inverse problem is regularized by employing a suitable mollification method.

## 2 Equivalent problem and its auxiliary spectral problem

In this section, we will introduce the equivalent inverse problem of the inverse problem (1.1)-(1.5) and examine its auxiliary spectral problem and its properties. We will also give two functional spaces which are Banach spaces, ([17]).

**Definition 2.1** *The pair $\{a(t), w(x,t)\}$ from the class $C[0,T] \times C^2(\overline{D}_T)$ for which the conditions (1.1)-(1.5) are satisfied, is called a classical solution of the inverse problem (1.1)-(1.5).*

From this definition, the consistency conditions

$$(A_0)\begin{cases} \varphi(0)=0,\ \varphi'(0)=\varphi'(1), \\ \psi(0)=0,\ \psi'(0)=\psi'(1), \\ h(0)=\varphi(1),\ h'(0)=\psi(1), \end{cases}$$

holds for the data $\varphi(x), \psi(x) \in C^1[0,1]$ and $h(t) \in C^1[0,T]$.

It is easy to verify that the following lemma holds ([26]):

**Lemma 2.1** *Let the consistency conditions $(A_0)$ holds. Moreover $f(x,t) \in C(D_T)$, and $g(h(t)) \neq 0$, $(\forall t \in [0,T])$ are satisfied. Then the problem of finding the classical solution $\{a(t), w(x,t)\}$ of the inverse problem (1.1)-(1.5) is equivalent to the inverse problem composed of the equations (1.1)-(1.4) and*

$$h''(t) = w_{xx}(1,t) + a(t)g(h(t)) + f(1,t),\ 0 \le t \le T. \tag{2.1}$$

Since the equation (1.1) is non-homogeneous, we may use Fourier (Eigenfunction Expansion) Method. For using this method to the inverse problem we need to obtain the auxiliary spectral problem from the homogeneous part of the equation (1.1), i.e. $w_{tt}(x,t) = w_{xx}(x,t)$. Thus, the auxiliary spectral problem of equivalent inverse problem (1.1)-(1.4) and (2.1) can be obtained as

$$\begin{cases} X''(x) + \lambda X(x) = 0, \quad 0 \le x \le 1, \\ X(0)=0,\ X'(0)=X'(1). \end{cases} \tag{2.2}$$

This spectral problem has the eigenvalues $\lambda_k = (\mu_k)^2$ with $\mu_k = 2\pi k$, $k = 0,1,2,...$ and corresponding eigenfunctions (see [10])

$$X_0(x) = x,\ X_{2k-1}(x) = x\cos(\mu_k x),\ X_{2k}(x) = \sin(\mu_k x),\ k = 1,2,.... \tag{2.3}$$

The spectral problem (2.2) is not self adjoint. The adjoint spectral problem of (2.2) is ([10])

$$\begin{cases} Y''(x) + \lambda Y(x) = 0, \quad 0 \le x \le 1, \\ Y'(1) = 0, \ Y(0) = Y(1). \end{cases} \tag{2.4}$$

This spectral problem has the same eigenvalues of (2.2), and corresponding eigenfunctions are

$$Y_0(x) = 2, \ Y_{2k-1}(x) = 4\cos(\mu_k x), \ Y_{2k}(x) = 4(1-x)\sin(\mu_k x), \ k = 1, 2, .... \tag{2.5}$$

It is easy to verify that the systems (2.3) and (2.5) are biorthonormal on $[0, 1]$, i.e.

$$(X_i(x), Y_j(x)) = \int_0^1 X_i(x) Y_j(x) dx = \begin{cases} 0, \ i \neq j, \\ 1, \ i = j. \end{cases} .$$

Moreover the system (2.3) forms a Riesz basis in $L_2[0, 1]$. Then any function $r(x) \in L_2[0, 1]$ can be expanded in biorthogonal series

$$r(x) = \sum_{k=0}^{\infty} r_k X_k(x)$$

where $r_k = \int_0^1 r(x) Y_k(x) dx$, $k = 0, 1, 2, ....$

Let us consider following spaces to investigate the inverse problem (1.1)-(1.4) and (2.1):

I

$$B^3_{2,T} = \left\{ w(x,t) = \sum_{k=0}^{\infty} w_k(t) X_k(x) : \ w_k(t) \in C[0,T], \ J_T(w) = \|w_o(t)\|_{C[0,T]} \right.$$
$$\left. + \left( \sum_{k=1}^{\infty} (\mu_k^3 \|w_{2k}(t)\|_{C[0,T]})^2 \right)^{1/2} + \left( \sum_{k=1}^{\infty} (\mu_k^3 \|w_{2k-1}(t)\|_{C[0,T]})^2 \right)^{1/2} < +\infty \right\},$$

with the norm $\|w(x,t)\|_{B^3_{2,T}} \equiv J_T(w)$ which is related with the Fourier coefficients of the function $w(x,t)$ by the eigenfunctions $X_k(x)$, $k = 0, 1, 2, ....$

II $E^3_T = B^3_{2,T} \times C[0,T]$ of the vector function $z(x,t) = \{a(t), w(x,t)\}$ with the norm

$$\|z(x,t)\|_{E^3_T} = \|a(t)\|_{C[0,T]} + \|w(x,t)\|_{B^3_{2,T}} .$$

## 3 Direct problem

Consider the one-dimensional direct non-linear hyperbolic equation in $D_T$ given by the equations (1.1)-(1.4). Because of the presence of the non-linearity $g(w)$, no analytical method is available and hence the finite difference method (FDM) is applied for numerical discretization. We divide the domain $(0, 1) \times (0, T)$ into $nx$ and $nt$ subintervals of equal length $hx$ and $ht$, where $hx = 1/nx$ and $ht = T/nt$, respectively. We denote by $W^n_j := W(x_j, t_n)$, $a^n := a(t_n)$ and $f^n_j := f(x_j, t_n)$, where $x_j = jhx$, $t_n = nht$ for $j = 0, ..., nx$, $n = 0, ..., nt$. Then, a central difference approximation to the equations (1.1)-(1.3) and (2.1) at the mesh points $(x_j, t_n)$ is

$$W^{n+1}_j = r^2 W^n_{j+1} + 2(1-r^2) W^n_j + r^2 W^n_{j-1} - W^{n-1}_j + (ht)^2 (a^n g(W^n_j) + f^n_j), \tag{3.1}$$

$$j = 1, ..., nx - 1,\ n = 1, ..., nt - 1,$$

$$W_j^0 = \varphi_j,\ \ j = 0, ..., nx,\ \frac{W_j^1 - W_j^{-1}}{2ht} = \psi_j,\ j = 1, ..., nx - 1, \tag{3.2}$$

$$W_0^n = 0,\ \frac{W_{nx}^n - W_{nx-1}^n}{hx} = \frac{W_1^n - W_0^n}{hx},\ n = 0, ..., nt, \tag{3.3}$$

where $r = \frac{ht^2}{hx^2}$. Putting $n = 0$ in the equation (3.1) and using (3.2), we obtain

$$W_j^1 = \frac{1}{2}(r^2\varphi_{j+1} + 2(1-r^2)\varphi_j + r^2\varphi_{j-1} + 2ht\psi_j + (ht)^2(a^0 g(\varphi_j) + f_j^0)),\ j = 1, ..., nx-1. \tag{3.4}$$

Equations (3.1)-(3.4) represent an explicit finite difference method which is stable for $r \le 1$ when $g(w) = w^p,\ p = 1, 2, 3$ and $g(w) = \sin w$.

