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A new characterization of simple groups $^2\mathrm{D_n}(3)$

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Abstract. In this paper, we prove that the simple groups $^2D_n(3)$, where $(n = 2^e + 2, e \geq 4)$ can be uniquely determined by its order and the largest elements order.

Keywords. Elements order, the largest elements order, Frobenius group, prime graph.

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1 Introduction

For a finite group G, the set of prime divisors of $|G|$ is denoted by $\pi(G)$ and the largest element of the set $\pi_e(G)$ of element orders of G is denoted by $k(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv{\in}\pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1$.

We also denote the set of all the primes dividing n by $\pi(n)$ where n is a natural number. Next, we know that $|G|$ is the product of m_1, m_2,... $m_t(G)$, where m_i is a positive integer with $\pi(m_i) = \pi_i$. All m_i are called the order components of G.
If H be a finite group such that $|G| = |H|$ and $k(G) = k(H)$ implies that $G \cong H$, then we say the group G is characterizable by using its order and the largest elements order. Next, for example the authors in ([2,4,5,7,13,9]) proved that the simple groups $L_3(q)$ and $U_3(q)$ where q is some special power of prime, the simple group $L_2(q)$ where $q = p^n < 125$, the simple K_4 -groups of type $L_2(p)$, where p is a prime but not 2^n-1, the projective general linear group $PGL(2, q)$ and suzuki group $Sz(q)$, where $q - 1$ or $q \pm \sqrt{2q} + 1$ is a prime number are characterizable by using the largest elements order and the order of the group.

In this paper, we prove that the simple groups $^2D_n(3)$, where $(n = 2^e + 2, e \geq 4)$, can be uniquely determined by its order and the largest elements order. We note that $^2D_n(3) \cong P\Omega^-_{2n}(3)$. In fact, we prove the following main theorem.
Main Theorem. Let G be a group with $|G| = |^2D_n(3)|$ and $k(G) = k(^2D_n(3))$, where $(n = 2^e + 2, e \geq 4)$. Then $G \cong^2 D_n(3)$.

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2 Notation and Preliminaries

In this section, we denote the several Lemmas and definition where we for proving the main theorem need them. Hence we have the following Lemmas.

Lemma 2.1 *[8] Let G be a Frobenius group of even order with kernel K and complement H. Then*

1 $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;
2 $|H|$ divides $|K| - 1$;
3 K is nilpotent.

Definition 2.1 *A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.*

Lemma 2.2 *[1] Let G be a 2-Frobenius group of even order. Then*

1 $t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;
2 G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.

Lemma 2.3 *[3] If $t(G) \geq 2$, H is a π_{i-}subgroup of G, and $H \trianglelefteq G$, then $\prod_{j=1, j\neq i}^{t(G)} m_i \mid (|H| - 1)$*

Lemma 2.4 *[16] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

1 G is a Frobenius group;
2 G is a 2-Frobenius group.
3 G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.

Lemma 2.5 *[17] Let q, k, l be natural numbers. Then*

1 $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.

2 $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

3 $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

3 Proof of the Main Theorem

In this section, we prove that the main theorem. To do this, we denote the simple groups ${}^2D_n(3)$ by D. To prove the main theorem we will prove several lemmas as follows. We note that $|D| = \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$ and $k(D) = 3^{n-1} - 1$.

Theorem 3.1 *Let G be a group and $D =^2 D_n(3)$ where $(n = 2^e + 2, e \geq 4)$. Then $k(G) = k(D)$ and $|G| = |D|$ if and only if $G \cong D$.*

Proof. First, we note that $m_1 = 3^{n(n-1)}(3^n + 1)(3^{n-1} - 1)\prod_{i=1}^{n-2}(3^{2i} - 1)$ and $m_2 = \frac{3^{n-1}+1}{2}$ are two components of D. Next, m_2 be odd order component of G, and also it is one of odd order components of K/H. It follows that $t(K/H) \geq 2$. Now Lemma 2.4 implies that G satisfies one of the following cases.

Lemma 3.1 *The group G is not a Frobenius group.*

Proof. We prove that G is not a Frobenius group. Opposite, we assume G be a Frobenius group with kernel K and complement H. Then by Lemma 2.4, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. So, $|K| = 3^{n(n-1)}(3^n + 1)(3^{n-1} - 1)\prod_{i=1}^{n-2}(3^{2i} - 1)$, $|H| = \frac{3^{n-1}+1}{2}$. Now, suppose that r is a prime divisor of $3^{2i} - 1$ and $G_r \in Syl_r(G)$. Thus, $|G_r| \mid \frac{3^n+1}{4}$ and $G_r \trianglelefteq G$ it follows that $|G_r| \equiv 1(\mod m_2)$. As a result there is the natural number s so that$|G_r| = s(\frac{3^{n-1}+1}{2}) + 1$. On the other hand, we have $|G_r| \leq \frac{3^n+1}{4}$, where that we deduce $s = 1$, so must be $\frac{(3^{n-1}+1)}{2} + 1$ divides $\frac{3^n+1}{4}$, which is impossible. Hence, G is not a Frobenius group.

