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Further results on the outer connected monophonic number of a graph

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Abstract. For a connected graph G, a *connected outer connected monophonic set* of a graph G is an outer connected monophonic set S such that the subgraph induced by S is connected. The minimum cardinality of a connected outer connected monophonic set of G is the *connected outer connected monophonic number* of G and is denoted by $cm_{co}(G)$. We determine bounds for it and characterize graphs which realize these bounds. Some realization results on the connected outer connected monophonic number of a graph are studied.

Keywords. monophonic set · outer connected monophonic set · connected outer connected monophonic set · connected outer connected monophonic number.

1 Introduction

By a graph $G = (V, E)$ we mean a finite simple undirected connected graph. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [1,7]. The *distance* $d(x, y)$ between two vertices x and y in a connected graph G is the length of a shortest $x - y$ path in G. A $x - y$ path of length $d(x, y)$ is called $x - y$ *geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v of G is called an *end-vertex* of G if its degree is 1. A vertex v of a connected graph G is called a *support vertex* of G if it is adjacent to an end-vertex of G. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on a $x - y$ monophonic path for some x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number*

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of G and is denoted by $m(G)$. The monophonic number of a graph, algorithmic aspects of monophonic concepts were studied by several authors in [2–4, 8, 11]. A set S of vertices in a graph G is said to be an *outer connected monophonic set* if S is a monophonic set of G and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected monophonic set of G is the *outer connected monophonic number* of G and is denoted by $m_{oc}(G)$. The outer connected monophonic number of a graph was introduced and further studied in [5, 6].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max \{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced in [9] and further studied in [10].

The following theorems will be used in the sequel.

Theorem 1 [11] Each extreme vertex of a connected graph G belongs to every monophonic set of G.

Theorem 2 [11] For the complete graph $K_p (p \geq 2)$, $m(K_p) = p$.

Theorem 3 [11] If T is a tree with k end-vertices, then $m(T) = k$.

Theorem 4 [5] Each extreme vertex of a connected graph G belongs to every outer connected monophonic set of G.

Theorem 5 [5] For the complete graph $K_p (p \geq 2)$, $m_{oc}(K_p) = p$.

Theorem 6 [5] Let G be a connected graph with a cut-vertex v and let S be an outer connected monophonic set of G. Then every component of $G - v$ contains an element of S.

Theorem 7 [5] If T is a tree with k end-vertices, then $m_{oc}(T) = k$.

Throughout this paper G denotes a connected graph with at least two vertices.

2 Main Results

Definition 1 A *connected outer connected monophonic set* of a graph G is an outer connected monophonic set S such that the subgraph induced by S is connected. The minimum cardinality of a connected outer connected monophonic set of G is the *connected outer connected monophonic number* of G and is denoted by $cm_{co}(G)$.

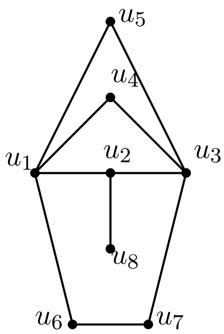

Figure 2.1: G

Example 1 For the graph G given in Figure 2.1, it is clear that no 2-element subset of $V(G)$ is a monophonic set of G. It is easily observed that $S = \{u_1, u_3, u_8\}$ is a minimum monophonic set of G and so $m(G) = 3$. Since the subgraph induced by $V - S$ is not connected, S is not an outer connected monophonic set. It is clear that no 2-element or 3-element subset of $V(G)$ is an outer connected monophonic set of G. It is easily verified that the set $S_1 = \{u_4, u_5, u_6, u_8\}$ is a minimum outer connected monophonic set of G and so $m_{oc}(G) = 4$. Since the subgraph induced by S_1 is not connected, S_1 is not a connected outer connected monophonic set of G. Also, it is easy to verify that no 2-element, 3-element or 4-element subset of $V(G)$ is a connected outer connected monophonic set of G. It is clear that the set $S_2 = \{u_2, u_3, u_6, u_7, u_8\}$ is a minimum connected outer connected monophonic set of G and so $cm_{co}(G) = 5$. Thus the monophonic number, the outer connected monophonic number and the connected outer connected monophonic number of a graph are different.