## 4 Existence and uniqueness of the classical solution of the inverse problem

In this section, we will examine the existence and uniqueness of the solution of the inverse initial-boundary value problem for the equation (1.1) with non-classical boundary condition.

Since the system (2.3) forms Riesz basis and the systems (2.3) and (2.5) are bi-orthogonal in $L_2[0, 1]$ and the function $a(t)$ is time dependent, seeking the solution of the problem (1.1)-(1.4) and (2.1) in the following form is suitable:

$$w(x, t) = \sum_{k=0}^{\infty} w_k(t) X_k(x), \tag{4.1}$$

where $w_k(t) = \int_0^1 w(x, t) Y_k(x) dx$, $k = 0, 1, 2, ....$

For an arbitrary $a(t) \in C[0, T]$, the solution of the problem (1.1)-(1.4) and (2.1) can be written as

$$w(x, t) = w_0(t) X_0(x) + \sum_{k=1}^{\infty} w_{2k-1}(t) X_{2k-1}(x) + \sum_{k=1}^{\infty} w_{2k}(t) X_{2k}(x).$$

By using the Fourier's method,from the equations (1.1)-(1.2) we obtain

$$\begin{cases} w_0''(t) = F_0(t; w, a), & \\ w_{2k-1}''(t) + \mu_k^2 w_{2k-1}(t) = F_{2k-1}(t; w, a), & k = 1, 2, ..., \\ w_{2k}''(t) + \mu_k^2 w_{2k}(t) = F_{2k}(t; w, a) - 2\mu_k w_{2k-1}(t), & k = 1, 2, ..., \\ w_k(0) = \varphi_k,\ \ w_k'(0) = \psi_k, & k = 0, 1, 2, ..., \end{cases} \tag{4.2}$$

where $F_k(t; w, a) = a(t) g_k(t) + f_k(t)$, $f_k(t) = \int_0^1 f(x, t) Y_k(x) dx$, $g_k(t) = \int_0^1 g(w) Y_k(x) dx$, $\varphi_k = \int_0^1 \varphi(x) Y_k(x) dx$, $\psi_k = \int_0^1 \psi(x) Y_k(x) dx$, $k = 0, 1, 2, ....$

Solving the Cauchy problems (4.2), we obtain

$$\begin{cases} \begin{array}{ll} w_0(t) = w_0(0) + tw_0'(0) + \int_0^t (t-\tau) w_0''(\tau) d\tau \\ \quad = \varphi_0 + t\psi_0 + \int_0^t (t-\tau) F_0(\tau; w, a) d\tau, & 0 \le t \le T, \\ w_{2k-1}(t) = \varphi_{2k-1} \cos \mu_k t + \frac{1}{\mu_k} \psi_{2k-1} \sin \mu_k t \\ \quad + \frac{1}{\mu_k} \int_0^t F_{2k-1}(\tau; w, a) \sin \mu_k (t-\tau) d\tau, & k = 1, 2, ..., \\ w_{2k}(t) = \varphi_{2k} \cos \mu_k t + \frac{1}{\mu_k} \psi_{2k} \sin \mu_k t + \frac{1}{\mu_k} \int_0^t F_{2k}(\tau; w, a) \sin \mu_k (t-\tau) d\tau \\ \quad -t\varphi_{2k-1} \sin \mu_k t - \frac{1}{\mu_k} \psi_{2k-1} \left[ \frac{1}{\mu_k} \sin \mu_k t - t \cos \mu_k t \right] & k = 1, 2, .... \\ \quad - \frac{2}{\mu_k} \int_0^t \int_0^\tau F_{2k-1}(\xi; w, a) \sin \mu_k (\tau - \xi) d\xi \sin \mu_k (t-\tau) d\tau, \end{array} \end{cases} \tag{4.3}$$

Let substitute the expressions (4.3) into (4.1) to determine $w(x,t)$. Then we get

$$\begin{aligned} w(x,t) &= \left( \varphi_0 + t\psi_0 + \int_0^t (t-\tau) F_0(\tau; w, a) d\tau \right) X_0(x) \\ &+ \sum_{k=1}^\infty \left( \varphi_{2k-1} \cos \mu_k t + \frac{1}{\mu_k} \psi_{2k-1} \sin \mu_k t \right. \\ &+ \left. \frac{1}{\mu_k} \int_0^t F_{2k-1}(\tau; w, a) \sin \mu_k (t-\tau) d\tau \right) X_{2k-1}(x) \\ &+ \sum_{k=1}^\infty \left( \varphi_{2k} \cos \mu_k t + \frac{1}{\mu_k} \psi_{2k} \sin \mu_k t + \frac{1}{\mu_k} \int_0^t F_{2k}(\tau; w, a) \sin \mu_k (t-\tau) d\tau \right. \\ &- t\varphi_{2k-1} \sin \mu_k t - \frac{1}{\mu_k} \psi_{2k-1} \left[ \frac{1}{\mu_k} \sin \mu_k t - t \cos \mu_k t \right] \\ &- \left. \frac{2}{\mu_k} \int_0^t \int_0^\tau F_{2k-1}(\xi; w, a) \sin \mu_k (\tau - \xi) d\xi \sin \mu_k (t-\tau) d\tau \right) X_{2k}(x). \end{aligned} \tag{4.4}$$

Consider $x = 1$ in the equation (1.1) to find the coefficient solution $a(t)$. Then by using the condition (2.1), we obtain

$$\begin{aligned} a(t) = \frac{1}{g(h(t))} \Bigg[ h''(t) - f(1,t) + \sum_{k=1}^\infty \mu_k^2 \Bigg( \varphi_{2k-1} \cos \mu_k t + \frac{\psi_{2k-1}}{\mu_k} \sin \mu_k t \\ + \frac{1}{\mu_k} \int_0^t F_{2k-1}(\tau; w, a) \sin \mu_k (t-\tau) d\tau \Bigg) \Bigg]. \end{aligned} \tag{4.5}$$

Thus, the solution of problem (1.1)-(1.4) and (2.1) is reduced to the solution of system (4.4) and (4.5) with respect to the unknown functions $\{a(t), w(x,t)\}$. Therefore to prove the uniqueness of the solution of the problem (1.1)-(1.4) and (2.1) is equivalent to prove the uniqueness of the solution of system (4.4) and (4.5).

