Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **41** (4), 1-11 (2021).

Existence and approximation of solutions for nonlinear hybrid Caputo-Hadamard fractional integro-differential equations via Dhage iteration principle

Abdelouaheb Ardjouni

Received: 11.11.2019 / Revised: 26.02.2021 / Accepted: 14.05.2021

Abstract. *This article concerns the existence and approximation of solutions of the initial value problems of nonlinear hybrid Caputo-Hadamard fractional integro-differential equations. The Dhage iteration principle in a partially ordered normed linear space is used to obtain the desired results. An example is presented to illustrate the main results.*

Keywords. Approximating solutions, initial value problems, Dhage iteration principle, hybrid fixed point theorem.

Mathematics Subject Classification (2010): 34A12, 34A38.

1 Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[8], [13], [15], [16], [17], [19]–[24] and the references therein.

Fractional differential equations involving Riemann-Liouville and Caputo type fractional derivatives have been studied extensively by several researchers. However, the literature on Hadamard type fractional differential equations is not yet as enriched. The fractional derivative due to Hadamard, introduced in 1892, differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent.

Hybrid differential equations arise from a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [2, 3, 9, 11, 13, 14, 24].

A. Ardjouni
Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria.
E-mail: abd_ardjouni@yahoo.fr

Dhage and Lakshmikantham [14] discussed the existence of solutions for the following first-order hybrid differential equation

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)}{g(t,x(t))}\right) = f(t,x(t)) \ \ a.e.\ t\in[t_0,t_0+T], \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where $t_0, T \in \mathbb{R}$ with $T > 0$, $g : [t_0, t_0 + T] \times \mathbb{R} \to \mathbb{R}\setminus\{0\}$ and $f : [t_0, t_0 + T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. By using the fixed point theorem in Banach algebra, the authors obtained the existence results.

Let $\mathfrak{J} = [0, \mathfrak{a}]$ be a closed and bounded interval of the real line $\mathbb{R}$ for some $\mathfrak{a} \in \mathbb{R}$ with $\mathfrak{a} > 0$. The hybrid fractional differential equation

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{g(t,x(t))}\right) = f(t,x(t)) \ \ a.e.\ t\in\mathfrak{J}, \\ x(0) = 0, \end{cases}$$

has been investigated in [24], where D^{α} is the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $g : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}\setminus\{0\}$ and $f : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. By employing the fixed point theorem in Banach algebra, the authors obtained the existence of a solution.

Ardjouni and Djoudi [5] studied the existence and approximation of the solutions of the following nonlinear fractional integro-differential equation

$$\begin{cases} {}^{C}D^{\alpha}\left(\dfrac{x(t)}{p(t)+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g(s,x(s))ds}\right) = f(t,x(t)),\ t\in\mathfrak{J}, \\ x(0) = p(0)\theta, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $\theta \in \mathbb{R}$, $p : \mathfrak{J} \to \mathbb{R}$ and $g, f : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. By using the Dhage iteration principle, the authors obtained the existence and approximation of solutions.

Ahmad and Ntouyas in [1] discussed the existence and uniqueness of solutions to the following boundary value problem

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(\mathfrak{D}_1^{\beta}u(t) - g(t,u_t)\right) = f(t,u_t),\ t\in[1,b], \\ u(t) = \phi(t),\ t\in[1-r,1], \\ \mathfrak{D}_1^{\beta}u(1) = \eta \in \mathbb{R}, \end{cases}$$

where $\mathfrak{D}_1^{\alpha}$ and $\mathfrak{D}_1^{\beta}$ are the Caputo-Hadamard fractional derivatives, $0 < \alpha, \beta < 1$. By employing the fixed point theorems, the authors obtained existence and uniqueness results.

Let $J = [1, a]$ with $a > 1$. Inspired and motivated by the works mentioned above and some recent studies on hybrid fractional differential equations, we consider the existence and approximation of solutions for the following initial value problem (in short IVP) of the nonlinear hybrid Caputo-Hadamard fractional integro-differential equation

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(\dfrac{x(t)}{p(t)+\frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}g(s,x(s))\frac{ds}{s}}\right) = f(t,x(t)),\ t\in J, \\ x(1) = p(1)\theta, \end{cases} \tag{1.1}$$

where $\mathfrak{D}_1^\alpha$ is the Caputo-Hadamard fractional derivative of order $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $\theta \in \mathbb{R}$, $g, f : J \times \mathbb{R} \to \mathbb{R}$ are given functions and $p : J \to \mathbb{R}$ is a given function.

