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Generating functions of the products of Gaussian numbers with Chebyshev polynomials of first and second kinds

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Abstract. *In this paper, we introduce a operator in order to derive some new symmetric properties of Gaussian Pell and Gaussian Pell Lucas numbers, and we give some new generating functions of the products of Gaussian Fibonacci numbers, Gaussian Lucas numbers, Gaussian Jacobsthal numbers, Gaussian Jacobsthal Lucas numbers, Gaussian Pell numbers and Gaussian Pell Lucas numbers with Chebyshev polynomials of first and second kinds.*

Keywords. symmetric functions, generating functions, Gaussian Fibonacci numbers, Gaussian Pell numbers.

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1 Introduction

The authors in [5] defined and studied the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. They gave generating functions, Binet's formulas, explicit formulas and Q matrix of these numbers. They also presented explicit combinatorial and determinantal expressions, study negatively subscripted numbers and gave various identities. Similar to the Jacobsthal and Jacobsthal Lucas numbers they gave some interesting results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers.

In the papers [13, 16], second-order linear recurrence sequence $(U_n(a, b; p, q))_{n\geq 0}$ or briefly $(U_n)_{n\geq 0}$ is considered by the recurrence relation:

$$U_{n+2} = pU_{n+1} + qU_n, \quad \text{for } n \geq 0,$$

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with the initial conditions $U_0 = a$ and $U_1 = b$, special cases are listed in the table below:

a	b	p	q	$(U_n)_{n\geq 0}$	
i	1	1	1	$(GF_n)_{n\geq 0}$	Gaussian Fibonacci numbers
$2-i$	$1+2i$	1	1	$(GL_n)_{n\geq 0}$	Gaussian Lucas numbers
$\frac{i}{2}$	1	1	2	$(GJ_n)_{n\geq 0}$	Gaussian Jacobsthal numbers
$2-\frac{i}{2}$	$1+2i$	1	2	$(Gj_n)_{n\geq 0}$	Gaussian Jacobsthal Lucas numbers
i	1	2	1	$(GP_n)_{n\geq 0}$	Gaussian Pell numbers
$2-2i$	$2+2i$	2	1	$(GQ_n)_{n\geq 0}$	Gaussian Pell Lucas numbers

Table 1. Gaussian numbers.

Definition 1.1 *[7] The Chebyshev polynomials of second kind say $(U_n(x))_{n\in\mathbb{N}}$ is defined by the recurrence relation*

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad \textit{for } n \geq 2$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$.

Definition 1.2 *[7]The Chebyshev polynomials of first kind say $(T_n(x))_{n\in\mathbb{N}}$ is defined by the recurrence relation*

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad \textit{for } n \geq 2$$

with initial conditions $T_0(x) = 1$ and $T_1(x) = x$.

Binet's formulas are well known formulas in the theory of Fibonacci numbers. These formulas can also be carried out to the Gaussian Pell numbers. Let α and β be the roots of the characteristic equation $t^2 - 2t - 1 = 0$ of the recurrence relationship of the Gaussian Pell numbers. In the following theorem we give the Binet's formulas for Gaussian Pell and Gaussian Pell-Lucas numbers.

Theorem 1.1 *[27] Binet's formulas for Gaussian Pell and Gaussian Pell-Lucas numbers are given by*

$$GP_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta}, \quad n \geq 0$$

and

$$GQ_n = (\alpha^n + \beta^n) - i\left(\alpha\beta^n + \beta\alpha^n\right), \quad n \geq 0$$

respectively.

2 Definitions and some Properties

In this section, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under permutation of the letters of A. Taking an extra indeterminate z, one has two fundamental series

$$\lambda_z(A) = \underset{a\in A}{\Pi}(1 + az), \; \sigma_z(A) = \frac{1}{\underset{a\in A}{\Pi}(1 - az)}$$

the expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$

$$\lambda_z(A) = \sum_{n=0}^{+\infty} \Lambda_n(A)z^n, \ \sigma_z(A) = \sum_{n=0}^{+\infty} S_n(A)z^n.$$

Let us now start at the following definition.

Definition 2.1 *(see [1]) Let A and B be any two alphabets, then we give $S_n(A-B)$ by the following form:*

$$\frac{\underset{b\in B}{\Pi}(1-bz)}{\underset{a\in A}{\Pi}(1-az)} = \sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A-B) \tag{2.1}$$

with the condition $S_n(A-B) = 0$ for $n < 0$.