The next theorem follows from the fact that every connected outer connected monophonic set is an outer connected monophonic set of G and by Theorem 4.

Theorem 8 *Each extreme vertex of a connected graph G belongs to every connected outer connected monophonic set of G.*

Observation 1 For the complete graph $G = K_p (p \geq 2), cm_{co}(G) = p$.

Theorem 9 *For any connected graph G of order $p \geq 2$, $2 \leq m(G) \leq m_{oc}(G) \leq cm_{co}(G) \leq p$.*

Proof. Any monophonic set needs at least two vertices and so $m(G) \geq 2$. Since every outer connected monophonic set of G is a monophonic set, $m(G) \leq m_{oc}(G)$. Also, since every connected outer connected monophonic set of G is an outer connected monophonic set, $m_{oc}(G) \leq cm_{co}(G)$. Note that $V(G)$ is a connected outer connected monophonic set of G and so $cm_{co}(G) \leq p$. Hence the theorem.

Remark 1 The bounds in Theorem 9 are sharp. For the complete graph $G = K_p(p \geq 2)$, $m(G) = m_{oc}(G) = cm_{co}(G) = p$. Also, all the inequalities in Theorem 9 can be strict. For the graph G given in Figure 2.1 of order 8, the monophonic number of G is 3, the outer connected monophonic number of G is 4 and the connected outer connected monophonic number of G is 5. Thus, we have $2 < m(G) < m_{oc}(G) < cm_{co}(G) < p$.

Corollary 1 *If G is any connected graph such that $cm_{co}(G) = 2$, then $m_{oc}(G) = 2$.*

The converse of Corollary 1 need not be true. For the graph G given in Figure 2.2, $m_{oc}(G) = 2$ and $cm_{co}(G) = 3$.

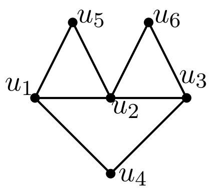

Figure 2.2: G

Corollary 2 *If G is any connected graph of order p such that $m_{oc}(G) = p$, then $cm_{co}(G) = p$.*

The converse of Corollary 2 need not be true. For a path $P_n(n \geq 4)$, $cm_{co}(P_n) = n$ and $m_{oc}(P_n) = 2$.

Theorem 10 *Let G be a connected graph with cut-vertices and let S be a connected outer connected monophonic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.*

Proof. The result follows from Theorem 6.

Theorem 11 *Each cut-vertex of a connected graph G belongs to every minimum connected outer connected monophonic set of G.*

Proof. Let S be a connected outer connected monophonic set of G and let v be a cut-vertex of G. Let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of $G - v$. By Theorem 10, S contains atleast one vertex from each $G_i (1 \leq i \leq r)$. Since the subgraph induced by S is connected, it follows that $v \in S$.

The following results are the consequences of Theorems 8 and 11.

Corollary 3 *For any connected graph G with k extreme vertices and l cut-vertices, $cm_{co}(G) \geq max\{2, k + l\}$.*

Corollary 4 *For any non-trivial tree T of order p, $cm_{co}(T) = p$.*

Now we proceed to characterize graphs G which realize the bounds in Theorem 9 are attained.

Theorem 12 *Let G be a connected graph. Then every vertex of G is either a cut-vertex or an extreme vertex if and only if $cm_{co}(G) = p$.*

Proof. Let G be a connected graph with every vertex of G is either a cut-vertex or an extreme vertex. Then the result follows from Theorems 8 and 11. Conversely, let $cm_{co}(G) = p$. Suppose that there is a vertex x in G which is neither a cut-vertex nor an extreme vertex.
Case (i): If x is not an extreme vertex then the subgraph induced by $N(x)$ is not complete. Then there exists $u, v \in N(x)$ such that $d(u, v) = 2$. Clearly x lies on a $u - v$ monophonic path in G.
Case(ii): If x is not a cut-vertex of G then $G - x$ is connected.

Hence in both cases $V - \{x\}$ is a connected outer connected monophonic set of G and so $cm_{co}(G) \leq |V - \{x\}| = p - 1$, which is a contradiction.