By a solution of the IVP (1.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the corresponding integral equation of (1.1), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The purpose of this paper is to use Dhage iteration principle to show the existence and approximation of solutions of (1.1) under weaker partially continuity and partially compactness type conditions.

The article is organized as follows. In Section 2 we give some preliminaries and key fixed point theorem that will be used in later sections. In Section 3 we prove some sufficient conditions of the existence and approximation of solutions of (1.1) by using Dhage iteration principle. For details on Dhage iteration principle we refer the reader to [10]. Finally, an example is given to illustrate our main results.

2 Preliminaries

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [19,23].

Definition 2.1 ([19]) *The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{I}_1^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} x(s) \frac{ds}{s},\ \alpha > 0.$$

Definition 2.2 ([19]) *The Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{D}_1^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n(x)(s) \frac{ds}{s},\ n - 1 < \alpha < n,$$

where $\delta^n = \left(t\frac{d}{dt}\right)^n$, $n \in \mathbb{N}$.

Lemma 2.1 ([19]) *Let $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$ and $x \in C^n([1, T])$. Then*

$$\left(\mathfrak{I}_1^\alpha \mathfrak{D}_1^\alpha x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(1)}{\Gamma(k+1)} (\log t)^k.$$

Lemma 2.2 ([19]) *For all $\mu > 0$ and $\nu > -1$,*

$$\frac{1}{\Gamma(\mu)} \int_1^t \left(\log \frac{t}{s}\right)^{\mu - 1} (\log s)^\nu \frac{ds}{s} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} (\log t)^{\mu+\nu}.$$

Let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is called regular if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [18] and the references therein.

Definition 2.3 *A mapping $\mathcal{A} : E \to E$ is called isotone or monotone nondecreasing if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{A}x \preceq \mathcal{A}y$ for all $x, y \in E$. Similarly, $\mathcal{A}$ is called monotone nonincreasing if $x \preceq y$ implies $\mathcal{A}x \succeq \mathcal{A}y$ for all $x, y \in E$. Finally, $\mathcal{A}$ is called monotonic or simply monotone if it is either monotone nondecreasing or monotone nonincreasing on E.*

Definition 2.4 *An operator $\mathcal{A}$ on a normed linear space E into itself is called compact if $\mathcal{A}(E)$ is a relatively compact subset of E. $\mathcal{A}$ is called totally bounded if for any bounded subset S of E, $\mathcal{A}(S)$ is a relatively compact subset of E. If $\mathcal{A}$ is continuous and totally bounded, then it is called completely continuous on E.*

Definition 2.5 (Dhage [10]) *A mapping $\mathcal{A}: E \to E$ is called partially continuous at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{A}x - \mathcal{A}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{A}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{A}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.6 (Dhage [9,10]) *An operator $\mathcal{A}$ on a partially normed linear space E into itself is called partially bounded if $\mathcal{A}(C)$ is bounded for every chain C in E. $\mathcal{A}$ is called uniformly partially bounded if all chains $\mathcal{A}(C)$ in E are bounded by a unique constant. $\mathcal{A}$ is called partially compact if $\mathcal{A}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{A}$ is called partially totally bounded if for any totally ordered and bounded subset C of E, $\mathcal{A}(C)$ is a relatively compact subset of E. If $\mathcal{A}$ is partially continuous and partially totally bounded, then it is called partially completely continuous on E.*

Definition 2.7 (Dhage [9]) *The order relation $\preceq$ and the metric d on a non-empty set E are said to be compatible if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function has this property.

Theorem 2.1 (Dhage [10]) *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain of E. Let $\mathcal{A}: E \to E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0$ or $x_0 \succeq \mathcal{A}x_0$, then the operator equation $\mathcal{A}x = x$ has a solution x^* in E and the sequence $\{\mathcal{A}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Remark 2.1 ([13]) The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E is held if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$.