Corollary 2.1 *(see [1]). Taking $A = 0$ in (2.1) gives*

$$\underset{b\in B}{\Pi}(1-bz) = \sum_{n=0}^{+\infty} S_n(-B)z^n = \lambda_z(-B). \tag{2.2}$$

Further, in the case $A = 0$ or $B = 0$, we have

$$\sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A) \times \lambda_z(-B). \tag{2.3}$$

Thus,

$$S_n(A-B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B).$$

Definition 2.2 *(see [34]) Let g be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1,...,x_i,x_{i+1},...,x_n) - g^\sigma(x_1,...,x_i,x_{i+1},...,x_n)}{x_i - x_{i+1}},$$

where g^σ is given by

$$g^\sigma(x_1,...,x_i,x_{i+1},...,x_n) = g(x_1,...,x_{i-1},x_{i+1},x_i,x_{i+2},...,x_n).$$

Definition 2.3 *[8] Given an alphabet $E = \{e_1, e_2\}$, the symmetrizing operator $\delta^k_{e_1 e_2}$ is defined by*

$$\delta^k_{e_1 e_2} f(e_1) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2}, \text{ for } k \in \mathbb{N}.$$

Definition 2.4 *[39] The generalized hypergeometric functions ${}_pF_q(\cdot)$ are defined by*

$${}_pF_q[\alpha_1,...,\alpha_p;\ \beta_1,...,\beta_q;\ x] = \sum_{n=0}^{+\infty} \frac{(\alpha_1)_n...(\alpha_p)_n}{(\beta_1)_n...(\beta_q)_n} \frac{x^n}{n!} \tag{2.4}$$

where $\alpha_1,...,\alpha_p,\beta_1,...,\beta_q,\ x \in \mathbb{C}$, $\beta_1,...,\beta_q$ are neither zero nor negative integers, and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, (n=0) \\ \lambda(\lambda-1)...(\lambda-n+1),\ (n \geq 1) \end{cases}.$$

For the special case that corresponds to $p = 2$ and $q = 1$ in (2.4), we can obtain ${}_2F_1(a, b; c; x)$ Gauss hypergeometric function

$$ {}_2F_1(a, b; c; x) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. $$

Theorem 2.1 *[38] The hypergeometric form of the Chebyshev polynomials of the first kind, can be written as follows:*

$$ T_n(x) ={}_2 F_1\big(- n, n; \frac{1}{2}; \frac{1-x}{2}\big). $$

Theorem 2.2 *[38] The hypergeometric form of the Chebyshev polynomials of the second kind, can be written as follows:*

$$ U_n(x) = (n + 1)_2\, F_1\big(- n, n + 2; \frac{3}{2}; \frac{1-x}{2}\big). $$

3 Generating Functions of the Products of Gaussian Numbers with Chebyshev Polynomials of First and Second Kinds

The following propositions are one of the key tools of the proof of our main result.

Proposition 3.1 *[8] Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then*

$$ \sum_{n=0}^{+\infty} S_n(A)S_n(E)z^n = \frac{1 - a_1a_2e_1e_2z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^n z^n\right)}. \tag{3.1} $$

Proposition 3.2 *[15] Given two alphabets $A = \{a_1, a_2\}$ and $E = \{e_1, e_2\}$, then*

$$ \sum_{n=0}^{+\infty} S_{n-1}(A)S_n(E)z^n = \frac{(e_1 + e_2)z - e_1e_2(a_1 + a_2)z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^n z^n\right)}. \tag{3.2} $$

In this part, we now derive the new generating functions of the products of Gaussian numbers and polynomials with Chebyshev polynomials of first and second kinds.

For the case $A = \{a_1, -a_2\}$, $E = \{2e_1, -2e_2\}$ with replacing $a_2$by $(-a_2)$, e_1 by $(2e_1)$ and e_2 by $(-2e_2)$ in (3.1) and (3.2) we have

$$ \sum_{n=0}^{+\infty} S_n\left(a_1 + [-a_2]\right) S_n\left(2e_1 + [-2e_2]\right) z^n = \frac{1 - 4a_1a_2e_1e_2z^2}{(1 - 2a_1e_1)(1 + 2a_2e_1)(1 + 2a_1e_2)(1 - 2a_2e_2)}. \tag{3.3} $$

$$ \sum_{n=0}^{+\infty} S_{n-1}(a_1{+}[-a_2])S_n(2e_1{+}[-2e_2])z^n = \frac{2(e_1 - e_2)z + 4e_1e_2(a_1 - a_2)z^2}{(1 - 2a_1e_1)(1 + 2a_2e_1)(1 + 2a_1e_2)(1 - 2a_2e_2)}. \tag{3.4} $$

This case consists of three related parts. **Firstly**, the substitutions

$$ \begin{cases} a_1 - a_2 = 1 \\ a_1a_2 = 1 \end{cases} \text{and} \begin{cases} e_1 - e_2 = x \\ 4e_1e_2 = -1, \end{cases} $$

in (3.3) and (3.4) we obtain

$$\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right) S_n\left(2e_1+[-2e_2]\right) z^n = \frac{1+z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}. \tag{3.5}$$

$$\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right) S_n\left(2e_1+[-2e_2]\right) z^n = \frac{2xz-z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} \tag{3.6}$$

and we have the following theorems.