Theorem 13 *For any connected graph G, $cm_{co}(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then $cm_{co}(G) = 2$. Conversely, let $cm_{co}(G) = 2$. Let $S = \{u, v\}$ be a minimum connected outer connected monophonic set of G. Then uv is an edge. It is clear that a vertex different from u and v cannot lie on a $u - v$ monophonic path and so $G = K_2$.

Observation 2 The connected outer connected monophonic number of some standard graphs:

- For the cycle $C_n (n \geq 4)$, $cm_{co}(C_n) = 3$.
- For the wheel $W_n (n \geq 5)$, $cm_{co}(W_n) = 3$.
- For the star $G = K_{1,p-1}$ of p vertices $cm_{co}(G) = p$.
- For the complement of the cycle $C_n (n \geq 5)$, $cm_{co}(\bar{C_n}) = 3$.

3 Some realization results on the connected outer connected monophonic number

For any connected graph G, $rad_m(G) \leq diam_m(G)$. It is shown in [9] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the connected outer connected monophonic number can be prescribed when $rad_m(G) < diam_m(G)$.

Theorem 14 *For any three positive integers r, d and $k \geq 5$ with $r < d$, there exists a connected graph G such that $rad_m(G) = r$, $diam_m(G) = d$ and $cm_{co}(G) = k$.*

Proof. Let $r = 1$ and $d \geq 2$. Let $W = K_1 + C_{d+2}$ be a wheel of order $d + 3$ with the vertex set $V(C_{d+2}) = \{v_1, v_2 \dots v_{d+2}\}$ and $V(K_1) = z$. The graph G is obtained from the wheel W by adding $k - 4$ new vertices $u_1, u_2, \dots, u_{k-4}$ and joining each $u_i (1 \leq i \leq k - 4)$ to the vertex z of W. The graph G is shown in Figure 3.1. It is clear that $e_m(z) = 1, e_m(v_i) = d$ $(1 \leq i \leq d + 2)$ and $e_m(u_j) < d$ $(1 \leq j \leq k - 4)$. Then $rad_m(G) = r$ and $diam_m(G) = d$. Let $S = \{u_1, u_2, \dots, u_{k-4}, z\}$ be the set of all extreme vertices and cut-vertex of G.

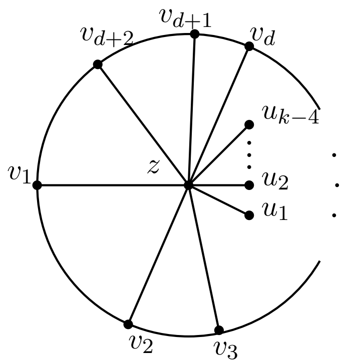

Figure 3.1: G

By Theorems 8 and 11, every connected outer connected monophonic set of G contains S. It is clear that S is not an outer connected monophonic set of G. Also for any two vertices $x, y \in V - S, S \cup \{x, y\}$ is not a connected outer connected monophonic set of G. It is easy to verify that $S \cup \{v_1, v_2, v_3\}$ is a minimum connected outer connected monophonic set of G and so $cm_{co}(G) = k$.

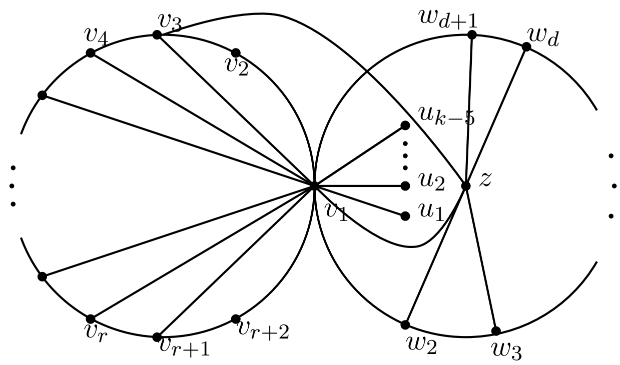

Figure 3.2: G

Now, let $r \geq 2$ and $r < d$. Let H be the graph obtained from the cycle C_{r+2} : $v_1, v_2, \cdots, v_{r+2}, v_1$ of order $r + 2$ by adding $k - 5$ new vertices $u_1, u_2, \cdots, u_{k-5}$ and joining each vertex $x \in \{u_1, u_2, \cdots, u_{k-5}, v_3, v_4, \cdots, v_{r+1}\}$ to the vertex v_1 of C_{r+2}.