Remark 2.2 ([9]) Note that every compact mapping in a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is continuous and partially totally bounded, but the converse may not be true. Then, the hypothesis concerning the partially continuous and partially compact operator in Theorem 2.1 may be replaced by the continuous and compact operator.

3 Main results

The equivalent integral formulation of the IVP (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)|, \tag{3.1}$$

$$x \leq y \iff x(t) \leq y(t), \tag{3.2}$$

for all $t \in J$. Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm, regular and also partially ordered with respect to the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ has some nice properties with respect to the above order relation in it.

Lemma 3.1 ([12]) *Let $(C(J,\mathbb{R}), \leq, \|\cdot\|)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(J,\mathbb{R})$.*

We need the following definition in what follows.

Definition 3.1 *A function $u \in C(J,R)$ is said to be a lower solution of the IVP (1.1) if it satisfies the corresponding integral inequality of*

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(\dfrac{u(t)}{p(t) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} g(s,u(s))\frac{ds}{s}}\right) \leq f(t,u(t)),\ t\in J, \\ u(1) \leq p(1)\,\theta. \end{cases} \tag{3.3}$$

Similarly, an upper solution $v \in C(J,R)$ for the IVP (1.1) is defined on J, by reversing the above inequalities.

We consider the following set of assumptions

(B1) $g, f : J\times\mathbb{R} \to \mathbb{R}$ are continuous functions and $p : J \to \mathbb{R}$ is a continuous function such that

$$\theta + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s} \geq 0,$$

and

$$p(t) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} g(s,x(s))ds > 0,$$

for all $t \in J$ and $x \in C(J,\mathbb{R})$.

(B2) There exist constants $K_g, K_f > 0$ such that

$$|g(t,x)| \leq K_g \text{ and } |f(t,x)| \leq K_f \text{ for all } t\in J \text{ and } x\in\mathbb{R}.$$

(B3) There exists constant $K_p > 0$ such that

$$|p(t_2) - p(t_1)| \leq K_p |t_2 - t_1| \text{ for all } t_1, t_2 \in J.$$

(B4) $g(t,x)$ and $f(t,x)$ are monotone nondecreasing functions in x for all $t \in J$.

(B5) The IVP (1.1) has a lower solution $u \in C(J,\mathbb{R})$.

Lemma 3.2 *Let $h \in C(J,\mathbb{R})$ and $q \in C(J,(0,\infty))$. If $\dfrac{x}{q} \in C^1(J,\mathbb{R})$, then the IVP*

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(\dfrac{x(t)}{q(t)}\right) = h(t),\ t \in J, \\ x(1) = q(1)\,\theta, \end{cases} \tag{3.4}$$

is equivalent to the integral equation

$$x(t) = q(t)\left(\theta + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} h(s)\frac{ds}{s}\right),\ t \in J. \tag{3.5}$$

Theorem 3.1 *Assume that hypotheses $(B1)-(B5)$ hold. Then the IVP (1.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$\begin{aligned} x_{n+1}(t) = &\left(p(t) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} g(s,x_n(s))\frac{ds}{s}\right) \\ &\times\left(\theta + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x_n(s))\frac{ds}{s}\right), \end{aligned} \tag{3.6}$$

for all $t \in J$, where $x_0 = u$ converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then by Lemma 3.1, every compact chain in E is compatible with respect to the norm $\|\cdot\|$ and order relation $\leq$. Define the operator $\mathcal{A}$ on E by

$$\begin{aligned} (\mathcal{A}x)(t) = &\left(p(t) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} g(s,x(s))\frac{ds}{s}\right) \\ &\times\left(\theta + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\right),\ t \in J. \end{aligned} \tag{3.7}$$

From the continuity of the integral, it follows that $\mathcal{A}$ defines the map $\mathcal{A}: E \to E$. Now, by Lemma 3.2, the IVP (1.1) is equivalent to the operator equation

$$(\mathcal{A}x)(t) = x(t),\ t \in J. \tag{3.8}$$

We shall show that the operators $\mathcal{A}$ satisfies all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: $\mathcal{A}$ is nondecreasing operator on E. Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (B4), we obtain

$$\begin{aligned} (\mathcal{A}x)(t) = &\left(p(t) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} g(s,x(s))\frac{ds}{s}\right) \\ &\times\left(\theta + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\right) \\ \leq &\left(p(t) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} g(s,y(s))\frac{ds}{s}\right) \\ &\times\left(\theta + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,y(s))\frac{ds}{s}\right) \\ = &(\mathcal{A}y)(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E.