Theorem 3.1 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Fibonacci numbers with Chebyshev polynomials of second kind is given by*

$$\sum_{n=0}^{+\infty} GF_n \ (n+1)_2 F_1\big(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\big) z^n = \frac{i+2x(1-i)z+(2i-1)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.$$

Proof. We know that

$$GF_n = iS_n(a_1+[-a_2]) + (1-i)S_{n-1}(a_1+[-a_2])$$

see [19]. We see that

$$\sum_{n=0}^{+\infty} GF_n \ (n+1)_2 F_1\big(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\big) z^n$$

$$= \sum_{n=0}^{+\infty} (iS_n(a_1+[-a_2]) + (1-i)S_{n-1}(a_1+[-a_2]))S_n(2e_1+[-2e_2])z^n$$

$$\begin{aligned} &= i\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\ &\quad +(1-i)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\ &= \frac{i+iz^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} \\ &\quad +\frac{2x(1-i)z-(1-i)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} \\ &= \frac{i+2x(1-i)z+(2i-1)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}. \end{aligned}$$

This completes the proof.

Theorem 3.2 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Fibonacci numbers with Chebyshev polynomials of first kind is given by*

$$\sum_{n=0}^{+\infty} GF_n \, {}_2F_1\big(-n, n; \frac{1}{2}; \frac{1-x}{2}\big) z^n = \frac{i+x\left(1-2i\right)z+\left(2i\left(1-x^2\right)-1\right)z^2-\left(1-i\right)xz^3}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.$$

Proof. We have

$$GF_n = iS_n\left(a_1 + [-a_2]\right) + (1-i)\, S_{n-1}\left(a_1 + [-a_2]\right),$$

see [19]. We see that

$$\begin{aligned}
&\sum_{n=0}^{+\infty} GF_n\, {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n \\
&= \sum_{n=0}^{+\infty} \left(iS_n\left(a_1 + [-a_2]\right) + (1-i)\, S_{n-1}\left(a_1 + [-a_2]\right)\right) \\
&\times \left(S_n\left(2e_1 + [-2e_2]\right) - xS_{n-1}\left(2e_1 + [-2e_2]\right)\right) z^n \\
&= i\sum_{n=0}^{+\infty} S_n\left(a_1 + [-a_2]\right) S_n\left(2e_1 + [-2e_2]\right) z^n \\
&-ix\sum_{n=0}^{+\infty} S_n\left(a_1 + [-a_2]\right) S_{n-1}\left(2e_1 + [-2e_2]\right) z^n \\
&+ (1-i)\sum_{n=0}^{+\infty} S_{n-1}\left(a_1 + [-a_2]\right) S_n\left(2e_1 + [-2e_2]\right) z^n \\
&- (1-i)\, x\sum_{n=0}^{+\infty} S_{n-1}\left(a_1 + [-a_2]\right) S_{n-1}\left(2e_1 + [-2e_2]\right) z^n \\
&= \frac{i\left(1+z^2\right)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} - \frac{ix\left(z+2xz^2\right)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} \\
&+ \frac{(1-i)\left(2xz-z^2\right)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} - \frac{x\,(1-i)\left(z+z^3\right)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} \\
&= \frac{i + x\,(1-2i)\, z + \left(2i\left(1-x^2\right)-1\right) z^2 - (1-i)\, xz^3}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.
\end{aligned}$$

This completes the proof.

Theorem 3.3 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Lucas numbers with Chebyshev polynomials of second kind is given by*

$$\sum_{n=0}^{+\infty} GL_n\ (n+1)_2\, F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n = \frac{2-i+2x(3i-1)z+(3-4i)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.$$

Proof. We know that

$$GL_n = (2-i)S_n(a_1 + [-a_2]) + (3i-1)S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\sum_{n=0}^{+\infty} GL_n\ (n+1)_2\, F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n$$

$$= \sum_{n=0}^{+\infty} ((2-i)S_n(a_1 + [-a_2]) + (3i-1)S_{n-1}(a_1 + [-a_2]))S_n(2e_1 + [-2e_2]) z^n$$

$$= (2-i)\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n$$
$$+(3i-1)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n$$
$$= \frac{(2-i)+(2-i)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}$$
$$+\frac{2x(3i-1)z-(3i-1)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}$$
$$= \frac{2-i+2x(3i-1)z+(3-4i)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.$$

This completes the proof.

