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The solvability degenerate elliptic-parabolic problem with nonlinear boundary conditions

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Abstract. We prove existence and uniqueness of weak solutions nonlinear elliptic-parabolic problem with nonlinear boundary conditions. These including heat transfer in a solid in contact with a moving fluid, thermoelasticity, diffusion phenomena, problems in fluid dynamics.

Keywords. nonlinear equations, elliptic-parabolic, nonlinear boundary condition, solvability.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $T > 0$. Let us denote $Q_T = \Omega \times [0, T)$. The purpose of paper is to establish the existence and uniqueness of a weak solution for a nonlinear degenerate elliptic-parabolic equations with nonlinear dynamical boundary condition

$$\begin{gathered} \frac{\partial \gamma(u)}{\partial t} - div\, a(x, Du) = f \;\; in \;\; Q_T, \\ u_t + a(x, Du)\eta = g \;\; on \;\; S_T = \partial\Omega \times (0,\, T)\,, \\ \gamma\left(u\left(0\right)\right) = u_0 \;\; in \;\; \partial\Omega, \end{gathered} \tag{1.1}$$

where $u_0 \in L_1(\Omega),\ f \in L_1\left(0,\, T;\, L_1(\Omega)\right),\ g \in L_1\left(0,\, T;\, L_1\left(\partial\Omega\right)\right)$ and η is unit outward normal on $\partial\Omega$. Hear $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Caratheodory function. For the function $\gamma : \mathbb{R} \to \mathbb{R}$ we assume that:

1. γ is increasing and Lipshitz;
2. $\gamma(s) = 0$, when $s = 0$;
3. $\gamma \in C(\mathbb{R}) \cap C^1([0, \infty))$.

The Caratheodory function $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies:

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a) there exists $\lambda > 0$ such that $a(x,\xi)\cdot\xi \geq \lambda\,|\xi|^p$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n,\ p > 1$;

b) there exist $C > 0$ such that $|a(x,\xi)| \leq C\left(1 + |\xi|^{p-1}\right)$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$;

c) $(a(x,\xi) - a(x,\eta))\cdot(\xi - \eta) > 0$ for a.e. $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^n$, $\xi \neq \eta$.

We denote $|E|$ the Lebesgue measure of a set $E \subset \mathbb{R}^n$ or its $(n-1)$-Hausdorff measure. For $1 \leq p < \infty,\ L_1(\Omega)$ and $W_p^1(\Omega)$ denote respectively the Lebesgue and Sobolev spaces, and $\overset{\circ}{W}{}_p^1(\Omega)$ is the closure $C^\infty(\Omega)$ functions which vanishing in $\partial\Omega$.

We also consider the problem

$$\frac{\partial u}{\partial t} - \psi(x,t)u_{tt} - div\, a(x, Du) = f(x,t)$$

$$u|_{\Gamma(Q_T)} = 0,$$

where

$$\Gamma\left(Q_T\right) = \left(\partial\Omega \times (0,T)\right) \cup \left(\Omega \times \{(x,t):\ \ t = 0\}\right)$$

is a parabolic boundary of the domain Q_T and $\psi(x,t) = \lambda(\rho)\,\omega(t)\,\varphi(T-t)$ is weight function. Here $\rho = \rho(x) = dist(x,\partial\Omega)$, $\lambda(\rho) \geq 0$, $\lambda(\rho) \in C^1[0, diam\,\Omega]$, $|\lambda'(\rho)| \leq c_1\sqrt{\lambda(\rho)}$, $\omega(t) \geq 0$, $\omega(t) \in C^1[0,T]$, $\varphi(z) \geq 0$, $\varphi'(z) \geq 0$, $\varphi(z) \in C^1[0,T]$, $\varphi(0) = \varphi'(0) = 0$, $\varphi(z) \geq c_2\, z\, \varphi'(z)$ and $c_1,\ c_2$ are positive constants.

The weak solution is the following. For given $f \in L_1\big(0,T,\ L_1(\Omega)\big)$, $g \in L_1\big(0,T;\ L_1(\partial\Omega)\big)$, $u_0 \in L_1(\Omega)$, a weak solution of (1.1) in Q_T such that $\gamma(u) \in C\left([0,T);\ L_1(\Omega)\right)$, $\gamma\left(u\left(0\right)\right) = u_0$ and there exists $u \in L_p\left(0,T;\ W_p^1(\Omega)\right)$ such that

$$\frac{d}{dt}\int\limits_\Omega \gamma(x)\xi dx + \frac{d}{dt}\int\limits_{\partial\Omega} u(s)\xi ds + \int\limits_\Omega a(x,Du)D\xi dx = \int\limits_\Omega f(x)\xi dx + \int\limits_{\partial\Omega} g(s)\xi ds \quad (1.2)$$

for any $\xi \in C^1\left(\overline{\Omega}\right)$.

We define $u^+ = \max\left(u,\,0\right),\ u^- = \min\left(u,\,0\right)$. For the monotone γ and u, we set

$$R^+_{\gamma,u} = \gamma_+\left|\Omega\right| + u_+\left|\partial\Omega\right|,\ R^-_{\gamma,u} = \gamma_-\left|\Omega\right| + u_-\left|\partial\Omega\right|,$$

where $v_- = \inf v,\ v_+ = \sup v$. We suppose $R^-_{\gamma,u} < R^+_{\gamma,u}$ and we write $R_{\gamma,u} = \left(R^-_{\gamma,u}, R^+_{\gamma,u}\right)$.