Step II: $\mathcal{A}$ is a partially continuous operator on E. Let $\{x_n\}$ be a sequence in a chain C in E such that $x_n \to x$ when $n \to \infty$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} (\mathcal{A}x_n)(t) &= \lim_{n\to\infty} \left[\left(p(t) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta-1} g(s, x_n(s)) \frac{ds}{s} \right) \right. \\
&\quad \left. \times \left(\theta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, x_n(s)) \frac{ds}{s} \right) \right] \\
&= \left(p(t) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta-1} \left[\lim_{n\to\infty} g(s, x_n(s)) \right] \frac{ds}{s} \right) \\
&\quad \times \left(\theta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \left[\lim_{n\to\infty} f(s, x_n(s)) \right] \frac{ds}{s} \right) \\
&= \left(p(t) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta-1} g(s, x(s)) \frac{ds}{s} \right) \\
&\quad \times \left(\theta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \right) \\
&= (\mathcal{A}x)(t),
\end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{A}x_n\}$ converges to $\mathcal{A}x$ pointwise on J.

Next, we show that $\{\mathcal{A}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
&|(\mathcal{A}x_n)(t_2) - (\mathcal{A}x_n)(t_1)| \\
&\leq \left(|p(t_1)| + \frac{1}{\Gamma(\beta)} \int_1^{t_1} \left(\log \frac{t_1}{s} \right)^{\beta-1} |g(s, x_n(s))| \frac{ds}{s} \right) \\
&\times \left(\frac{1}{\Gamma(\alpha)} \left| \int_1^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} f(s, x_n(s)) \frac{ds}{s} - \int_1^{t_1} \left(\log \frac{t_1}{s} \right)^{\alpha-1} f(s, x_n(s)) \frac{ds}{s} \right| \right) \\
&+ \left(|p(t_2) - p(t_1)| + \frac{1}{\Gamma(\beta)} \left| \int_1^{t_2} \left(\log \frac{t_2}{s} \right)^{\beta-1} g(s, x_n(s)) \frac{ds}{s} \right. \right. \\
&\left. \left. - \int_1^{t_1} \left(\log \frac{t_1}{s} \right)^{\beta-1} g(s, x_n(s)) \frac{ds}{s} \right| \right) \\
&\times \left(|\theta| + \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} |f(s, x_n(s))| \frac{ds}{s} \right).
\end{aligned}$$

So,

$$
\begin{aligned}
&\left|\left(\mathcal{A}x_n\right)(t_2)-\left(\mathcal{A}x_n\right)(t_1)\right| \\
&\leq\left(\left|p\left(t_1\right)\right|+\frac{K_g\left(\log a\right)^{\beta}}{\Gamma\left(\beta+1\right)}\right)\left(\frac{1}{\Gamma\left(\alpha\right)}\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\alpha-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right)\right.\\
&\left.\times\left|f\left(s,x_n(s)\right)\right|\frac{ds}{s}+\frac{1}{\Gamma\left(\alpha\right)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\left|f\left(s,x_n(s)\right)\right|\frac{ds}{s}\right)\\
&+\left(K_p\left|t_2-t_1\right|+\frac{1}{\Gamma\left(\beta\right)}\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\beta-1}-\left(\log\frac{t_2}{s}\right)^{\beta-1}\right)\left|g\left(s,x_n(s)\right)\right|\frac{ds}{s}\right.\\
&\left.+\frac{1}{\Gamma\left(\beta\right)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\beta-1}\left|f\left(s,x_n(s)\right)\right|\frac{ds}{s}\right)\left(\left|\theta\right|+\frac{K_f\left(\log a\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)\\
&\leq\left(\left|p\left(t_1\right)\right|+\frac{K_g\left(\log a\right)^{\beta}}{\Gamma\left(\beta+1\right)}\right)\frac{2K_f}{\Gamma\left(\alpha+1\right)}\left(\log\frac{t_2}{t_1}\right)^{\alpha}\\
&+\left(K_p\left|t_2-t_1\right|+\frac{2K_g}{\Gamma\left(\beta+1\right)}\left(\log\frac{t_2}{t_1}\right)^{\beta}\right)\left(\left|\theta\right|+\frac{K_f\left(\log a\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)\\
&\to 0 \text{ as } t_2-t_1\to 0,
\end{aligned}
$$