Theorem 3.4 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Lucas numbers with Chebyshev polynomials of first kind is given by*

$$\sum_{n=0}^{+\infty} GL_n\, {}_2F_1(-n,n;\frac{1}{2};\frac{1-x}{2})z^n$$
$$= \frac{(2-i)+(4i-3)\,xz+\left(3-4x^2+2i\left(x^2-2\right)\right)z^2+(1-3i)\,xz^3}{1-2xz+(3-4x^2)\,z^2+2xz^3+z^4}.$$

Proof. We know that

$$GL_n = (2-i)S_n(a_1+[-a_2])+(3i-1)S_{n-1}(a_1+[-a_2]),$$

see [19]. We see that

$$\sum_{n=0}^{+\infty} GL_n\ {}_2F_1(-n,n;\frac{1}{2};\frac{1-x}{2})z^n = \sum_{n=0}^{+\infty}\left((2-i)\,S_n\left(a_1+[-a_2]\right)+(3i-1)\,S_{n-1}\left(a_1+[-a_2]\right)\right)$$
$$\times\left(S_n\left(2e_1+[-2e_2]\right)-xS_{n-1}\left(2e_1+[-2e_2]\right)\right)z^n$$
$$= (2-i)\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right)S_n\left(2e_1+[-2e_2]\right)z^n$$
$$-(2-i)\,x\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right)S_{n-1}\left(2e_1+[-2e_2]\right)z^n$$
$$+(3i-1)\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right)S_n\left(2e_1+[-2e_2]\right)z^n$$
$$+(1-3i)\,x\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right)S_{n-1}\left(2e_1+[-2e_2]\right)z^n$$

$$= \frac{(2-i)\left(1+z^2\right)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}-\frac{x\,(2-i)\left(z+2xz^2\right)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}$$
$$+\frac{(3i-1)\left(2xz-z^2\right)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}-\frac{x\,(3i-1)\left(z+z^3\right)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}$$

$$= \frac{2 - i + (4i - 3)\, xz + \left(3 - 4x^2 + 2i\left(x^2 - 2\right)\right) z^2 + (1 - 3i)\, xz^3}{1 - 2xz + (3 - 4x^2)\, z^2 + 2xz^3 + z^4}.$$

This completes the proof.

Secondly, the substitutions

$$\begin{cases} a_1 - a_2 = 1 \\ a_1 a_2 = 2 \end{cases} \text{ and } \begin{cases} e_1 - e_2 = x \\ 4e_1 e_2 = -1, \end{cases}$$

in (3.3) and (3.4) we give

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{1 + 2z^2}{1 - 2xz + (5 - 8x^2)z^2 + 2xz^3 + 4z^4}. \quad (3.7)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{2xz - z^2}{1 - 2xz + (5 - 8x^2)z^2 + 2xz^3 + 4z^4}, \quad (3.8)$$

and we have the following theorems.

Theorem 3.5 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal numbers with Chebyshev polynomials of second kind is given by*

$$\sum_{n=0}^{+\infty} GJ_n \ (n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n = \frac{i + 2x(2 - i)z + (3i - 2)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}.$$

Proof. We know that

$$GJ_n = \frac{i}{2} S_n(a_1 + [-a_2]) + (1 - \frac{i}{2}) S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\sum_{n=0}^{+\infty} GJ_n \ (n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n$$

$$= \sum_{n=0}^{+\infty} (\frac{i}{2} S_n(a_1 + [-a_2]) + (1 - \frac{i}{2}) S_{n-1}(a_1 + [-a_2])) \times S_n(2e_1 + [-2e_2]) z^n$$

$$= \frac{i}{2} \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n$$

$$+ (1 - \frac{i}{2}) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n$$

$$= \frac{i + 2iz^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4} + \frac{2x(2 - i)z - (2 - i)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}$$

$$= \frac{i + 2x(2 - i)z + (3i - 2)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}.$$

This completes the proof.

Theorem 3.6 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal numbers with Chebyshev polynomials of first kind is given by*

$$\sum_{n=0}^{+\infty} GJ_n\, {}_2F_1(-n,n;\frac{1}{2};\frac{1-x}{2})z^n$$

$$= \frac{i+(2x-3ix)\,z-4\left(4x+i\left(1+2x+2x^2\right)\right)z^2-4\,(2-i)\,xz^3}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}.$$

