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On the completeness of the system of Weber functions

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Abstract. *Weber functions* $D_{\frac{\lambda_n-1}{2}}\left(\sqrt{2}x\right)$, $n=0,1,2,...$, *are considered, where* λ_n *is the eigenvalue of the perturbed harmonic oscillator on the semi-axis with finite potential and with the Dirichlet boundary condition at zero. The completeness in the space* $L_2(0,\infty)$ *of a system of functions* $\left\{D_{\frac{\lambda_n-1}{2}}\left(\sqrt{2}x\right)\right\}_{n=0}^{\infty}$ *is proved.*

Keywords. Weber function · perturbed harmonic oscillator · eigenvalues · completeness of a system of functions.

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1 Introduction and main result

In the space $L_2(0,\infty)$ we consider self-adjoint operator

$$L_0=-\frac{d^2}{dx^2}+x^2,$$

generated by the left side of the equation

$$-y''+x^2y=\lambda y,\ 0<x<\infty,\quad \lambda\in C, \tag{1.1}$$

and boundary condition

$$y(0)=0. \tag{1.2}$$

The equation (1.1) has [1] the solution $f_0(x,\lambda)$ in the form

$$f_0(x,\lambda)=D_{\frac{\lambda-1}{2}}\left(\sqrt{2}x\right),$$

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where $D_\nu(x)$ is the Weber function. It is well known (see $[1,8]$) that for each $x \in [0,\infty)$ the function $f_0(x,\lambda)$ is entire and the following asymptotic is fulfilled:

$$f_0(x,\lambda) = \left(\sqrt{2}x\right)^{\frac{\lambda-1}{2}} e^{-\frac{x^2}{2}} \left(1 + O\left(x^{-2}\right)\right), \quad x \to \infty. \tag{1.3}$$

Since $q_0(x) = x^2 \to +\infty$ for $x \to +\infty$, then the spectrum of the operator L_0 consists of simple real eigenvalues. From (1.3) it follows that for each fixed λ from the complex plane, the relation $f_0(x,\lambda) \in L_2(0,\infty)$ holds. Therefore, the spectrum of the problem (1.1) - (1.2), i.e. of the operator L_0, coincide with the zeros of the function $f_0(0,\lambda) = D_{\frac{\lambda-1}{2}}(0)$. It is shown in the work [8], that the spectrum of L_0 is purely discrete and consists of simple eigenvalues $\hat{\lambda}_n = 4n+3$, $n = 0, 1, \ldots$. The corresponding eigenfunctions $\left\{f_0\left(x,\hat{\lambda}_n\right)\right\}_{n=0}^{\infty}$ form an orthogonal basis in the space $L_2(0,\infty)$. Consequently, the following relation is true:

$$\sum_{n=0}^{\infty} \frac{f_0\left(x,\hat{\lambda}_n\right)}{\alpha_n^0} \frac{f_0\left(y,\hat{\lambda}_n\right)}{\alpha_n^0} = \delta(x-y), \tag{1.4}$$

where $\left(\alpha_n^0\right)^2 = (2n+1)!\frac{\sqrt{\pi}}{2}$, $\delta(x)$ is Dirac's delta.

We now consider the self-adjoint operator

$$L = L_0 + q(x)$$

in space $L_2(0,\infty)$, where the potential $q(x)$ is real and satisfies the condition

$$\int_0^\infty \left(1+x^3\right)|q(x)|\,dx < \infty.$$

It is well known that the equation

$$-y'' + x^2 y + q(x)\,y = \lambda y,\ 0 < x < \infty, \quad \lambda \in C \tag{1.5}$$

under the condition (1.6) has a solution $f(x,\lambda)$ such that

$$f(x,\lambda) = f_0(x,\lambda)(1+o(1)), \quad x \to \infty.$$

The spectrum of the operator L, i.e. of the problem (1.5), (1.2) coincides with the roots of the function $f(0,\lambda)$. In the paper [8] the asymptotic behavior of the eigenvalues of the operator $L$is studied. It was also proved there that the spectrum of the operator L consists of a sequence of simple real eigenvalues $\lambda_n, n \geq 0$, and the following asymptotic formula is valid

$$\lambda_n = 4n+3+O\left(n^{-\frac{1}{2}}\right), n \to \infty. \tag{1.6}$$

Of particular interest is the question of the completeness of the system of functions $\{f_0(x,\lambda_n)\}_{n=0}^{\infty}$. Studying the last question can be very useful in studying the inverse spectral problem for the operator T, since the completeness of the mentioned system of functions plays a key role in the unique solvability of the Gelfand-Levitan equation (see [2]- [5]).