uniformly for all $n\in\mathbb{N}$. This shows that the convergence $\mathcal{A}x_n\to\mathcal{A}x$ is uniformly and hence $\mathcal{A}$ is partially continuous on E.

Step III: $\mathcal{A}$ is a partially compact operator on E. Let C be an arbitrary chain in E. We show that $\mathcal{A}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{A}(C)$ is uniformly bounded. Let $x\in C$ be arbitrary. Then

$$
\begin{aligned}
\left|\left(\mathcal{A}x\right)(t)\right| &\leq\left(\left|p\left(t\right)\right|+\frac{1}{\Gamma\left(\beta\right)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\beta-1}\left|g(s,x(s))\right|\frac{ds}{s}\right)\\
&\times\left(\left|\theta\right|+\frac{1}{\Gamma\left(\alpha\right)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left|f\left(s,x(s)\right)\right|\frac{ds}{s}\right)\\
&\leq\left(K_p\left(t-1\right)+\left|p\left(1\right)\right|+\frac{K_g}{\Gamma\left(\beta\right)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\beta-1}\frac{ds}{s}\right)\\
&\times\left(\left|\theta\right|+\frac{K_f}{\Gamma\left(\alpha\right)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\\
&\leq\left(K_pa+\left|p\left(1\right)\right|+\frac{K_g\left(\log a\right)^{\beta}}{\Gamma\left(\beta+1\right)}\right)\left(\left|\theta\right|+\frac{K_f\left(\log a\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)=r,
\end{aligned}
$$

for all $t\in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\|\leq r$ for all $x\in C$. Hence $\mathcal{A}$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{A}(C)$ is an equicontinuous set in

E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned} &\left|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\right| \\ &\leq \left(|p(t_1)| + \frac{K_g (\log a)^{\beta}}{\Gamma(\beta+1)}\right) \frac{2K_f}{\Gamma(\alpha+1)} \left(\log \frac{t_2}{t_1}\right)^{\alpha} \\ &+ \left(K_p |t_2 - t_1| + \frac{2K_g}{\Gamma(\beta+1)} \left(\log \frac{t_2}{t_1}\right)^{\beta}\right) \left(|\theta| + \frac{K_f (\log a)^{\alpha}}{\Gamma(\alpha+1)}\right) \\ &\to 0 \text{ as } t_2 - t_1 \to 0, \end{aligned}$$

uniformly for all $x \in C$. Hence $\mathcal{A}(C)$ is a compact subset of E and consequently $\mathcal{A}$ is a partially compact operator on E into itself.

Step IV: u satisfies the operator inequality $u \leq \mathcal{A}u$. By hypothesis $(B5)$, the IVP (1.1) has a lower solution u on J. Then we have

$$\mathfrak{D}_1^{\alpha} \left(\frac{u(t)}{p(t) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} g(s, u(s)) \frac{ds}{s}} \right) \leq f(t, u(t)),\ t \in J, \tag{3.9}$$

satisfying

$$u(1) \leq p(1)\,\theta.$$

Applying $\mathfrak{I}_1^{\alpha}$ to both sides of (3.9) and by using Lemma 2.1, we obtain

$$\begin{aligned} u(t) \leq &\left(p(t) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} g(s, u(s)) \frac{ds}{s}\right) \\ &\times \left(\theta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, u(s)) \frac{ds}{s}\right),\ t \in J. \end{aligned}$$

The definition of the operator $\mathcal{A}$ implies that $u(t) \leq (\mathcal{A}u)(t)$ for all $t \in J$. Consequently, u is a lower solution to the operator equation $x = \mathcal{A}x$.