Lemma 3.2 *The group G is not a 2-Frobenius group.*

Proof. We prove that that G is not a 2-Frobenius group. Opposite, we assume G be a 2-Frobenius group, so there a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius grops with kernel K/H and H, respectively. As a result, $t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$ and also G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$. Now, assume r is a prime divisor of $3^{2n} - 1$. Hence, we deduce that $r \mid \frac{3^n+1}{4}$ and $r \nmid (\frac{3^{n-1}-1}{2}$. As a result, $r \nmid |G/K|$, therefore $r \mid |H|$, which is impossible. Hence, G is not a 2-Frobenius group.

Lemma 3.3 *The group G is isomorphic to the group D.*

Proof. By the third case of Lemma 2.4,G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and also K/H is a non-abelian simple group. On the other hand, every odd order components of G are the odd order component of K/H. So, $t(K/H) \geq 2$. According to the classification of the finite simple groups we know that the possibilites for K/H are alternating group A_m, $m \geq 5$, 26 sporadic groups, simple groups Lie types.
First, we assume $G \cong D$. Then, we can see easily prove that . Now, we need prove sufficient condition, that is if $k(G) = k(D)$ and $|G| = |D|$, then $G \cong D$. Now, by [11] we have $k(D) = 3^{n-1} - 1$ and also $|D| = \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. Since that K/H is a non-abelian simple group. So, K/H is isomorphic one of the following groups.
Step 1. Let $K/H \cong A_m$, where $m \geq 5$ and $m = r, r + 1, r + 2$. Then by [11] $\pi(A_m) = m$ and $|A_m| \mid |G|$. For this purpose, we consider, $3^{n-1} - 1 = m$. Since that $m \geq 5$, so we deduce $3^{n-1} - 1 \geq 5$. As a result, $m \leq 3^{n-1} - 1 \leq 3^{n-1} \leq 3^n$, so $m \leq 3^n$, where this is impossible. Hence, $K/H \ncong A_m$.
Step 2. If K/H is isomorphic to sporadic simple groups, then by [11], we have $k(S) = \{5,7,11,17,19,23,31,37,59\}$. Now, we consider $3^{n-1} - 1 = 5, 7, 11, 17, 19,$
$23, 31, 37, 59$.Next, for example if $3^{n-1} - 1 = 5$, then we deduce $3^{n-1} = 6$. So, we can see easily this equation is impossible. For other groups, we have a contradiction, similarly.
Step 3. In this case, we consider K/H is isomorphic to a the group of Lie-types.
3.1. $K/H \ncong B'_n(q')$, where $n' > 2$ and $C'_n(q')$ with $n' > 3$ and also q' is power of prime number. For this purpose, we consider $K/H \cong B'_n(q')$. Now by [11], $k(B'_n(q')) = q'^{n'} + q'$ and also $|B'_n(q')| = \frac{q'^{n'^2}\prod_{i=1}^{n'}\prod_{i=1}^{n'}(q'^{2i}-1)}{(2,q'-1)}$. Since that $|B'_n(q')| \mid |G|$. So, $|\frac{q'^{n'^2}\prod_{i=1}^{n'}\prod_{i=1}^{n'}(q'^{2i}-1)}{(2,q'-1)} \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. Now, we consider, $q'^{n'} + q' = 3^{n-1} - 1$, it follows that $q'^{n'} + q' + 1 = 3^{n-1}$, where this is impossible. For example if $n = 3$, then the equation $q'^3 + q' + 1 = 3^{n-1} - 1$ has not any solution. The another value of n, also we have a contradiction. For $K/H \ncong C'_n(q')$, we have a contradiction, similarly.

3.2. If $K/H \cong ^3 D_4(q')$, then by [11], $k(^3D_4(q')) = (q'^3 - 1)(q' + 1)$. Also we know that $|^3D_4(q')| \mid |G|$, so $q'^{12}(q'^8 + q'^4 + 1)(q'^6 - 1)(q'^2 - 1) \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. Now, we consider $3^{n-1} - 1 = (q'^3 - 1)(q' + 1)$ it follows that $3^{n-1} - 1 = q'^4 + q'^3 - q' - 1$. Thus, $3(3^{n-2} = q'(q'^3 + q'^2 - 1)$ so $q' = 3$ and $q'^3 + q'^2 - 1 = 3^{n-2}$ which is a contradiction.