Let us denote $z = [a(t), u(x,t)]^T$ and rewrite the system of equations (4.4) and (4.5) in the following operator equation

$$z = \Phi(z) \tag{4.6}$$

where $\Phi(z) \equiv [\phi_0, \phi_1]^T$ and $\phi_1$ and $\phi_0$ are equal to the right hand sides of (4.4) and (4.5), respectively as:

$$\begin{aligned} \phi_0(z) = \frac{1}{g(h(t))} \Bigg[ h''(t) - f(1,t) + \sum_{k=1}^\infty \mu_k^2 \Bigg( \varphi_{2k-1} \cos \mu_k t + \frac{\psi_{2k-1}}{\mu_k} \sin \mu_k t \\ + \frac{1}{\mu_k} \int_0^t F_{2k-1}(\tau; w, a) \sin \mu_k (t-\tau) d\tau \Bigg) \Bigg], \end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&\phi_1(z) = \left(\varphi_0 + t\psi_0 + \textstyle\int_0^t (t-\tau) F_0(\tau; w, a) d\tau\right) X_0(x) \\
&+ \textstyle\sum_{k=1}^{\infty} \left(\varphi_{2k-1} \cos \mu_k t + \frac{1}{\mu_k} \psi_{2k-1} \sin \mu_k t\right. \\
&\left.+ \tfrac{1}{\mu_k} \textstyle\int_0^t F_{2k-1}(\tau; w, a) \sin \mu_k (t-\tau) d\tau\right) X_{2k-1}(x) \\
&+ \textstyle\sum_{k=1}^{\infty} \left(\varphi_{2k} \cos \mu_k t + \frac{1}{\mu_k} \psi_{2k} \sin \mu_k t + \frac{1}{\mu_k} \int_0^t F_{2k}(\tau; w, a) \sin \mu_k (t-\tau) d\tau\right. \\
&-t\varphi_{2k-1} \sin \mu_k t - \tfrac{1}{\mu_k} \psi_{2k-1} \left[\tfrac{1}{\mu_k} \sin \mu_k t - t \cos \mu_k t\right] \\
&\left.- \tfrac{2}{\mu_k} \textstyle\int_0^t \int_0^\tau F_{2k-1}(\xi; w, a) \sin \mu_k (\tau - \xi) d\xi \sin \mu_k (t-\tau) d\tau\right) X_{2k}(x). \qquad (4.8)
\end{aligned}
$$

Let us demonstrate that $\Phi$ maps $E_T^3$ onto itself continuously. In other words, we need to show $\phi_0(z) \in C[0,T]$ and $\phi_1(z) \in B_{2,T}^3$ for arbitrary $z = [a(t), u(x,t)]^T$ with $a(t) \in C[0,T]$, $u(x,t) \in B_{2,T}^3$. We will use the following assumptions on the data of problem (1.1)-(1.4) and (2.1):

$(A_1)$ $\varphi(x) \in C^3[0,1]$, $\varphi(0) = \varphi''(0) = 0$, $\varphi'(0) = \varphi'(1)$,
$(A_2)$ $\psi(x) \in C^2[0,1]$, $\psi(0) = 0$, $\psi'(0) = \psi'(1)$,
$(A_3)$ $h(t) \in C^1[0,T]$, $h(0) = \varphi(1)$, $h'(0) = \psi(1)$,
$(A_4)$ $f(x,t) \in C(\overline{D}_T)$, $f_x, f_{xx} \in C[0,1]$, $\forall t \in [0,T]$, $f(0,t) = 0$, $f_x(0,t) = f_x(1,t)$,

$$
(A_5) \begin{cases}
g(w) \in C^2(\mathbb{R}),\ g(0) = 0,\ (g(w))_x|_{x=1} = (g(w))_x|_{x=0}, \\
g'(h(t)) = g'(0),\ g(h(t)) \neq 0\ \forall t \in [0,T], \\
\left|g(w^1) - g(w^2)\right| \leq d_0 \left|w^1 - w^2\right|, \\
\left|g'(w^1) - g'(w^2)\right| \leq d_1 \left|w^1 - w^2\right|,\ d_i, i = 0,1,2 \text{ are positive constants.} \\
\left|g''(w^1) - g''(w^2)\right| \leq d_2 \left|w^1 - w^2\right|,
\end{cases}
$$

By using integration by parts under the assumptions $(A_0) - (A_4)$ and the first condition of $(A_5)$, we have

$$
\begin{cases}
\varphi_{2k-1} = \frac{-\sqrt{8}}{\mu_k^3} \int_0^1 \varphi'''(x) \sqrt{2} \sin(\mu_k x) dx, \\
\psi_{2k-1} = \frac{-\sqrt{8}}{\mu_k^2} \int_0^1 \psi''(x) \sqrt{2} \cos(\mu_k x) dx, \\
f_{2k-1}(t) = \frac{-\sqrt{8}}{\mu_k^2} \int_0^1 f_{xx}(x,t) \sqrt{2} \cos(\mu_k x) dx, \\
g_{2k-1}(t) = \frac{-\sqrt{8}}{\mu_k^2} \int_0^1 \left[g''(w) w_x^2(x,t) + g'(w) w_{xx}(x,t)\right] \sqrt{2} \cos(\mu_k x) dx \\
\varphi_{2k} = \frac{-\sqrt{8}}{\mu_k^3} \int_0^1 \left[\varphi'''(x)(1-x) - 3\varphi''(x)\right] \sqrt{2} \cos(\mu_k x) dx, \\
\psi_{2k} = \frac{-\sqrt{8}}{\mu_k^2} \int_0^1 \left[\varphi''(x)(1-x) - 2\varphi'(x)\right] \sqrt{2} \sin(\mu_k x) dx, \\
f_{2k}(t) = \frac{-\sqrt{8}}{\mu_k^2} \int_0^1 \left[f_{xx}(x,t)(1-x) - 2f_x(x,t)\right] \sqrt{2} \sin(\mu_k x) dx, \\
g_{2k}(t) = \frac{-\sqrt{8}}{\mu_k^2} \int_0^1 \left[\left\{g''(w) w_x^2(x,t) + g'(w) w_{xx}(x,t)\right\}(1-x)\right. \\
\left.-2g'(w) w_x(x,t)\right] \sqrt{2} \sin(\mu_k x) dx.
\end{cases} \qquad (4.9)
$$

First, let us show that $\phi_0(z) \in C[0,T]$. Consider the equalities (4.9) into (4.7), we obtain

$$
\begin{aligned}
|\phi_0(z)| \leq \frac{1}{|g(h(t))|} &\left\{\left|h''(t)\right| + |f(1,t)| + \sum_{k=1}^{\infty} \left(\frac{1}{\mu_k} |\alpha_{2k-1}| + \frac{1}{\mu_k} |\beta_{2k-1}|\right.\right. \\
&\left.\left.+ \frac{1}{\mu_k} \int_0^T \left[|\gamma_{2k-1}(t)| + |a(t)| \, |\eta_{2k-1}(t)|\right] dt\right)\right\} \qquad (4.10)
\end{aligned}
$$

where $\eta_{2k-1}(t) = -\sqrt{8}\int_0^1 \left[g''(w)w_x^2(x,t) + g'(w)w_{xx}(x,t)\right]\sqrt{2}\cos(\mu_k x)dx$, $\alpha_{2k-1} = -\sqrt{8}\int_0^1 \varphi'''(x)\sqrt{2}\sin(\mu_k x)dx$, $\beta_{2k-1} = -\sqrt{8}\int_0^1 \psi''(x)\sqrt{2}\cos(\mu_k x)dx$, and $\gamma_{2k-1}(t) = -\sqrt{8}\int_0^1 f_{xx}(x,t)\sqrt{2}\cos(\mu_k x)dx$. Using Cauchy-Schwartz inequality

$$|\phi_0(z)| \le \frac{1}{|g(h(t))|}\left\{|h''(t)| + |f(1,t)| + \left(\sum_{k=1}^{\infty}\left(\frac{1}{\mu_k}\right)^2\right)^{1/2}\left[\left(\sum_{k=1}^{\infty}|\alpha_{2k-1}|^2\right)^{1/2}\right.\right.$$
$$\left.\left.+\left(\sum_{k=1}^{\infty}|\beta_{2k-1}|^2\right)^{1/2} + \int_0^T\left(|a(t)|\left(\sum_{k=1}^{\infty}|\eta_{2k-1}(t)|^2\right)^{1/2} + \left(\sum_{k=1}^{\infty}|\gamma_{2k-1}(t)|^2\right)^{1/2}\right)dt\right]\right\}$$

is obtained from the equation (4.10). Since $\sqrt{2}\sin\mu_n x$ (or $\sqrt{2}\cos\mu_n x$) forms a biorthogonal system of functions on $[0,1]$, by using Bessel's inequality we get

$$\sum_{k=1}^{\infty}|\alpha_{2k-1}|^2 \le \left\|\varphi'''\right\|^2_{L_2[0,1]}, \quad \sum_{k=1}^{\infty}|\beta_{2k-1}|^2 \le \left\|\psi''\right\|^2_{L_2[0,1]},$$

$$\sum_{k=1}^{\infty}|\eta_{2k-1}(t)|^2 \le \left\|g''w_x^2 + g'w_{xx}\right\|^2_{L_2(D_T)}, \quad \sum_{k=1}^{\infty}|\gamma_{2k-1}(t)|^2 \le \left\|f_{xx}\right\|^2_{L_2(D_T)}.$$