The graph G is obtained from H and the wheel $W = K_1 + C_{d+1}$ of order $d+2$ with the vertex set $V(C_{d+1}) = \{w_1, w_2, \cdots, w_{d+1}\}$ and $V(K_1) = z$, by identifying the vertex v_1 of H and the vertex w_1 of W; and also joining the vertex v_3 of H to the vertex z of W. The graph G is shown in Figure 3.2. It is easily verified that $r \le e_m(u) \le d$ for any vertex u in G with $e_m(z) = r$ and $e_m(v_2) = d$. Then $rad_m(G) = r$ and $diam_m(G) = d$. Let $S = \{u_1, u_2, \cdots, u_{k-5}, v_2, v_{r+2}, v_1\}$ be the set of all extreme vertices and cut-vertex of G. By Theorems 8 and 11, every connected outer connected monophonic set of G contains S. It is clear that S is not a connected outer connected monophonic set of G. Also for any vertex $x \in V - S$, $S \cup \{x\}$ is not a connected outer connected monophonic set of G. It is easy to verify that $S_1 = S \cup \{w_2, w_3\}$ is a minimum connected outer connected monophonic set of G and so $cm_{co}(G) = k$.

We leave the following problem as an open question.

Problem 1 For any three positive integers r, d and $k \ge 5$ with $r = d$, does there exist a connected graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $cm_{co}(G) = k$?

Theorem 15 *If p, d and k are positive integers such that $2 \le d \le p-2$, $3 \le k < p$ and $p - d - k + 2 \ge 0$, then there exists a connected graph G of order p, monophonic diameter $diam_m(G) = d$ and $cm_{co}(G) = k$.*

Proof. We prove this theorem by considering two cases.
Case 1. Let $d = 2$ and $k \ge 3$. Add $p-3$ new vertices $w_1, w_2, \ldots, w_{k-3}, u_1, u_2, \ldots, u_{p-k}$ to the path $P_3 : x, y, z$ of order 3 and join each vertex $w_i (1 \le i \le k-3)$ to the vertex y of P_3; and join each vertex $u_i (1 \le i \le p-k)$ to the vertices x, y and z of P_3; and also join each vertex $u_i (1 \le i \le p-k-1)$ to the vertex $u_j (i+1 \le j \le p-k)$, thereby producing the graph G of order p is shown in Figure 3.3. It is easily verified that $1 \le e_m(u) \le 2$ for any vertex u in G with $e_m(x) = 2$ and $e_m(y) = 1$. Then the monophonic diameter of G is $diam_m(G) = 2$. For $k = 3$, let $S = \{x, z\}$ be the set of all extreme vertices of G. Clearly, S is not a connected outer connected monophonic set of G. It is easy to verify that for any vertex $u \in V - S$, $S \cup \{u\}$ is a minimum connected outer connected monophonic set of G and so $cm_{co}(G) = 3$. For $k \ge 4$, let $S = \{w_1, w_2, \ldots, w_{k-3}, x, z, y\}$ be the set of all extreme vertices and cut-vertex of G. By Theorems 8 and 11, every connected outer connected monophonic set of G contains S. It is clear that S is the unique minimum connected outer connected monophonic set of G and so $cm_{co}(G) = k$.