3.3. $K/H \ncong E_6(q')$, $E_7(q')$, $E_8(q')$, $F_4(q')$. For example if $K/H \cong F_4(q')$, then by [11] $k(F_4(q')) = (q'^3 - 1)(q' + 1)$. On the other hand, $|F_4(q')| = q'^{24}(q'^{12} - 1)(q'^8 - 1)(q'^6 - 1)(q'^2 - 1)$. Since that $|F_4(q')| \mid |G|$, so $q'^{24}(q'^{12} - 1)(q'^8 - 1)(q'^6 - 1)(q'^2 - 1) \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. For this purpose, we consider $3^{n-1} - 1 = (q'^3 - 1)(q' + 1)$. As a result like to proof 3.2, we have a contradiction. For $K/H \ncong E_6(q'), E_7(q'), E_8(q')$, we have a contradiction, similarly.

3.4. If $K/H \cong ^2 E_6(q')$, then by [11], $k(^2E_6(q')) = \frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}$. Also, we know that $|^2E_6(q')| = \frac{q'^{36}(q'^{12}-1)(q'^9+1)(q'^8-1)(q'^6-1)(q'^5+1)(q'^2-1)}{(3,q'+1)}$. Now, we consider $\frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)} = 3^{n-1} - 1$. First, if $(3, q' + 1) = 1$ then $(q' + 1)(q'^2 + 1)(q'^3 - 1) = 3^{n-1} - 1$. It follows that $q'^6 + q'^5 + q'^4 - q'^2 - q' = 3^{n-1}$, so $3(3^{n-2}) = q'(q'^5 + q'^4 - q' - 1)$. As a result $q' = 3$ and $3^{n-2} = q'^5 + q'^4 - q' - 1$, which is a contradiction. Now, if $(3, q' + 1) = 3$ then we deduce $\frac{(q'+1)(q'^2+1)(q'^3-1)}{3} = 3^{n-1} - 1$. As a result, $3(3^{n-1} = q'(q'^5 + q'^4 + q'^3 - q' - 1)$, which is a contradiction, similarly.

3.5. If $K/H \cong ^2 G_2(3^{2m+1})$, where $m \geq 1$ then by [11], $k(^2G_2(3^{2m+1})) = 3^{2m+1} + 3^{m+1}+1$. Also, we know that $|^2G_2(3^{2m+1})| = q'^3(q'^3+1)(q'-1)$. Since that $|^2G_2(3^{2m+1})| \mid |G|$. Hence, $q'^3(q'^3 + 1)(q' - 1) \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. For this purpose, we consider $3^{2m+1} + 3^{m+1} + 1 = 3^{n-1} - 1$. Now, since $m \geq 1$, so $38 \geq 3^{2m+1} + 3^{m+1} + 2 = 3^{n-1}$. As a result $3^{n-1} \geq 38$, so $n \geq 5$. On the other hand, we know that $n = 2^e + 2$, $e \geq 4$, so which is a contradiction.

3.6. If $K/H \cong ^2 B_2(q')$, where $q' = 2^{2m+1}$, $m \geq 1$, then by [11], $k(^2B_2(q')) = q' + \sqrt{2q'} + 1$, also $|^2B_2(q')| = q'^2(q'^2 + 1)(q' - 1)$. Since that $|^2B_2(q')| \mid |G|$ so $q'^2(q'^2 + 1)(q' - 1) \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. Now, we consider, $q' + \sqrt{2q'} + 1 = 3^{n-1} - 1$. Hence $2^{2m+1} + 2^{m+1} + 2 = 3(3^{n-2})$. It follows that $2(2^{2m} + 2^m + 1) = 3(3^{n-2})$. As a result we deduce $2 \mid 3^{n-2}$ and $2^{2m} + 2^m + 1 = 3$, this is impossible, because we deduce $m = 0$ where $m \geq 1$.

3.7. If $K/H \cong G_2(q')$,then by [11], $k(G_2(q')) = q'^2 + q' + 1$ and also $|G_2(q')| = q'^6(q'^6 - 1)(q'^2 - 1)$. Since $|G_2(q')| \mid |G|$, so $q'^6(q'^6 - 1)(q'^2 - 1) \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. For this purpose, we consider $q'^2 + q' + 1 = 3^{n-1} - 1$ so $q'^2 + q' + 1 = 3^{n-1} - 1 < 3^{n-1} < 3^n$. It follows that $q'^2 \leq 3^n$ thus $q'^6 \leq 3^{3n}$. On the other hand, we have $q'^6 < q'^6(q'^6 - 1)(q'^2 - 1) \leq \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)} \leq 3^n+1$. As a result $3^{3n} \leq 3^n+1$, which this is impossible.