The main result of this paper is the following theorem.

Theorem 1.1 *Let numbers λ_n, $\lambda_n \neq \lambda_k$, $(n \neq k)$ of the form (1.6). Then the system of functions $\{f_0(x,\lambda_n)\}_{n=0}^{\infty}$ is complete in $L_2(0,\infty)$.*

2 Proof of the theorem

Let $h(x) \in L_2(0,\infty)$ be such that

$$\int\limits_0^\infty h(x)\, f_0(x,\lambda_n)\, dx = 0,\; n \geq 0.$$

Consider a function $H(\lambda) = \int\limits_0^\infty h(x)\, f_0(x,\lambda)\, dx$, which is obviously an entire function. As shown in the work [6], the function $D_{\frac{\mu^2-1}{2}}\left(\mu\, t\sqrt{2}\right)$ satisfies the following asymptotic equality uniformly in t

$$D_{\frac{\mu^2-1}{2}}\left(\mu\, t\sqrt{2}\right) \sim C\,(2e)^{-\frac{1}{4}\mu^2}\,\mu^{\frac{1}{2}\mu^2-\frac{1}{2}}\,\frac{e^{-\mu^2\xi}}{(t^2-1)^{\frac{1}{4}}},$$

where

$$\xi = \frac{\pi i}{4} + \int\limits_0^t \sqrt{u^2-1}du$$

and the regular branch of the radical is chosen from the condition $\left.\sqrt{u^2-1}\right|_{u=2} > 0$. Setting $\mu\, t = x,\ \lambda = \mu^2$, we find that the function $f_0(x,\lambda)$ has the asymptotics uniformly for all real x:

$$f_0(x,\lambda) \sim C\,(2e)^{-\frac{\lambda}{4}}\,\frac{\left(\sqrt{\lambda}\right)^{\frac{\lambda}{2}}}{(x^2-\lambda)^{\frac{1}{4}}}e^{-\left(\frac{\pi i\lambda}{4}+\int_0^x\sqrt{y^2-\lambda}dy\right)},\ \lambda\to\infty. \tag{2.1}$$

Using (2.1), we establish (see [6, 7]) the following estimate for the function $f_0(x,\lambda)$

$$|f_0(x,\lambda)| \leq C a(\lambda)\,\rho^{-1}(x,\lambda)\exp\{-\sigma(x,\lambda)\}, \tag{2.2}$$

where

$$a(\lambda) = \left|\frac{\lambda}{2e}\right|^{\frac{Re\lambda}{4}} e^{\frac{\pi-\varphi}{4}Im\lambda},\ \lambda = |\lambda|\, e^{i\varphi},\ \varphi\in[0,2\pi), \tag{2.3}$$

$$\rho(x,\lambda) = 1 + |\lambda|^{\frac{1}{12}} + \left|x^2-\lambda\right|^{\frac{1}{4}}, \tag{2.4}$$

$$\sigma(x,\lambda) = Re\int\limits_0^x \sqrt{y^2-\lambda}dy \tag{2.5}$$

and the regular branch of the radical is chosen from the condition $\left.\sqrt{y^2-\lambda}\right|_{\lambda=y^2-1} = 1$. It should be noted that the last condition implies the inequality $Re\sqrt{y^2-\lambda} \geq 0$ for all values of λ.

Now let $|\lambda| = R$. We represent the function $H(\lambda) = \int\limits_0^\infty h(x) f_0(x,\lambda)\, dx$ in the form

$$H(\lambda) = \int\limits_0^R h(x) f_0(x,\lambda)\, dx + \int\limits_R^\infty h(x) f_0(x,\lambda)\, dx = H_1(\lambda) + H_2(\lambda).$$

Using (2.2) - (2.5) taking into account the relation

$$\sigma(x,\lambda) = \tfrac{1}{2}Re\left[x\sqrt{x^2-\lambda} - \lambda\ln\left(x^2+\sqrt{x^2-\lambda}\right) + \lambda\ln\sqrt{-\lambda}\right]$$
$$= \tfrac{x^2}{2} - \tfrac{Re\lambda}{2}\ln x + \tfrac{Re\lambda}{4} + \ln\left|\tfrac{\lambda}{4e}\right| + \tfrac{\pi-\varphi}{4}Im\lambda + O\left(x^{-2}\right),\ x\to+\infty$$

we find that

$$|H_1(\lambda)| \le C\|h\|\,a(R)\,R^{\frac{1}{2}},$$
$$|H_2(\lambda)| \le Ca(R)\int_R^\infty \left|h(x)e^{-\frac{x^2}{4}}\right|dx \le Ca(R)\|h\|\int_R^\infty e^{-\frac{x^2}{2}}dx.$$