Proof. We know that

$$GJ_n = \frac{i}{2}S_n(a_1+[-a_2])+(1-\frac{i}{2})S_{n-1}(a_1+[-a_2]),$$

see [19]. We see that

$$\sum_{n=0}^{+\infty} GJ_n\, {}_2F_1(-n,n;\frac{1}{2};\frac{1-x}{2})z^n = \sum_{n=0}^{+\infty}\left(\frac{i}{2}S_n\left(a_1+[-a_2]\right)+\left(1-\frac{i}{2}\right)S_{n-1}\left(a_1+[-a_2]\right)\right)$$
$$\times\left(S_n\left(2e_1+[-2e_2]\right)-xS_{n-1}\left(2e_1+[-2e_2]\right)\right)z^n$$

$$= \frac{i}{2}\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right)S_n\left(2e_1+[-2e_2]\right)z^n - \frac{ix}{2}\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right)S_{n-1}\left(2e_1+[-2e_2]\right)z^n$$

$$+\left(1-\frac{i}{2}\right)\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right)S_n\left(2e_1+[-2e_2]\right)z^n$$
$$-x\left(1-\frac{i}{2}\right)\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right)S_{n-1}\left(2e_1+[-2e_2]\right)z^n$$

$$= \frac{i\left(1+2z^2\right)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4} - \frac{ix(z+4x)z^2}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}$$

$$+\frac{(2-i)\left(2xz-z^2\right)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4} - \frac{x\,(2-i)\left(z+2z^3\right)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}$$

$$= \frac{i+2x\,(1-i)\,z+(i\left(3-4x^2\right)-2)z^2+2x\,(i-2)\,z^3}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}.$$

This completes the proof.

Theorem 3.7 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal-Lucas numbers with Chebyshev polynomials of second kind is given by*

$$\sum_{n=0}^{+\infty} Gj_n\ (n+1)_2\, F_1(-n,n+2;\frac{3}{2};\frac{1-x}{2})z^n = \frac{4-i+2x(5i-2)z+(10-7i)z^2}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}.$$

Proof. We know that

$$Gj_n = \Big(2 - \frac{i}{2}\Big) S_n(a_1 + [-a_2]) + \Big(\frac{5}{2}i - 1\Big) S_{n-1}(a_1 + [-a_2])$$

see [19]. We see that

$$\sum_{n=0}^{+\infty} Gj_n \ (n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n = \sum_{n=0}^{+\infty} \Big(\Big(2 - \frac{i}{2}\Big) S_n(a_1 + [-a_2])\Big)$$

$$+\Big(\frac{5}{2}i - 1\Big) S_{n-1}(a_1 + [-a_2])) S_n(2e_1 + [-2e_2]) z^n$$

$$= \Big(2 - \frac{i}{2}\Big) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n$$

$$+\Big(\frac{5}{2}i - 1\Big) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n$$

$$= \frac{4 - i + (8 - 2i)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}$$

$$+\frac{2x(5i - 2)z - (5i - 2)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}$$

$$= \frac{4 - i + 2x(5i - 2)z + (10 - 7i)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}.$$

This completes the proof.

Theorem 3.8 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal-Lucas numbers with Chebyshev polynomials of first kind is given by*

$$\sum_{n=0}^{+\infty} Gj_{n\ 2}F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n$$

$$= \frac{4 - i + 6x(i-1)z + \left((10 - 16x^2) + i\left(4x^2 - 7\right)\right) z^2 + 2x\left(2 - 5i\right) z^3}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}.$$

Proof. We know that

$$Gj_n = \big(2 - \frac{i}{2}\big) S_n(a_1 + [-a_2]) + \big(\frac{5}{2}i - 1\big) S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\sum_{n=0}^{+\infty} Gj_{n\ 2}F_1\big(-n, n; \frac{1}{2}; \frac{1-x}{2}\big) z^n = \sum_{n=0}^{+\infty} \big(\big(2 - \frac{i}{2}\big) S_n(a_1 + [-a_2]) + \big(\frac{5}{2}i - 1\big) S_{n-1}(a_1 + [-a_2])\big)$$

$$\times \left(S_n\left(2e_1 + [-2e_2]\right) - xS_{n-1}\left(2e_1 + [-2e_2]\right)\right) z^n$$

$$= \big(2 - \frac{i}{2}\big) \sum_{n=0}^{+\infty} S_n\left(a_1 + [-a_2]\right) S_n\left(2e_1 + [-2e_2]\right) z^n$$

$$-\big(2-\frac{i}{2}\big)\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right) S_{n-1}\left(2e_1+[-2e_2]\right) z^n$$

$$+\big(\frac{5}{2}i-1\big)\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right) S_n\left(2e_1+[-2e_2]\right) z^n$$

$$-x\big(\frac{5}{2}i-1\big)\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right) S_{n-1}\left(2e_1+[-2e_2]\right) z^n$$

$$=\frac{(4-i)\left(1+2z^2\right)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}-\frac{x\,(4-i)\,(z+4xz^2)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}$$

$$+\frac{(5i-2)\left(2xz-z^2\right)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}-\frac{x\,(5i-2)\left(z+2z^3\right)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}$$

$$=\frac{4-i+6x(i-1)z+\left((10-16x^2)+i\left(4x^2-7\right)\right)z^2+2x\,(2-5i)\,z^3}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}.$$

This completes the proof.