Taking these estimates into account we conclude that the majorizing series (4.10) is convergent. This implies that by the Weierstrass-M test, the series (4.7) are uniformly convergent in $[0,T]$. Thus $\phi_0(z)$ is continuous in $[0,T]$.

Now, let us show that $\phi_1(z) \in B^3_{2,T}$, i.e. we need to show

$$J_T(\phi_1) = \|\phi_{1,0}(t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}(\mu_k^3\|\phi_{1,2k}(t)\|_{C[0,T]})^2\right)^{1/2}$$
$$+\left(\sum_{k=1}^{\infty}(\mu_k^3\|\phi_{1,2k-1}(t)\|_{C[0,T]})^2\right)^{1/2} < +\infty,$$

where $\phi_{1,0}(t)$, $\phi_{1,2k}(t)$ and $\phi_{1,2k-1}(t)$ are the equal to the right hand side of $w_0(t)$, $w_{2k}(t)$ and $w_{2k-1}$ as in (4.3), respectively. After some manipulations under the assumptions $(A_0)-(A_5)$, we obtain

$$\|\phi_{1,0}(t)\|_{C[0,T]} \le |\varphi_0| + T|\psi_0| + T^2\left(\max_{0\le t\le T}|f_0(t)| + |a(t)|\max_{0\le t\le T}|g_0(t)|\right),$$

$$\textstyle\sum_{k=1}^{\infty}(\mu_k^3\|\phi_{1,2k-1}(t)\|_{C[0,T]})^2 \le 2\sum_{k=1}^{\infty}|\alpha_{2k-1}|^2 + 2\sum_{k=1}^{\infty}|\beta_{2k-1}|^2$$
$$+4T^2\left(\max_{0\le t\le T}|a(t)|\right)^2\textstyle\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|\eta_{2k-1}(t)|\right)^2 + 4T^2\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|\gamma_{2k-1}(t)|\right)^2,$$

$$\textstyle\sum_{k=1}^{\infty}(\mu_k^3\|\phi_{1,2k}(t)\|_{C[0,T]})^2 \le 4\sum_{k=1}^{\infty}|\alpha_{2k}|^2 + 4\sum_{k=1}^{\infty}|\beta_{2k}|^2$$
$$+4T^2\left(\max_{0\le t\le T}|a(t)|\right)^2\textstyle\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|\eta_{2k}(t)|\right)^2 + 4T^2\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|\gamma_{2k}(t)|\right)^2$$
$$+4T^2\textstyle\sum_{k=1}^{\infty}|\alpha_{2k-1}|^2 + 4(1+T)^2\sum_{k=1}^{\infty}|\beta_{2k-1}|^2 + 4T^2\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|\gamma_{2k-1}(t)|\right)^2$$
$$+4T^2\left(\max_{0\le t\le T}|a(t)|\right)^2\textstyle\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|\eta_{2k-1}(t)|\right)^2,$$

where

$$\eta_{2k-1}(t) = -\sqrt{8}\int_0^1 \left[g''(w)w_x^2(x,t) + g'(w)w_{xx}(x,t)\right]\sqrt{2}\cos(\mu_k x)dx,$$

$$\eta_{2k}(t) = -\sqrt{8}\int_0^1 \left[\left\{g''(w)w_x^2(x,t) + g'(w)w_{xx}(x,t)\right\}(1-x)\right.$$

$$\left.-2g'(w)w_x(x,t)\right]\sqrt{2}\sin(\mu_k x)dx,$$

$$\alpha_{2k-1} = -\sqrt{8}\int_0^1 \varphi'''(x)\sqrt{2}\sin(\mu_k x)dx,$$

$$\alpha_{2k} = -\sqrt{8}\int_0^1 \left[\varphi'''(x)(1-x) - 3\varphi''(x)\right]\sqrt{2}\cos(\mu_k x)dx,$$

$$\gamma_{2k-1}(t) = -\sqrt{8}\int_0^1 f_{xx}(x,t)\sqrt{2}\cos(\mu_k x)dx,$$

$$\gamma_{2k}(t) = -\sqrt{8}\int_0^1 \left[f_{xx}(x,t)(1-x) - 2f_x(x,t)\right]\sqrt{2}\sin(\mu_k x)dx,$$

$$\beta_{2k-1} = -\sqrt{8}\int_0^1 \psi''(x)\sqrt{2}\cos(\mu_k x)dx,$$

$$\beta_{2k} = -\sqrt{8}\int_0^1 \left[\varphi''(x)(1-x) - 2\varphi'(x)\right]\sqrt{2}\sin(\mu_k x)dx.$$

The series on the right side of above equations are convergent from the Bessel inequalities

$$\sum_{k=1}^{\infty}|\alpha_{2k-1}|^2 \le \left\|\varphi'''\right\|_{L_2[0,1]}^2, \quad \sum_{k=1}^{\infty}|\beta_{2k-1}|^2 \le \left\|\psi''\right\|_{L_2[0,1]}^2,$$

$$\sum_{k=1}^{\infty}|\eta_{2k-1}(t)|^2 \le \left\|g''w_x^2 + g'w_{xx}\right\|_{L_2(D_T)}^2, \quad \sum_{k=1}^{\infty}|\gamma_{2k-1}(t)|^2 \le \left\|f_{xx}\right\|_{L_2(D_T)}^2,$$

$$\sum_{k=1}^{\infty}|\alpha_{2k}|^2 \le \left\|\varphi'''(1-x) - 3\varphi''\right\|_{L_2[0,1]}^2, \quad \sum_{k=1}^{\infty}|\beta_{2k}|^2 \le \left\|\varphi''(1-x) - 2\varphi'\right\|_{L_2[0,1]}^2,$$

$$\sum_{k=1}^{\infty}|\eta_{2k}(t)|^2 \le \left\|\left\{g''w_x^2 + g'w_{xx}\right\}(1-x) - 2g'w_x\right\|_{L_2(D_T)}^2,$$

$$\sum_{k=1}^{\infty}|\gamma_{2k}(t)|^2 \le \left\|f_{xx}(1-x) - 2f_x\right\|_{L_2(D_T)}^2.$$

Thus $J_T(\phi_1) < +\infty$ and $\phi_1$ belongs to the space $B_{2,T}^3$.