3.8. If $K/H \cong ^2 A'_n(q')$, where $n' \geq 2$, then by [11], $k(^2A'_n(q')) = \frac{q'^{n'+1}-1}{(3,q'+1)}$. On the other hand, we have $|^2A'_n(q')| = \frac{q'^{n'(n'+1)/2}\prod_{i=1}^{n'}(q'^{i+1}-(1^{i+1})}{(n'+1,q'+1)}$. Since that $|^2A'_n(q')| \mid |G|$. So, we have $\frac{q'^{n'(n'+1)/2}\prod_{i=1}^{n'}(q'^{i+1}-(1^{i+1})}{(n'+1,q'+1)} \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. For this purpose, we consider $\frac{q'^{n'+1}-1}{(3,q'+1)} = 3^{n-1} - 1$, so $q'^{n'+1} = 3^{n-1}$. As a result $q' = 3$ and $n = n' + 2$. On ther hand $n = 2^e + 2$ thus $n' = 2^e$ which is impossible. The another case $(n', q' + 1) = n'$ is impossible, similarly.

3.9. If $K/H \cong L_{n'+1}(q')$, where $n \geq 1$, then by [11], $k(L_{n+1}(q')) = \frac{q'^{n'+1}-1}{q'-1(n'+1,q'-1)}$. Also we know that $|L_{n'+1}(q')| = \frac{q'^{n'(n'+1)/2}(q'^{n'}-1)\prod_{i=1}^{n'}(q'^{i+1}-1)}{(n'+1,q'+1)}$. Since that $|L_{n+1}(q')| \mid |G|$. So, $\frac{q'^{n'(n'+1)/2}(q'^{n'}-1)\prod_{i=1}^{n'}(q'^{i+1}-1)}{(n'+1,q'+1)} \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. For this purpose, we consider $\frac{q'^{n'+1}-1}{q'-1(n'+1,q'-1)} = 3^{n-1}-1$. First if $(q'-1,n'-1)=1$ then $\frac{q'^{n'+1}-1}{q'-1} = 3^{n-1}-1$. As a result $q'^{n'} + q'^{n'-1} + ... = 3^{n-1}-1$ where this is impossible. For example if $q'=3$, then we see that impossible. Now, if $(q'-1,n'-1)=n'$ then we have a contradiction, similarly.

3.10. If $K/H \cong D_{n'}(q')$, where $n \geq 4$. Then, by [11], $k(D_{n'}(q')) = \frac{q'^{n'-1}+1(q'+1)}{(4,q'-1)}$. On the other hand, we know that $|D_{n'}(q')| = \frac{q'^{n'(n'-1)}(q'^{n'}-1)\prod_{i=1}^{n'-1}(q'^{2i}-1)}{(4,q'^{n'}-1)}$. Since that $|D_{n'}(q')| \mid |G|$. So, $\frac{q'^{n'(n'-1)}(q'^{n'}-1)\prod_{i=1}^{n'-1}(q'^{2i}-1)}{(4,q'^{n'}-1)} \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. Hence, we consider $\frac{q'^{n'-1}+1(q'+1)}{(4,q'-1)} = 3^{n-1}-1$. Now, if $(4,q'-1)=1$, then we deduce $q'^{n'-1}+1(q'+1) = 3^{n-1}-1$. Thus $q'^{n'} + q'^{n'-1} + q' + 2 = 3^{n-1}$, this is impossible. The another case is impossible, similarly.

3.11. If $K/H \cong^2 D_{n'}(q')$, where $q' > 3$ then by [11], $k(^2D_{n'}(q')) = \frac{q'(q'+1)(q'^{2n'-2}+1)}{2}$. On the other hand, we know $|^2D_{n'}(q')| = \frac{q'^{n'(n'-1)}(q'^{n'}+1)\prod_{i=1}^{n'-1}(q'^{2i}-1)}{(4,q'^{n'}+1)}$. Now, since that $|^2D_{n'}(q')| \mid |G|$ so $\frac{q'^{n'(n'-1)}(q'^{n'}+1)\prod_{i=1}^{n'-1}(q'^{2i}-1)}{(4,q'^{n'}+1)} \mid \frac{3^{n(n-1)}(3^n+1)\prod_{i=1}^{n-1}(3^{2i}-1)}{(4,3^n+1)}$. For this purpose, we consider $\frac{q'(q'+1)(q'^{2n'-2}+1)}{2} = 3^{n-1}-1$. It follows that $3^{n-1}-1 \leq q'^{2n'}$. Since that $q'^{2n'} \mid |G|$ but $3^{n-1}-1 \nmid |G|$, which is impossible. Hence, we have the following isomorphic.

3.12. $K/H \cong^2 D_{n'}(3)$. As a result $|K/H| = |D|$. On the other hand, we know that $H \trianglelefteq K \trianglelefteq G$, and also $k(K/H) \mid k(D)$ so $3^{n-1}-1 = 3^{n'-1}-1$. As a result $n = n'$. Now, since that $|K/H| = |D|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$ and $G = K \cong D$.

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