Thirdly, the substitutions

$$\begin{cases} a_1-a_2=2 \\ a_1a_2=1 \end{cases} \text{and} \begin{cases} e_1-e_2=x \\ 4e_1e_2=-1, \end{cases}$$

in (3.3) and (3.4) we give

$$\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n=\frac{1+z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}. \quad (3.9)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n=\frac{2xz-2z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}, \quad (3.10)$$

and we have the following theorems.

Theorem 3.9 *For $n\in\mathbb{N}$, the new generating function of the product of Gaussian Pell numbers with Chebyshev polynomials of second kind is given by*

$$\sum_{n=0}^{+\infty} GP_n\,(n+1)_2\,F_1(-n,n+2;\frac{3}{2};\frac{1-x}{2})z^n=\frac{i+2x(1-2i)z+(5i-2)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Proof. We know that

$$GP_n=iS_n(a_1+[-a_2])+(1-2i)S_{n-1}(a_1+[-a_2]),$$

see [19]. We see that

$$\sum_{n=0}^{+\infty} GP_n\ (n+1)_2\,F_1(-n,n+2;\frac{3}{2};\frac{1-x}{2})z^n$$

$$=\sum_{n=0}^{+\infty}(iS_n(a_1+[-a_2])+(1-2i)S_{n-1}(a_1+[-a_2]))\times S_n(2e_1+[-2e_2])z^n$$

$$= i\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n$$

$$+(1-2i)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n$$

$$= \frac{i+iz^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4} + \frac{2x(1-2i)z-2(1-2i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}$$

$$= \frac{i+2x(1-2i)z+(5i-2)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

This completes the proof.

Theorem 3.10 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell numbers with Chebyshev polynomials of first kind is given by*

$$\sum_{n=0}^{+\infty} GP_{n2}F_1(-n,n;\frac{1}{2};\frac{1-x}{2})z^n = \frac{i+(x-4xi)z+\left(i\left(5-2x^2\right)-2\right)z^2+(2i-1)xz^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Proof. We know that

$$GP_n = iS_n(a_1+[-a_2])+(1-2i)S_{n-1}(a_1+[-a_2]) \text{ [19]}.$$

We see that

$$\sum_{n=0}^{+\infty} GP_n\, {}_2F_1(-n,n;\frac{1}{2};\frac{1-x}{2})z^n = \sum_{n=0}^{+\infty}\left(iS_n\left(a_1+[-a_2]\right)+(1-2i)\,S_{n-1}\left(a_1+[-a_2]\right)\right)$$

$$\times\left(S_n\left(2e_1+[-2e_2]\right)-xS_{n-1}\left(2e_1+[-2e_2]\right)\right)z^n$$

$$= i\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right)S_n\left(2e_1+[-2e_2]\right)z^n - ix\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right)S_{n-1}\left(2e_1+[-2e_2]\right)z^n$$

$$+(1-2i)\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right)S_n\left(2e_1+[-2e_2]\right)z^n$$

$$-(1-2i)\,x\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right)S_{n-1}\left(2e_1+[-2e_2]\right)z^n$$

$$= \frac{i\left(1+z^2\right)}{1-4xz+(6-4x^2)z^2+4xz^3+z^4} - \frac{ix\left(2z+2xz^2\right)}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}$$

$$+\frac{(1-2i)\left(2xz-2z^2\right)}{1-4xz+(6-4x^2)z^2+4xz^3+z^4} - \frac{x\,(1-2i)\left(z+z^3\right)}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}$$

$$= \frac{i+(x-4xi)z+\left(i\left(5-2x^2\right)-2\right)z^2+(2i-1)xz^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

This completes the proof.

Theorem 3.11 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell-Lucas numbers with Chebyshev polynomials of second kind is given by*

$$\sum_{n=0}^{+\infty} GQ_n \ (n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})z^n = \frac{2-2i+2x(6i-2)z+(6-14i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Proof. We know that

$$GQ_n = (2-2i)S_n(a_1+[-a_2]) + (6i-2)S_{n-1}(a_1+[-a_2]),$$

see [19]. We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} GQ_n\,(n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})z^n &= \sum_{n=0}^{+\infty}((2-2i)S_n(a_1+[-a_2]) \\ &\quad +(6i-2)S_{n-1}(a_1+[-a_2]))S_n(2e_1+[-2e_2])z^n \\ &= (2-2i)\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\ &\quad +(6i-2)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\ &= \frac{2-2i+(2-2i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4} \\ &\quad +\frac{2x(6i-2)z-2(6i-2)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4} \\ &= \frac{2-2i+2x(6i-2)z+(6-14i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.\end{aligned}$$

This completes the proof.