Thus $\mathcal{A}$ satisfies all the conditions of Theorem 2.1 with $x_0 = u$ and we apply it to conclude that the operator equation $\mathcal{A}x = x$ has a solution. Consequently the integral equation and the IVP (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.6) converges monotonically to x^*. This completes the proof.

Remark 3.1 The conclusion of Theorem 3.1 also remains true if we replace the hypothesis $(B5)$ with the following one

(B5') The IVP (1.1) has an upper solution $v \in C(J, R)$.

Example 1 Given a closed and bounded interval $J = [1, e]$ in $\mathbb{R}$, consider the IVP,

$$\begin{cases} \mathfrak{D}_1^{1/3} \left(\dfrac{x(t)}{\pi + \cos t + \frac{1}{\Gamma(1/5)} \int_1^t \left(\log \frac{t}{s}\right)^{-4/5} \arctan x(s) \frac{ds}{s}} \right) = \tanh x(t),\ t \in J, \\ x(1) = \pi + \cos 1, \end{cases} \tag{3.10}$$

where $\alpha = 1/3$, $\beta = 1/5$, $\theta = 1$, $g(t,x) = \arctan x$, $f(t,x) = \tanh x$ and $p(t) = \pi + \cos t$. Clearly, the functions g and f are continuous on $J \times \mathbb{R}$, p is continuous on J and

$$\pi + \cos t + \frac{1}{\Gamma(1/5)} \int_1^t \left(\log \frac{t}{s}\right)^{-4/5} \arctan x(s) \frac{ds}{s} > 0, \ \forall t \in J.$$

The functions g and f satisfy the hypothesis $(B2)$ with $K_g = \pi/2$ and $K_f = 1$. The function p satisfies the hypothesis $(B3)$ with $K_p = 1$. Moreover, the functions g and f are nondecreasing in x for each $t \in J$ and so the hypothesis $(B4)$ is satisfied. Finally the IVP (3.10) has a lower solution

$$u(t) = \left(\pi + \cos t - \frac{\pi (\log t)^{1/5}}{2\Gamma(6/5)}\right)\left(1 - \frac{(\log t)^{1/3}}{\Gamma(4/3)}\right),$$

defined on J. Thus all hypotheses of Theorem 3.1 are satisfied. Hence we apply Theorem 3.1 and conclude that the IVP (3.10) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_{n+1}(t) &= \left(\pi + \cos t + \frac{1}{\Gamma(1/5)} \int_1^t \left(\log \frac{t}{s}\right)^{-4/5} \arctan x(s) \frac{ds}{s}\right) \\ &\times \left(1 + \frac{1}{\Gamma(1/3)} \int_1^t \left(\log \frac{t}{s}\right)^{-2/3} \tanh x_n(s) \frac{ds}{s}\right), \end{aligned} \tag{3.11}$$

for all $t \in J$, where $x_0 = u$, converges monotonically to x^*.

Remark 3.2 In view of Remark 3.1, the existence of the solutions x^* of the IVP (3.10) may be obtained under the upper solution

$$v(t) = \left(\pi + \cos t + \frac{\pi (\log t)^{1/5}}{2\Gamma(6/5)}\right)\left(1 + \frac{(\log t)^{1/3}}{\Gamma(4/3)}\right),$$

defined on J and the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_{n+1}(t) &= \left(\pi + \cos t + \frac{1}{\Gamma(1/5)} \int_1^t \left(\log \frac{t}{s}\right)^{-4/5} \arctan x(s) \frac{ds}{s}\right) \\ &\times \left(1 + \frac{1}{\Gamma(1/3)} \int_1^t \left(\log \frac{t}{s}\right)^{-2/3} \tanh x_n(s) \frac{ds}{s}\right), \end{aligned} \tag{3.12}$$

for all $t \in J$, where $x_0 = v$, converges monotonically to x^*.