By virtue of the definition of the space $B_{2,T}^3$, it is easy to obtain that

$$|w_x| \leq c_1 \|w\|_{B_{2,T}^3} \text{ and } |w_{xx}| \leq c_2 \|w\|_{B_{2,T}^3} \tag{4.11}$$

where $c_i$, $i = 1, 2$ are real constants.

Now, let $z_1$ and $z_2$ be any two elements of $E_T^3$. We know that $\|\Phi(z_1) - \Phi(z_2)\|_{E_T^3} = \|\phi_0(z_1) - \phi_0(z_2)\|_{C[0,T]} + \|\phi(z_1) - \phi(z_2)\|_{B_{2,T}^3}$. Here $z_i = \left[a^{(i)}(t), w^{(i)}(x,t)\right]^T$, $i = 1, 2$.

From the equations (4.7) and (4.8), we get

$$\phi_0(z_1) - \phi_0(z_2) = \frac{1}{g(h(t))} \sum_{k=1}^{\infty} \mu_k \int_0^t \left(F_{2k-1}(\tau; w^{(1)}, a^{(1)}) - F_{2k-1}(\tau; w^{(2)}, a^{(2)})\right)$$
$$\times \sin \mu_k (t - \tau) d\tau,$$

$$\phi_1(z_1) - \phi_1(z_2) = \int_0^t (t - \tau) \left(F_0(\tau; w^{(1)}, a^{(1)}) - F_0(\tau; w^{(2)}, a^{(2)})\right) d\tau X_0(x)$$

$$+ \sum_{k=1}^{\infty} \frac{1}{\mu_k} \left[\int_0^t \left(F_{2k-1}(\tau; w^{(1)}, a^{(1)}) - F_{2k-1}(\tau; w^{(2)}, a^{(2)})\right) \sin \mu_k (t - \tau) d\tau\right] X_{2k-1}(x)$$

$$+ \sum_{k=1}^{\infty} \left[\frac{1}{\mu_k} \int_0^t \left(F_{2k}(\tau; w^{(1)}, a^{(1)}) - F_{2k}(\tau; w^{(2)}, a^{(2)})\right) \sin \mu_k (t - \tau) d\tau\right.$$

$$\left. - \frac{2}{\mu_k} \int_0^t \int_0^\tau \left(F_{2k-1}(\xi; w^{(1)}, a^{(1)}) - F_{2k-1}(\xi; w^{(2)}, a^{(2)})\right) \sin \mu_k (\tau - \xi) d\xi \sin \mu_k (t - \tau) d\tau\right]$$
$$\times X_{2k}(x).$$

After some manipulations in last equations and using the estimates (4.11) and the Lipschitz continuities in $(A_5)$, we obtain

$$\|\phi_0(z_1) - \phi_0(z_2)\|_{C[0,T]} \leq \frac{T}{\min\limits_{0 \leq t \leq T} |g(h(t))|} \left[C_1(a^{(1)}, w^{(1)}, w^{(2)}, T) \left\|w^{(1)} - w^{(2)}\right\|_{B_{2,T}^3}\right.$$
$$\left. + C_2(w^{(2)}) \left\|a^{(1)} - a^{(2)}\right\|_{C[0,T]}\right],$$

$$\|\phi_1(z_1) - \phi_1(z_2)\|_{B_{2,T}^3} \leq T \left[C_3(a^{(1)}, w^{(1)}, w^{(2)}, T) \left\|w^{(1)} - w^{(2)}\right\|_{B_{2,T}^3}\right.$$
$$\left. + C_4(w^{(2)}, T) \left\|a^{(1)} - a^{(2)}\right\|_{C[0,T]}\right],$$

where $C_k$, $k = \overline{1,4}$ are the constants depend on the norms $\|a^{(i)}\|_{C[0,T]}$, $\|w^{(i)}\|_{B_{2,T}^3}$, $i = 1, 2$, and $T$. From the last inequalities it follows that

$$\|\Phi(z_1) - \Phi(z_2)\|_{E_T^3} \leq A(T) C(a^{(1)}, a^{(2)}, w^{(1)}, w^{(2)}) \|z_1 - z_2\|_{E_T^3}$$

where $A(T) = T\left(1 + \frac{1}{\min\limits_{0\le t\le T}|g(h(t))|}\right)$ with $g(h(t)) \neq 0, \forall t \in [0,T]$ and $C(a^{(1)}, a^{(2)}, w^{(1)}, w^{(2)}) = \max\{C_1, C_2, C_3, C_4\}$ is the constant depends on the norms $\left\|a^{(i)}\right\|_{C[0,T]}$ and $\left\|w^{(i)}\right\|_{B^3_{2,T}}$, $i = 1, 2$.

For sufficiently small $T$, $0 < A(T)C(a^{(1)}, a^{(2)}, w^{(1)}, w^{(2)}) < 1$. This implies that the operator $\Phi$ is contraction mapping which maps $E^3_T$ onto itself continuously. Then according to Banach fixed point theorem there exists unique solution of (4.6) .

Thus, we proved the following theorem:

**Theorem 4.1** *Let the assumptions ($A_0$)-($A_5$) be satisfied. Then, the inverse problem (1.1)-(1.4) and (2.1) has unique solution for small $T$.*

## 5 Numerical solution of the inverse problem

In this section,we study the numerical solution of the inverse problem (1.1)-(1.4) and (2.1) for non-linear hyperbolic equation.

The discrete form of direct problem is given in Section 3 with the equations (3.1)-(3.4). By using the condition (2.1), we obtain

$$a(t) = \frac{h''(t) - w_{xx}(1,t) - f(1,t)}{g(h(t))}.$$

After discretizing last equation, we have

$$a^n = \frac{(h^{n+1} - 2h^n + h^{n-1})/(ht)^2 - (W^n_{nx} - 2W^n_{nx-1} + W^n_{nx-2})/(hx)^2 - f^n_{nx}}{g(h^n)},\ n = 1, ..., nt{-}1 \tag{5.1}$$

$$a^{nt} = \frac{(h^{nt} - 2h^{nt-1} + h^{nt-2})/(ht)^2 - (W^{nt}_{nx} - 2W^{nt}_{nx-1} + W^{nt}_{nx-2})/(hx)^2 - f^{nt}_{nx}}{g(h^{nt})}, \tag{5.2}$$

$$a^0 = \frac{(h^2 - 2h^1 + h^0)/(ht)^2 - (W^0_{nx} - 2W^0_{nx-1} + W^0_{nx-2})/(hx)^2 - f^0_{nx}}{g(h^0)}. \tag{5.3}$$

Now let us consider (5.1)-(5.3) in (3.1), we obtain the system with respect to $W^n_j$ , $j = 0, ..., nx$, $n = 0, ..., nt$ which can be solved explicitly. Then using the calculated values of $W^n_j$ in (5.1)-(5.3), we obtain the values of $a^n$ , $n = 0, ..., nt$.

### 51 Numerical examples and discussion

In this section,we perform numerical experiments to validate the FDM in solving the direct problem of determining $w(x,t)$ and inverse problem of determining $a(t)$. In all examples in this section we take, for simplicity $T = 1$.