Theorem 3.12 *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell Lucas numbers with Chebyshev polynomials of first kind is given by*

$$\sum_{n=0}^{+\infty} GQ_n \, {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2})z^n$$

$$= \frac{2\,(1-i) + x\,(10i-6)\,z + \left(6-4x^2+i\left(4x^2-14\right)\right)z^2 + x\,(2-6i)\,z^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Proof. We know that

$$GQ_n = (2-2i)S_n(a_1+[-a_2]) + (6i-2)S_{n-1}(a_1+[-a_2]),$$

see [19]. We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} GQ_n \, {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2})z^n &= \sum_{n=0}^{+\infty}\left((2-2i)\,S_n\,(a_1+[-a_2]) + (6i-2)\,S_{n-1}\,(a_1+[-a_2])\right) \\ &\quad \times \left(S_n\,(2e_1+[-2e_2]) - xS_{n-1}\,(2e_1+[-2e_2])\right) z^n\end{aligned}$$

$$= (2-2i)\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right) S_n\left(2e_1+[-2e_2]\right) z^n$$
$$-x\left(2-2i\right)\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right) S_{n-1}\left(a_1+[-a_2]\right) z^n$$
$$+\left(6i-2\right)\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right) S_n\left(2e_1+[-2e_2]\right) z^n$$
$$-x\left(6i-2\right)\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right) S_{n-1}\left(2e_1+[-2e_2]\right) z^n$$
$$= \frac{(2-2i)\left(1+z^2\right)}{1-4xz+(6-4x^2)z^2+4xz^3+z^4} - \frac{x\left(2-2i\right)\left(2z+2xz^2\right)}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}$$
$$+\frac{(6i-2)\left(2xz-2z^2\right)}{1-4xz+(6-4x^2)z^2+4xz^3+z^4} - \frac{x\left(6i-2\right)\left(z+z^3\right)}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}$$
$$= \frac{2\left(1-i\right)+x\left(10i-6\right)z+\left(6-4x^2+i\left(4x^2-14\right)\right)z^2+x\left(2-6i\right)z^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

This completes the proof.

4 Conclusion

In this paper, by making use of Eqs. (3.1) and (3.2), we have derived some new generating functions of the products of Gaussian Fibonacci numbers, Gaussian Lucas numbers, Gaussian Jacobsthal numbers, Gaussian Jacobsthal Lucas numbers, Gaussian Pell numbers and Gaussian Pell Lucas numbers with Chebyshev polynomials of first and second kinds. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

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References

1. Abderrezzak, A.: *Généralisation de la transformation d'Euler d'une série formelle*, Adv. Math. **103**, 180–195 (1994).
2. Abderrezzak, A.: *Généralisation d'identité de Carlitz. Howard et Lehmer*, Aequ. Math. **49**, 36–46 (1995).
3. Asci, M., Gurel, E.: *On Bivariate Complex Fibonacci and Lucas Polynomials*, Conference on Mathematical Sciences ICM 2012, March 11–14 (2012).
4. Asci, M., Gurel, E.: *Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials*, Notes Number Theory Discrete Math. **19**, 25–36 (2013).