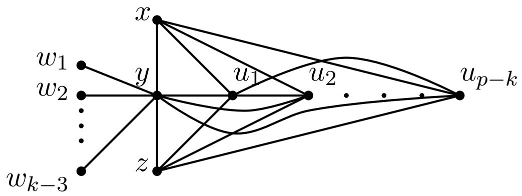

Figure 3.3: G

Case 2. $d \ge 3$ and $k \ge 3$. First, let $k = 3$. Let $C_{d+2} : v_1, v_2, \ldots, v_{d+2}, v_1$ be the cycle of order $d+2$. Add $p-d-2$ new vertices $w_1, w_2, \ldots, w_{p-d-2}$ to C_{d+2} and join each vertex $w_i (1 \le i \le p-d-2)$ to both v_1 and v_3, thereby producing the graph G of order p. It is easily verified that $3 \le e_m(u) \le d$ for any vertex u in G, $e_m(v_i) = d (1 \le i \le d+2)$ and hence the monophonic diameter of G is d. It is clear that, the set $S = \{v_3, v_4, v_5\}$ is a minimum connected outer connected monophonic set of G and so $cm_{co}(G) = 3 = k$. Now, let $k \ge 4$. The required graph G_1 is obtained from the cycle $C_{d+1} : v_1, v_2, \ldots, v_{d+1}, v_1$ of order $d+1$ by adding $p-d-1$ new vertices $u_1, u_2, \ldots, u_{k-3}, w_1, w_2, \ldots, w_{p-d-k+2}$ and joining each vertex $u_i (1 \le i \le k-3)$ to the vertex v_1 of C_{d+1}; and joining each vertex $w_j (1 \leq j \leq p-d-k+2)$ to the vertices v_2, v_3 and v_4 of C_{d+1}. The graph G_1 of order p is shown in Figure 3.4.

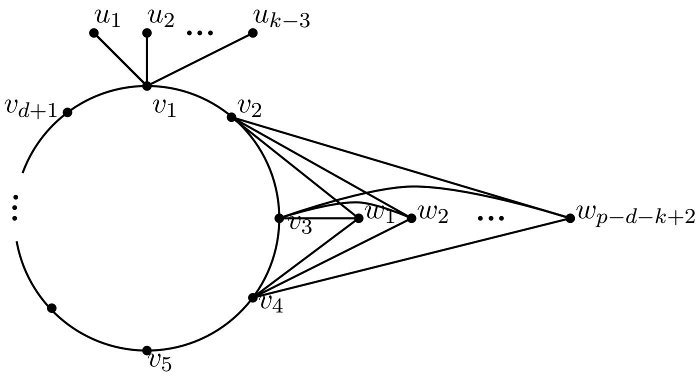

Figure 3.4: G_1

It is easily verified that $3 \leq e_m(u) \leq d$ for any vertex u in G_1 with $e_m(u_i) = d (1 \leq i \leq k-3)$ and hence the monophonic diameter of G_1 is d. Let $S = \{u_1, u_2, \ldots, u_{k-3}, v_1\}$ be the set of all extreme vertices and cut-vertex of G_1. By Theorems 8 and 11, every connected outer connected monophonic set of G_1 contains S. It is clear that S is not a connected outer connected monophonic set of G_1. Also, for any vertex $x \in V - S$, $S \cup \{x\}$ is not a connected outer connected monophonic set of G_1. It is easily seen that $S \cup \{v_2, v_{d+1}\}$ is a minimum connected outer connected monophonic set of G_1 and so $cm_{co}(G_1) = k$.

Theorem 16 *For any integer k with $3 \leq k \leq p$ there is a connected graph G of order p such that $cm_{co}(G) = k$.*

Proof. For the graph G of order p given in Figure 3.3 of Theorem 15, it is proved that $cm_{co}(G) = k$.

In view of Theorem 9, we have the following realization result.

Theorem 17 *For any three positive integers a, b and c such that $2 \leq a \leq b \leq c$, there exists a connected graph G with $m(G) = a$, $m_{oc}(G) = b$ and $cm_{co}(G) = c$.*

Proof. We prove this theorem by considering four cases.
Case 1. $2 \leq a = b = c$. Its follows from Theorems 2, 5 and Observation 1, the complete graph $G = K_a$ has the desired property.
Case 2. $2 < a = b < c$. Its follows from Theorems 3, 7 and Corollary 4, any tree of order c with a end-vertices has the desired property.