*Example 1 (sine-Gordon equation case)* Consider first the direct IBVP (1.1)-(1.4) with the input data

$$\varphi(x) = \cos(\pi x) - 1,\ \psi(x) = \frac{1}{2}\cos(\pi x) - 1,\ a(t) = \exp(t),$$
$$f(x,t) = \frac{1}{4}((1 + 4\pi^2)\cos(\pi x) - 1)exp(t/2) - \sin((\cos(\pi x) - 1)exp(t/2))\exp(t),$$
$$g(w(x,t)) = \sin(w(x,t)),\ x \in [0,1],\ t \in [0,1].$$

The analytical solution of the direct problem (1.1)-(1.4) is

$$w(x,t) = (\cos(\pi x) - 1)exp(t/2)$$

The numerical solution of the direct problem given by Equations (1.1)-(1.4) is obtained using the FDM described in Section 3. It can be seen from the Figure 1 that the mesh $ht = 0.01$ and $hx = 0.01$,i.e. $nt = nx = 100$, is sufficiently fine for accurately solving the direct problem.

For the inverse problem consider the equations (1.1)-(1.4) and (2.1) with the over determination condition data

$$h(t) = -2\exp(t/2).$$

One can easily check that $h(t) \in C^1[0,1]$, $\psi(x) \in C^2[0,1]$ and $\varphi(x) \in C^3[0,1]$ satisfy the conditions $(A_0) - (A_5)$ except $\varphi''(0) = 0$. As the condition of Theorem 1 is not satisfied we can not conclude the unique solvability of the inverse problem. However, the solution at least exists and is given by

$$\{a(t), w(x,t)\} = \{\exp(t), (\cos(\pi x) - 1)exp(t/2)\}$$

which can easily be checked by direct substitution. Figure 2 shows the exact and numerical inverse solutions of $\{a(t), w(1/2,t)\}$ for $nt = nx = 100$.

*Example 2 (Klein-Gordon equation with cubic non-linearity case)* As in Example 1, consider first the direct IBVP (1.1)-(1.4) with the input data

$$g(w(x,t)) = w^3(x,t),\ f(x,t) = 2x - x^3(t^2+t+1)^3\,|t-1/2|\,,$$
$$\varphi(x) = x,\ \psi(x) = x,\ a(t) = |t-1/2|\,,$$
$$x \in [0,1],\ t \in [0,1].$$

The solution of the direct problem (1.1)-(1.4) is

$$w(x,t) = x(t^2+t+1)$$

It can be seen from the Figure 3 that the mesh $ht = 1/80$ and $hx = 1/80$,i.e. $nt = nx = 80$, is sufficiently fine for accurately solving the direct problem.

For the inverse problem consider the equations (1.1)-(1.4) and (2.1) with the over determination condition data

$$h(t) = t^2+t+1.$$

Note that for this example, the force $a(t)$ is continuous but non-differentiable at the peak $t = 1/2$. Since $a(t)$ is non-differentiable, Theorem 1 is not satisfied. Thus we can not conclude the unique solvability of the inverse problem. Nevertheless, we can find the numerical solution of the inverse problem. Figure 4 shows numerical solution $\{a(t), w(1/2,t)\}$ of the inverse problem (1.1)-(1.4) and (2.1) with $nt = nx = 80$.

*Example 3 (Klein-Gordon equation with quadratic non-linearity case)* As in Examples above, consider first the direct IBVP (1.1)-(1.4) with the input data

$$g(w(x,t)) = w^2(x,t),\ f(x,t) = (2\pi x - (1+4\pi^2)\sin(2\pi x))\exp(t) - (2\pi x - \sin(2\pi x))^2,$$
$$\varphi(x) = 2\pi x - \sin(2\pi x),\ \psi(x) = 2\pi x - \sin(2\pi x),\ a(t) = \exp(-2t),$$
$$x \in [0,1],\ t \in [0,1].$$

The analytical solution of the direct problem (1.1)-(1.4) is

$$w(x,t) = (2\pi x - \sin(2\pi x))\exp(t)$$

It can be seen from the Figure 5 that the mesh $ht = 0.01$ and $hx = 0.01$,i.e. $nt = nx = 100$, is sufficiently fine for accurately solving the direct problem.

For the inverse problem consider the equations (1.1)-(1.4) and (2.1) with the over determination condition data

$$h(t) = 2\pi \exp(t).$$

It is easy to check that $h(t) \in C^1[0,1]$, $\psi(x) \in C^2[0,1]$ and $\varphi(x) \in C^3[0,1]$ satisfy the conditions $(A_0) - (A_3)$. Moreover, the conditions $(A_4)$ and $(A_5)$ are also satisfied. Hence, according to the Theorem 1 the solution of the inverse problem exists and unique. In fact, it can easily be checked by direct substitution that the analytical solution is given by

$$\{a(t), w(x,t)\} = \{\exp(-2t), (2\pi x - \sin(2\pi x))\exp(t)\}.$$

Figure 6 shows the exact and numerical solutions of $\{a(t), w(1/2,t)\}$ for $nt = nx = 100$.

Next, we investigate the stability of numerical solution with respect to the noisy overdetermination data (1.5) (or (2.1)), defined by the function

$$(h)_\gamma(t) = h(t) + \gamma\theta, \tag{5.4}$$

where $\gamma$ is the percentage of noise and $\theta$ are random variables generated from a uniform distribution in the interval $[-0.5, 0.5]$ which are generated using rand command in MATLAB.

Figures 7 and 8 show the exact and numerical solutions of $\{a(t), w(1/2,t)\}$ when the input data (5.4) is contaminated by $\gamma = 0.01$ and $0.05$ noise for Examples 3, respectively. From the Figure 7, it can be seen that the numerical coefficient solutions become unstable as the input data is contaminated with noise, because the derivatives in (5.1)-(5.3) are unstable under the random noisy input (5.4) if they are calculated using simply finite differences. But from the Figure 8, it can be seen clearly that the agreement between the numerical results and the analytical solutions $w(1/2,t)$ is good for exact data. In order to obtain a stable numerical derivative we employ the mollification method with a Gaussian mollifier ([18]), given by

$$J_\delta = \frac{1}{\delta\sqrt{\pi}} \exp(-t^2/\delta^2)$$

where $\delta > 0$ is the radius of mollification acting as an averaging filter. Its choice is based on standard methods for choosing regularization parameter in ill-posed problems such as the generalized cross-validation criterion. The mollification of the noisy data (5.4) is performed through the convolution

$$(J_\delta * h)(t) = \int_{-\infty}^{+\infty} J_\delta(\tau)h(t-\tau)d\tau, .$$

We notice that the mollifier $J_\delta$ is always positive and becomes close to zero outside the interval centred at the origin and of radius $3\delta$. Good results for the derivative $h''(t)$ are therefore expected in the interval $[3\delta, T - 3\delta]$. Notice that although $(h)_\gamma(t)$, given by (5.4) is non-smooth, its mollification $(J_\delta * h)(t)$, is $C^\infty$ functions, hence differentiable. The mollified derivative is then computed using that

$$(J_\delta * (h)_\gamma)''(t) = (J_\delta * (h)''_\gamma)(t) = (J''_\delta * (h)_\gamma)(t).$$

We use these mollified data to approximate (5.1)-(5.3), i.e. we replace the finite difference quotients $\frac{1}{(ht)^2}(h^{n+1} - 2h^n + h^{n-1})$, in (5.1) by $(J_\delta * (h)_\gamma)''(t)$ for $n = 0, ..., nt$. This mollification of the first-order derivative has been performed using MATLAB version of the computational program supplied by D. A. Murio in ([19]). For $\gamma = 0.01$ noise, error of mollified input data (5.4) is $0.00494$ and the radius of mollification $\delta = 0.03402$. For $\gamma = 0.05$ noise, error of mollified input data (5.4) is $0.00624$ and the radius of mollification $\delta = 0.03271$.