5. Asci, M., Gurel, E.: *Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Numbers,* Ars Comb. **111**, 53–63 (2013).
6. Araci, S.: *Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus*, Appl. Math. Comput. **233**, 599–607 (2014).
7. Benoit, A.: *Algorithmique Semi-Numérique Rapide des Séries de Tchebychev*, Thèse Doctorat, Ecole doctorale de Mathématiques et Informatiques de Bordeaux. (2012).
8. Boussayoud, A., Kerada, M.: *Symmetric and Generating Functions*, Int. Electron. J. Pure Appl. Math. **7**, 195–203 (2014).
9. Boussayoud, A., Kerada, M., Sahali, R., Rouibah, W.: *Some Applications on Generating Functions*, J.Concr. Appl. Math. **12**, 321–330 (2014).
10. Boussayoud, A., Kerada, M., Sahali, R.: *Symmetrizing Operations on Some Orthogonal Polynomails*, Int. Electron. J. Pure Appl. Math. **9**, 191–199 (2015).
11. Boussayoud, A., Sahali, R.: *The application of the operator* $L_{b_1b_2}^{-k}$ *in the series* $\sum_{j=0}^{+\infty} a_j b_1^j z^j$, J. Adv. Res. Appl. Math. **7**, 68–75 (2015).
12. Boussayoud, A., Harrouche, N.: *Complete symmetric functions and k-Fibonacci numbers*, Commun. Appl. Anal. **20**, 457-465, (2016).
13. Boussayoud, A.,Kerada, M., Boulyer, M.: *A simple and accurate method for determination of some generalized sequence of numbers*, Int.J.Pure Appl Math. **108**, 503–511, (2016).
14. Boussayoud, A., Kerada, M., Harrouche, N.: *On the k-Lucas numbers and Lucas Polynomials*, Turkish Journal of Analysis and Number. **5**, 121–125, (2017).
15. Boussayoud, A.: L'action de l'opérateur $\delta_{e_1e_2}^k$ sur la série $\sum\limits_{n=0}^{+\infty} S_n(A)e_1^n z^n$. *Doctoral dissertation, Mohamed Seddik Ben Yahia University, Jijel, Algeria* (2017).
16. Boussayoud, A.: *On some identities and generating functions for Pell-Lucas numbers*, Online.J. Anal. Comb. **12**(01), 1–10,(2017).
17. Boughaba, S., Boussayoud, A.: *On Some Identities And Generating Function of Both* K*-jacobsthal Numbers and Symmetric Functions in Several Variables*, Konuralp J. Math. **7**(02), 235–242. (2019).
18. Boughaba, S., Boussayoud, A.: *Construction of Symmetric Functions of Generalized Fibonacci Numbers*. Tamap Journal of Mathematics and Statistics. **3**, 1–8, (2019).
19. Boughaba,S. , Boussayoud, A., Araci, S., Kerada, M.: *Construction of Generating Functions of Gaussian Fibonacci Numbers and Polynomials*, (Submitted).
20. Bolat, C., Kose, H.: *On the Properties of k-Fibonacci numbers*, Int. J. Contemp. Math. Sciences. **5**, 1097–1105, (2010).
21. Djordjevic, G. B.: *Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers*, Fibonacci Q. **42**, 106–113, (2004).
22. Djordjevic, G. B., Srivastava, H. M.: *Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers*, Math. Comput. Modelling. **42**, 1049–1056, (2005).
23. Falcon, S., Plaza,A.: *On the Fibonacci* k*-numbers*, Chaos,Sulutions & Fractals. **32**, 1615–1624, (2007).
24. Falcon, S., Plaza, A.: *The* k*-Fibonacci sequence and the Pascal* 2*-triangle*, Chaos, Sulutions & Fractals. **33**, 38–49, (2008).
25. Falcon, S., Plaza,A.: *k-Fibonacci sequences and Polynomials and their derivatives*, Chaos, Sulutions & Fractals. **39**, 1005–1019, (2009).
26. Foata, D., Han, G-N.: *Nombres de Fibonacci et polynômes orthogonaux*, Leonardo Fibonacci: il tempo, le opere, l'eredità scientifica. 179–200, (1994).
27. Halici, S., Oz, S.: *On Some Gaussian Pell and Pell-Lucas Numbers*, Ordu Univ. J. Sci. Tech. **6**(01), 8–18, (2016).
28. Halici, S., Oz, S.: *On Gaussian Pell Polynomials and Their Some Properties*, Palest. J. Math. **7**(01), 251–256, (2018).

29. Horadam, A.F.: *Generating functions for powers of a certain generalized sequence of numbers*, Duke Math. J. **32**, 437–446, (1965).
30. Horadam, A. F. Horadam, E. M.: *Roots of recurrence-generated polynomials*, Fibonacci Q. **20**(03), 219–226, (1982).
31. Horadam, A.F., Mahon, J.M.: *Pell and Pell-Lucas Polynomials*, Fibonacci Q. **23**, 7–20, (1985).
32. Jordan, J. H.: *Gaussian Fibonacci and Lucas numbers*, Fibonacci Q. **3**, 315–318, (1965).
33. Kruchinin, D.V., Kruchinin, V.V.: *Application of a composition of generating functions for obtaining explicit formulas of polynomials*, J. Math. Anal. Appl. **404**, 161.171, (2013).
34. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, *Oxford University Press, Oxford* (1979).
35. Mezo, I.: *Several Generating Functions for second-order recurrence sequences*, J. Integer Seq. **12**, 1–16, (2009).
36. Pintér, A.: Srivastava, H. M., *Generating functions of the incomplete Fibonacci and Lucas numbers*, Rend. Circ. Mat. Palermo. **48**, 591–596, (1999).
37. Tasci, D., Cetin Firengiz, M.: *Incomplete Fibonacci and Lucas p-numbers*, Math. Comput. Modelling. **52**, 1763–1770, (2010).
38. Kim,T., Kim, D. S., Jang, L.C., Dolgy,D. V.:*Representation by Chebyshev Polynomials for Sums of Finite Products of Chebyshev Polynomials*, Symmetry. **10**, 742, (2018).
39. Yılmaz, O.G., Aktaş, R., Taşdelen, F.:*On Some Formulas for the k-Analogue of Appell Functions and Generating Relations via k-Fractional Derivative*, Fractal Fract. **4**, 48,(2020).