It follows from the last estimates that $H(\lambda)$ is an entire function of order $\rho_1 = 1$ and for sufficiently large $|\lambda|$ the estimate

$$|H(\lambda)| \le C\|h\|\,a(|\lambda|)\,|\lambda|^{\frac{1}{2}}$$

is valid. Moreover, for $\lambda\to\infty$, the following relation holds (see [3], [6], [7])

$$\left|f_0^{-1}(0,\lambda)\right| \le Ca^{-1}(\lambda)\,|\lambda|^{-\frac{1}{4}}.$$

Whence it follows that for all $\lambda\to\infty$ the function $f_0^{-1}(0,\lambda)\int_0^\infty h(x)f_0(x,\lambda)\,dx$ admits the following estimate

$$\left|f_0^{-1}(0,\lambda)\int_0^\infty h(x)f_0(x,\lambda)\,dx\right| \le C\|h\|\,|\lambda|^{\frac{1}{4}}.$$

Since $f_0(0,\lambda)$ is an entire function of order $\rho = 1$, then consider the Hadamard factorization of this function:

$$f_0(0,\lambda) = C_1e^{p\lambda}\prod_{n=0}^\infty\left(1-\frac{\lambda}{\hat\lambda_n}\right)e^{\frac{\lambda}{\hat\lambda_n}},$$

where, $C_1 = f_0(0,0)$, $p = \dot f_0(0,0)$ where us agree to denote differentiation with respect to λ a dot. We also introduce the function

$$G(\lambda) = C_2e^{p\lambda}\prod_{n=0}^\infty\left(1-\frac{\lambda}{\lambda_n}\right)e^{\frac{\lambda}{\hat\lambda_n}}, \tag{2.6}$$

the set of roots of which coincides with the sequence λ_n, here $C_2 = C_1\prod_{n=0}^\infty \frac{\lambda_n}{\hat\lambda_n}$. It follows from (1.6), (2.6) that $G(\lambda)$ is an entire function of order $\rho = 1$. This implies that $G^{-1}(\lambda)\int_0^\infty h(x)f_0(x,\lambda)\,dx$ is an entire function of order $\rho = 1$.

On the other hand, inside the corner $\delta \le \arg\lambda \le 2\pi-\delta$, $\delta > 0$, the relation

$$\left|\frac{\lambda_n-\hat\lambda_n}{\lambda-\lambda_n}\right| \le \frac{Cn^{-\frac{1}{2}}}{|\lambda_n\sin\delta|} \le \frac{C}{n^{\frac{3}{2}}}$$

is satisfy. Then from the formula

$$\frac{f_0(0,\lambda)}{G(\lambda)} = \prod_{n=1}^\infty\left(1+\frac{\lambda_n-\hat\lambda_n}{\lambda-\lambda_n}\right)$$

follows that

$$\left|\frac{f_0(0,\lambda)}{G(\lambda)}\right| \le C\,.$$

Using the last relations, we obtain

$$\left|G^{-1}(\lambda)\int\limits_0^{\infty} h(x) f_0(x,\lambda)\,dx\right| \le M\,\|h\|\,R^{\frac{1}{2}}\,, \qquad (2.7)$$

where $R = |\lambda|$, $\delta \le \arg\lambda \le 2\pi - \delta$. $R = |\lambda| > R_0$, $\delta \le \arg\lambda \le 2\pi - \delta$. Now let $\delta > 0$ be such that the sector angle $-\delta \le \arg\lambda \le \delta$ is smaller π. Applying the Phragmen-Lindelof theorem [9] to the function $(1+\lambda)^{-\frac{1}{2}} G^{-1}(\lambda) \int\limits_0^{\infty} h(x) f_0(x,\lambda)\,dx$, we find that estimate (2.7) also holds in the sector $-\delta \le \arg\lambda \le \delta$. From this, using the Liouvilles theorem [9] we conclude that $G^{-1}(\lambda)\int\limits_0^{\infty} h(x) f_0(x,\lambda)\,dx \equiv 0$, i.e.

$$H(\lambda) = \int\limits_0^{\infty} h(x) f_0(x,\lambda)\,dx \equiv 0.$$

Setting $\lambda = \hat{\lambda}_n = 4n+3$ in the last equality and taking into account the expansion formula (1.4), we obtain $h(x) = 0$. This completes the proof of the theorem.

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