References

1. Ahmad, B., Ntouyas, S. K.: *Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations*, Electron. J. Differential Equations, **2017** (36), 1–11 (2017).
2. Ahmad, B., Ntouyas, S. K.: *Nonlocal boundary value problems for hybrid fractional differential equations and inclusions of Hadamard type*, Frac. Differ. Calc. **5** (2), 107–123 (2015).
3. Ahmad, B., Ntouyas, S. K.: *Initial-value problems for hybrid Hadamard fractional differential equations*, Electron. J. Differential Equations, **2014** (161), 1–8 (2014).

4. Ardjouni, A.: *Positive solutions for nonlinear Hadamard fractional differential equations with integral boundary conditions*, AIMS Math. **4** (4), 1101–1113 (2019).
5. Ardjouni, A., Djoudi, A.: *Approximating solutions of nonlinear hybrid Caputo fractional integro-differential equations via Dhage iteration principle*, Ural Math. J. **5** (1), 3–12 (2019).
6. Ardjouni, A., Djoudi, A.: *Positive solutions for first-order nonlinear Caputo-Hadamard fractional relaxation differential equations*, Kragujevac J. Math. **45** (6), 897–908 (2021).
7. Boulares, H., Ardjouni, A., Laskri, Y.: *Positive solutions for nonlinear fractional differential equations*, Positivity **21**, 1201–1212 (2017).
8. Boulares, H., Ardjouni, A., Laskri, Y.: *Stability in delay nonlinear fractional differential equations*, Rend. Circ. Mat. Palermo **65**, 243–253 (2016).
9. Dhage, B. C.: *Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations*, Differ. Equ. Appl. **5**, 155–184 (2013).
10. Dhage, B. C.: *Partially condensing mappings in ordered normed linear spaces and applications to functional integral equations*, Tamkang J. Math. **45** (4), 397–426 (2014).
11. Dhage, B. C., Dhage, S. B., Ntouyas, S. K.: *Approximating solutions of nonlinear hybrid differential equations*, Appl. Math. Lett. **34**, 76–80 (2014).
12. Dhage, B. C., Dhage, S. B., Ntouyas, S. K.: *Approximating solutions of nonlinear second order ordinary differential equations via Dhage iteration principle*, Malaya J. Mat. **4** (1), 8–18 (2016).
13. Dhage, B. C., Khurpe, G. T., Shete, A. Y., Salunke, J. N.: *Existence and approximate solutions for nonlinear hybrid fractional integro-differential equations*, Int. J. Anal. Appl. **11** (2), 157–167 (2016).
14. Dhage, B. C., Lakshmikantham, V.: *Basic results on hybrid differential equations*, Nonlinear Anal. Hybrid Syst. **4**, 414–424 (2010).
15. Ge, F., Kou, C.: *Stability analysis by Krasnoselskii's fixed point theorem for nonlinear fractional differential equations*, Appl. Math. Comput. **257**, 308–316 (2015).
16. Ge, F., Kou, C.: *Asymptotic stability of solutions of nonlinear fractional differential equations of order* $1 < \alpha < 2$, J. Shanghai Normal Univ. **44** (3), 284–290 (2015).
17. Gomoyunov, M. I.: *Fractional derivatives of convex Lyapunov functions and control problems in fractional order systems*, Fract. Calc. Appl. Anal. **21** (5), 1238–1261 (2018).
18. Heikkilä, S., Lakshmikantham, V.: Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, *Marcel Dekker inc., New York*, (1994).
19. Kilbas, A. A., Srivastava, H. M., Trujillo, J. J.: Theory and Applications of Fractional Differential Equations, *Elsevier Science B. V., Amsterdam*, (2006).
20. Kou, C., Zhou, H., Yan, Y.: *Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis*, Nonlinear Anal. **74**, 5975–5986 (2011).
21. Lakshmikantham, V., Vatsala, A. S.: *Basic theory of fractional differential equations*, Nonlinear Anal. **69**, 2677–2682 (2008).
22. Li, N., Wang, C.: *New existence results of positive solution for a class of nonlinear fractional differential equations*, Acta Math. Sci. **33**, 847–854 (2013).
23. Podlubny, I.: Fractional Differential Equations, *Academic Press, San Diego*, (1999).
24. Zhao, Y., Sun, S., Han, Z., Li, Q.: *Theory of fractional hybrid differential equations*, Comput. Math. Appl. **62**, 1312–1324 (2011).