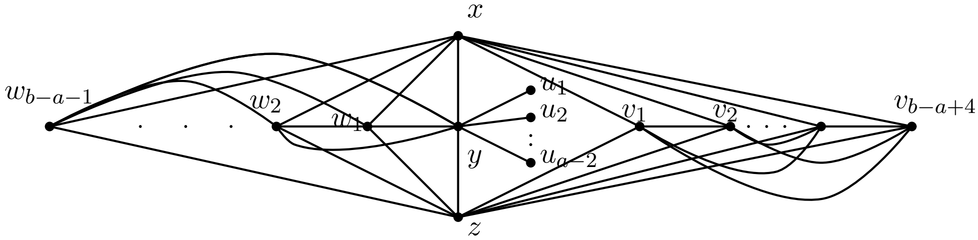

Figure 3.5 : G

Case 3. $2 < a < b = c$. Let $P_3 : x, y, z$ be a path of order 3 and $P_{b-a+4} : v_1, v_2, \ldots, v_{b-a+4}$ be a path of order $b-a+4$. Let H be the graph obtained from P_3 and P_{b-a+4} by joining each vertex v_i $(1 \leq i \leq b-a+4)$ of P_{b-a+4} to both the vertices x and z of P_3; and also joining each $v_i (1 \leq i \leq b-a+3)$ to the vertex

$v_j(i+1 \le j \le b-a+4)$. Let G be the graph obtained from H by adding $b-3$ new vertices $u_1, u_2, \ldots, u_{a-2}, w_1, w_2, \ldots, w_{b-a-1}$ and joining each $u_i(1 \le i \le a-2)$ to the vertex y of H; and joining each $w_i(1 \le i \le b-a-1)$ to the vertices x, y and z of H; and also joining each $w_i(1 \le i \le b-a-2)$ to the vertex $w_j(i+1 \le j \le b-a-1)$. The graph G is shown in Figure 3.5. Let $S = \{u_1, u_2, \ldots, u_{a-2}\}$ be the set of all extreme vertices of G. By Theorems 1, 4 and 8, every monophonic set, every outer connected monophonic set and every connected outer connected monophonic set of G contain S. It is clear that S is not a monophonic set of G. Also, for any vertex $u \in V-S, S \cup \{u\}$ is not a monophonic set of G. Clearly, $S_1 = S \cup \{x, z\}$ is a minimum monophonic set of G and so $m(G) = a$. Since the subgraph induced by $V - S_1$ is not connected, S_1 is not an outer connected monophonic set of G. The subgraph induced by $V - S_1$ is disconnected, it has two components say B_1 and B_2. Note that $V(B_1) = \{y, w_1, w_2, \ldots, w_{b-a-1}\}$ and $V(B_2) = \{v_1, v_2, \ldots, v_{b-a+4}\}$. It is easy to observe that any minimum outer connected monophonic set of G contains all the vertices of B_1. It is clear that the set $S_2 = S_1 \cup V(B_1)$ is a minimum outer connected monophonic set of G and so $m_{oc}(G) = b$. Since the subgraph induced by S_2 is connected, S_2 is a minimum connected outer connected monophonic set of G and so $cm_{co}(G) = b$.

Case 4. Subcase (i) $2 < a < b < c$ and $c = b+1$. Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Let H be the graph obtained from P_3 by adding $b-2$ new vertices $v_1, v_2, \ldots, v_{a-2}, w_1, w_2, \ldots, w_{b-a}$ and joining each $v_i(1 \le i \le a-2)$ to the vertex u_2; and joining each $w_i(1 \le i \le b-a)$ to the vertices u_1 and u_3. The graph G is obtained from the graph H and the complete graph K_2 with the vertex set $V(K_2) = \{x, y\}$ by joining the vertices x and y to all the vertices of P_3 of H. The graph G is shown in Figure 3.6. Let $S = \{v_1, v_2, \ldots, v_{a-2}\}$ be the set of all extreme vertices of G. By Theorems 1, 4 and 8, every monophonic set, every outer connected monophonic set and every connected outer connected monophonic set of G contain S. It is clear that S is not a monophonic set of G. Also, for any vertex $u \in V-S, S \cup \{u\}$ is not a monophonic set of G. Clearly, $S_1 = S \cup \{u_1, u_3\}$ is a minimum monophonic set of G and so $m(G) = a$. Since the subgraph induced by $V - S_1$ is not connected, S_1 is not an outer connected monophonic set of G. It is easy to observe that every outer connected monophonic set of G contains $\{w_1, w_2, \ldots, w_{b-a}\}$, the set $S_2 = S_1 \cup \{w_1, w_2, \ldots, w_{b-a}\}$ is a minimum outer connected monophonic set of G, and so $m_{oc}(G) = b$. Since the subgraph induced by S_2 is not connected, S_2 is not a connected outer connected monophonic set of G. Since the vertex u_2 is a cut-vertex of G, it follows from Theorems 8 and 11 that $S_3 = S_2 \cup \{u_2\}$ is a minimum connected outer connected monophonic set of G and so $cm_{co}(G) = b+1 = c$.