Figures 9 shows the exact and numerical solutions of $a(t)$ obtained after mollification, when the input data (1.5) is contaminated by $\gamma = 0.01$, and $0.05$ noise for Example 3, respectively. From this figure it can be seen that the application of the mollification to stabilize the derivative of the noisy function $(h)_\gamma(t)$, produce stable numerical solutions for $a(t)$.

## 6 Conflicts of interest

There are no conflicts of interest to this work.

## 7 Conclusion

In this paper, we solve the inverse problem to determine the unknown time-dependent coefficient of a non-linear source from an additional measurement. The existence and uniqueness of a classical solution to the inverse IBVP are proved under the assumptions $(A_0)$-$(A_5)$, which include that $g(w)$ and its derivatives of up to the second order are Lipschitz continuous functions. The inverse problem is also numerically solved by applying finite difference method. The examples illustrate how to implement the numerical method. Figures demonstrate that this method is effective for non-linear hyperbolic equation with sine, quadratic and cubic non-linearity.

## References

1. Böhme C., Michael R.:*A scale-invariant Klein-Gordon model with time-dependent potential*, Ann Univ Ferrara, **58(2)** 229-250 (2012).
2. Böhme C., Michael R.:*Energy bounds for Klein-Gordon equations with time-dependent potential*, Ann Univ Ferrara, **59(1)** 31-55 (2013).
3. Cannon JR, DuChateau P.:*An inverse problem for an unknown source term in a wave equation*, SIAM Journal on Applied Mathematics, **43(3)** 553-564 (1983).
4. Cavaterra C.:*An inverse problem for a semilinear wave equation*, Quad./Univ. degli studi di Milano, (1987).
5. Denisov AM.:*Inverse problem for a semilinear functional-differential wave equation*, Journal of Inverse and Ill-posed Problems, **16(9)** 837-848 (2008).
6. Grasselli M.:*Local existence and uniqueness for a quasilinear hyperbolic inverse problem*, Applicable Analysis, **32(1)** 15-30 (1989).
7. Greiner W.: Relativistic Quantum Mechanics-Wave Equations. *Berlin: Springer* (2000).
8. Hussein, S.O., Lesnic D., Yamamoto M.:*Reconstruction of space-dependent potential and/or damping coefficients in the wave equation*, Computers and Mathematics with Applications, **74.6** 1435-1454 (2017).
9. Imanuvilov O, Yamamoto M.:*Global uniqueness and stability in determining coefficients of wave equations*, Comm. Part. Diff. Equat., **26** 1409-1425 (2001).
10. Ionkin, N. I.:*The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition*, Differ. Uravn., **13(2)** 294-304 (1977).
11. Isakov V.: Inverse problems for partial differential equations. *Applied mathematical sciences. New York (NY): Springer* (2006).

12. Janno J., Šeletski A.:*Reconstruction of coefficients of higher order nonlinear wave equations by measuring solitary waves*, Wave Motion, **52** 15-25 (2015).
13. Jiménez S, Vazquez L.:*Analysis of four numerical schemes for a nonlinear Klein-Gordon equation*, Applied Mathematics and Computation, **35(1)** 61-94 (1990).
14. Kozhanov AI, Nikolai AL.:*On solvability of boundary-value problems for the wave equation with a nonlinear dissipation in noncylindrical domains*, Siberian Mathematical Journal, **42(6)** 1062-1081 (2001).
15. Lesnic, D.,Hussein S.O. , and Tomas Johansson B.: *Inverse space-dependent force problems for the wave equation*, Journal of Computational and Applied Mathematics, **306** 10-39 (2016).
16. Lin Z, Gilbert RP.:*Numerical algorithm based on transmutation for solving inverse wave equation*, Mathematical and Computer Modelling, **39(13)** 1467-1476 (2004).
17. Megraliev Ya.T.: *On solvability of a single inverse boundary value problem for a second order elliptic equation*, Vestn. Tver State. unta Ser. Attach maths., **23** 25-38 (2011).
18. Murio DA.: The Mollification Method and the Numerical Solution of Ill-Posed Problems, *Wiley Interscience, New York* (1993).
19. Murio DA.: Mollification and space marching, in: K.A. Woodbury (Ed.), Inverse Engineering Handbook, *CRC Press, Boca Raton, Florida* 219-326 (2002).
20. Nakamura G, Michiyuki W, Barbara K.:*On the identification of a coefficient function in a nonlinear wave equation.*, Inverse Problems, **25(3)** 035007 (2009)
21. Namazov GK.: Inverse Problems of the Theory of Equations of Mathematical Physics, *Baku, Azerbaijan* (1984). (in Russian).
22. Perring, J. K., and T. H. R. Skyrme.: *A model unified field equation*, Nuclear Physics, **31** 550-555 (1962).
23. Prilepko AI, Orlovsky DG, Vasin IA.: Methods for solving inverse problems in mathematical physics. Vol. 231, *Pure and applied mathematics. New York (NY): Marcel Dekker* (2000).
24. Qi GB, Wang JT.: *An inverse problem for a one-dimensional semilinear hyperbolic equation with an unknown source*, J. Harbin Inst. Tech, **4** 14-20 (1989)
25. Romanov VG.: Inverse Problems of Mathematical Physics, *VNU Science Press BV, Utrecht, Netherlands*, (1987).
26. Samarskii, A. A.:*On some problems of the theory of differential equations*, Differ. Uravn., **16(11)** 1925-1935 (1980).
27. Shcheglov A Yu.:*A method for finding coefficients of a quasilinear hyperbolic equation*, Computational Mathematics and Mathematical Physics, **46(5)** 776-795 (2006).
28. Shcheglov A Yu, Viktorova IV.:*Numerical Reconstitution of a Nonlinear Source in a Hyperbolic Equation*,Numerical Functional Analysis and Optimization, 137-150 (2004).
29. Shcheglov A Yu.: *Inverse coefficient problem for a quasilinear hyperbolic equation with final overdetermination*, Computational Mathematics and Mathematical Physics, **46(4)** 616-635 (2006).
30. Slodicka M, Seliga L.:*Determination of a time-dependent convolution kernel in a nonlinear hyperbolic equation*, Inverse Probl. Sci. Eng. **24(6)** 1011-1029 (2016).
31. Strauss W., Vazquez L.: *Numerical solution of a nonlinear Klein-Gordon equation*, Journal of Computational Physics, **28(2)** 271-278 (1978) .
32. Tekin, I., Mehraliyev, Y. T., and Ismailov, M. I.: *Existence and uniqueness of an inverse problem for nonlinear KleinGordon equation*, Math Meth Appl Sci., **42(10)** 3739-3753 (2019) .
33. Wazwaz, Abdul-Majid: *The tanh method: exact solutions of the sine-Gordon and the sinh-Gordon equations*, Applied Mathematics and Computation, **16(2)** 1196-1210 (2005).

34. Wazwaz, Abdul-Majid: *The integrable time-dependent sine-Gordon equation with multiple optical kink solutions*, Optik, **182** 605-610 (2019).
35. Wazwaz, Abdul-Majid: *New travelling wave solutions to the Boussinesq and the KleinGordon equations*, Communications in Nonlinear Science and Numerical Simulation, **13(5)** 889-901 (2008).
36. Whitham, G.B. Linear and Nonlinear Waves, *WileyInterscience, New York* (1999).
37. Zhengping, Y., Zhong, Wei-Ping: *Analytical solutions to sine-Gordon equation with variable coefficient*, Romanian Reports in Physics, **66(2)** 262-273 (2014).