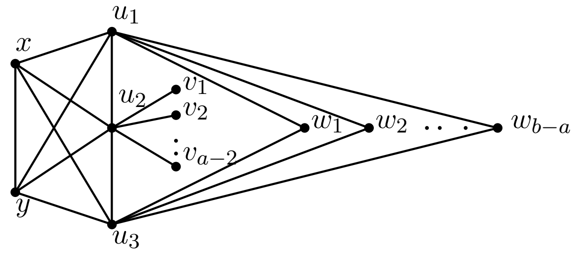

Figure 3.6 : G

Subcase ii) $2 < a < b < c$ and $c \ge b+2$. Let $P_3 : x, y, z$ be a path of order 3 and $P_{c-b-1} : v_1, v_2, \ldots, v_{c-b-1}$ be a path of order $c-b-1$. Let H be the graph obtained from P_3 and P_{c-b-1} by adding $a-2$ new vertices $u_1, u_2, \ldots, u_{a-2}$ and joining each u_i $(1 \le i \le a-2)$ to the vertex v_{c-b-1}; and also identifying the vertex y of P_3 and v_1 of P_{c-b-1}. The graph G is obtained from H by adding $b-a+2$ new vertices $w_1, w_2, \ldots, w_{b-a+2}$ and joining each $w_i(1 \le i \le b-a+2)$ to the vertices x and z.

The graph G is shown in Figure 3.7. Let $S = \{u_1, u_2, \ldots, u_{a-2}\}$ be the set of all extreme vertices of G. By Theorems 1, 4 and 8, every monophonic set, every outer connected monophonic set and every connected outer connected monophonic set of G contain S. It is clear that S is not a monophonic set of G. Also, for any vertex $u \in V - S$, $S \cup \{u\}$ is not a monophonic set of G. It is clear that $S_1 = S \cup \{x, z\}$ is a minimum monophonic set of G and so $m(G) = a$.

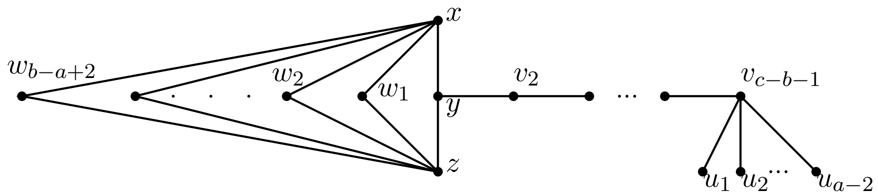

Figure 3.7: G

Since the subgraph induced by $V - S_1$ is not connected, S_1 is not an outer connected monophonic set of G. It is easy to observe that every outer connected monophonic set of G contains $\{w_1, w_2, \ldots, w_{b-a+2}\}$. The set $S_2 = S \cup \{w_1, w_2, \ldots, w_{b-a+2}\}$ is a minimum outer connected monophonic set of G and so $m_{oc}(G) = b$. Since the subgraph induced by S_2 is not connected, S_2 is not a connected outer connected monophonic set of G. Let $S' = S \cup \{v_1, v_2, \ldots, v_{c-b-1}\}$ be the set of all extreme vertices and cut-vertices of G. By Theorems 8 and 11, every connected outer connected monophonic set of G contains S'. Clearly, S' is not a connected outer connected monophonic set of G. It is easily verified that $S_3 = S' \cup \{x, w_1, w_2, \ldots, w_{b-a+2}\}$ is a minimum connected outer connected monophonic set of G and so $cm_{co}(G) = c$.

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