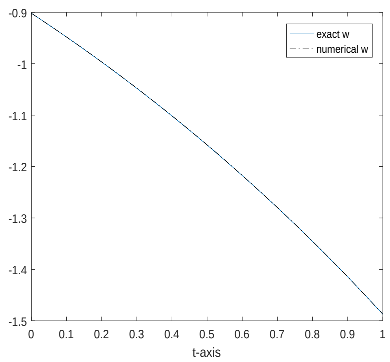


**Fig. 1** Exact and direct numerical solutions $w(1/2,t)$ for Example 1.

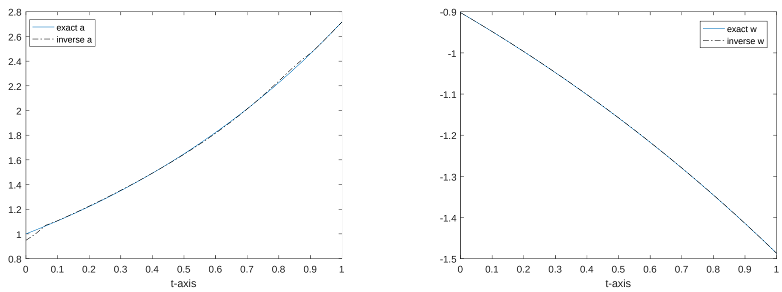


**Fig. 2** Exact and inverse numerical solutions $\{a(t), w(1/2,t)\}$ for Example 1.

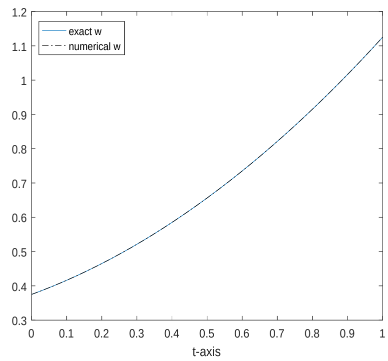


**Fig. 3** Exact and direct numerical solutions $w(1/2, t)$ for Example 2.

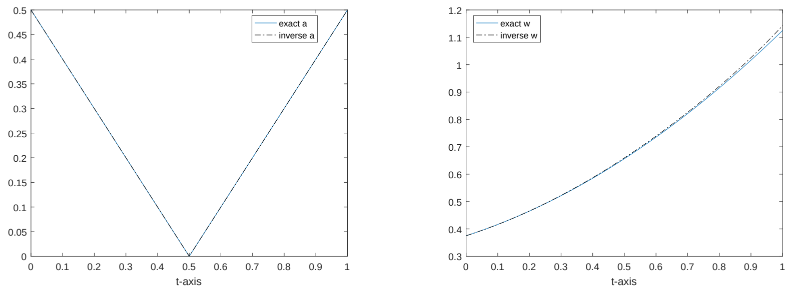


**Fig. 4** Exact and inverse numerical solutions $\{a(t), w(1/2, t)\}$ for Example 2.

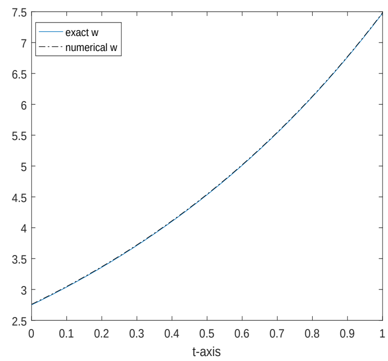


**Fig. 5** Exact and direct numerical solutions $w(1/2, t)$ for Example 3.

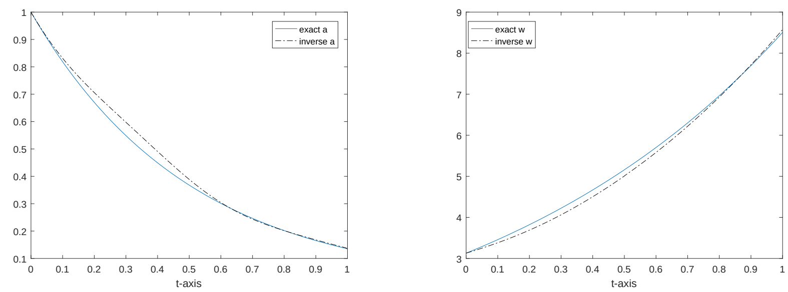


**Fig. 6** Exact and inverse numerical solutions $\{a(t), w(1/2, t)\}$ for Example 3.

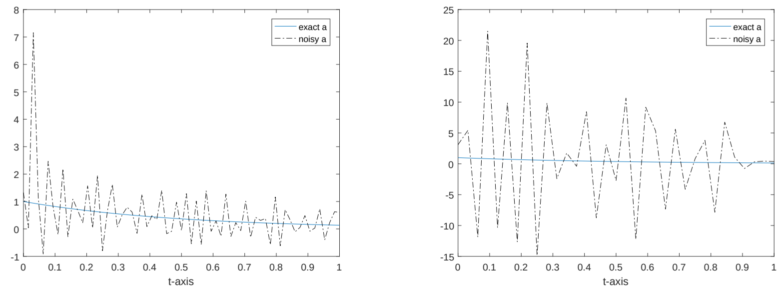


**Fig. 7** Exact and numerical coefficient solutions $a(t)$ for Example 3 with 0.01, and 0.05 noise.

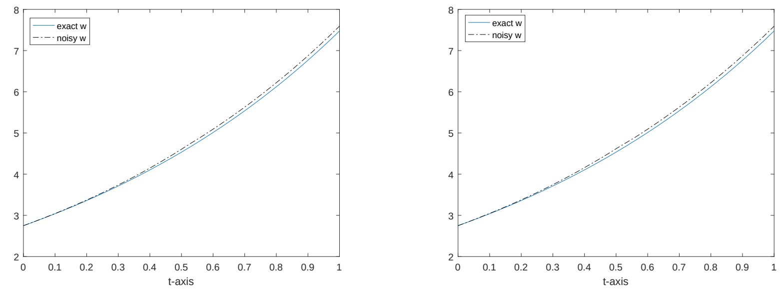


**Fig. 8** Exact and numerical $w(1/2, t)$ solutions for Example 3 with 0.01, and 0.05 noise.

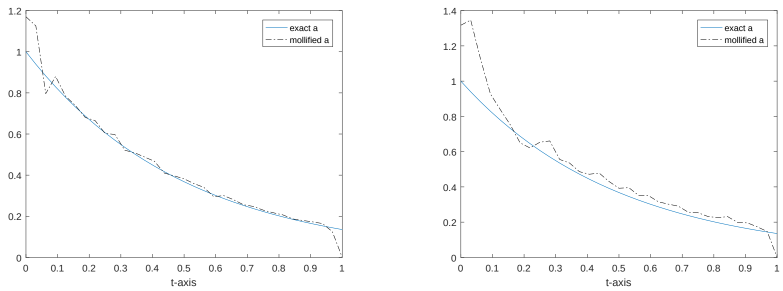


**Fig. 9** Exact and numerical coefficient solutions $a(t)$ for Example 3 after mollification with 0.01, and 0.05 noise.