The nonlinear dynamical boundary conditions, although not too widely considered in the mathematical literature, are very natural in many mathematical models including heat trunsfer in a solid in contact with a moving fluid, thermoelasticity, diffusion phenomena, problems in fluid dynamics, ets. (see [5,6]). The nonlinear boundary conditions also appear in the study of the Stefan problem when the boundary material has large thermal conductivity and sufficiently small thickness. Hence, the boundary material is regarded as the boundary of the domain. For instance, this is the case if one considers an iron ball in which water and ice coexist, in the study of the Hele-Shaw problem. Notice that general nonlinear diffusion operators of Leray-Lions type, different from the Laplacian, appear when one deals whith non-Newtonian fluids (see, [2,14] and the references therein for the case of the Hele-Shaw problem with non-Newtonian fluids).

Another application in mind concerns the filtration equation with dynamical nonlinear boundary conditions, which appears in the study of rainfall infiltration through soil, when accumulation of water on the ground surfaces caused by saturation of the surface layer is taken info account.

In contrast to the Dirichlet boundary condition, for the nonhomogeneous Neumann and dynamical boundary conditions, the problem is noncoercive, and moreover, the conservation of mass exhibits a necessary condition for the existence of solution related to the ranges of the nonlinearities γ. See also papers [7]-[13]. In [1] are consider this problem for maximal monoton graphs in $\mathbb{R}^2$.

Note. Our main tool to prove the contraction principle the concept due to Ph.Benilan (see [3,4]).

2. Main results

The main results of this paper are the following contraction principe and the existence and uniqueness theorem.

Theorem 2.1 *Let $u(x,t)$ be a weak solution in Q_T of problem (1.1) and $T > 0$. For $i = 1,2$ let $f_i \in L_1(0,\, T;\, L_1(\Omega))$, $g_i \in L_1(0,\, T;\, L_1(\partial\Omega))$, $u_{i0} \in L_1(\Omega)$. Then*

$$\int\limits_{\Omega} (\gamma_1(u) - \gamma_2(u))^+ \, dx + \int\limits_{\partial\Omega} (u_1(s) - u_2(s))^+ \, ds$$

$$\leq \int\limits_{\Omega} (\gamma_1(u_1(0)) - \gamma_2(u_2(0)))^+ \, dx + \int\limits_{\partial\Omega} (u_{10} - u_{20})^+ \, ds$$

$$+ \int\limits_0^t \int\limits_{\Omega} (f_1(\tau) - f_2(\tau))^+ \, d\tau + \int\limits_0^t \int\limits_{\partial\Omega} (g_1(\tau) - g_2(\tau))^+ \, d\tau \qquad (2.1)$$

for a.e. $t \in (0,\, T)$.

Following relation for $u, v \in L_1(\Omega)$ holds: $u \leq v$ if

$$\int\limits_{\Omega} (u-k)^+ \, dx \leq \int\limits_{\Omega} (v-k)^+ \, dx \quad and \quad \int\limits_{\Omega} (u+k)^- \, dx \leq \int\limits_{\Omega} (v+k)^- \, dx$$

for any $k > 0$.

Theorem 2.2 *Assume $R^-_{\gamma,u} < R^+_{\gamma,u}$ and $T > 0$. Let $f \in L_{p'}\left(0,\, T;\, L_{p'}(\Omega)\right)$, $g \in L_{p'}\left(0,\, T;\, L_{p'}(\partial\Omega)\right)$, $u_0 \in L_{p'}(\Omega)$ be such that*

$$\gamma_- \leq u_0 \leq \gamma_+ \qquad (2.2)$$

and

$$\int\limits_{\Omega} \gamma(u_0) \, dx + \int\limits_{\gamma\Omega} u_0 ds + \int\limits_0^T \left(\int\limits_{\Omega} f dx + \int\limits_{\partial\Omega} g ds \right) dt \in R_{\gamma,u}\,,\ \forall\, t \in [0,T). \qquad (2.3)$$

Then there exists a unique weak solution of problem (1.1)

The uniqueness part of Theorem 2.2 follows from Theorem 2.1. To prove Theorem 2.1 and the existence part of Theorem 2.2 we shall use the theory of nonlinear semigroups (see [11,12]). For some questions consider [8–10]. To study problem (1.1) from the point of view of nonlinear semigroup theory is $X = L_1(\Omega) \times L_1(\partial\Omega)$ provided with the natural norm

$$\|(f,\, g)\| = \|f\|_{L_1(\Omega)} + \|g\|_{L_1(\partial\Omega)}.$$

n this space we define the operator

$$B_{\gamma,\,u} = \left\{ \left(\gamma(u),\, u\big|_{\partial\Omega}\right),\, \left(\hat{\gamma}(u),\, \hat{u}\big|_{\partial\Omega}\right) \in X \times X : \right.$$

$: \exists u \in W_p^1(\Omega)$ *such that* $\left[u,\, \gamma(u),\, u\big|_{\partial\Omega}\right]$ *is a weak solution of elliptic problem*$\Big\}$,

in other words, $\left(\hat{\gamma}(u),\, \hat{u}\big|_{\partial\Omega}\right) \in B_{\gamma,u}\left(\left(\gamma(u),\, u\big|_{\partial\Omega}\right)\right)$ if and only if there exist $u \in W_p^1(\Omega)$ such that

$$\int\limits_{\Omega} a(x, Du)\cdot D\varphi = \int\limits_{\Omega} \hat{\gamma}(u)\varphi dx + \int\limits_{\partial\Omega} \hat{u}\,|_{\partial\Omega}\,\varphi ds, \tag{2.4}$$

for all $\varphi \in L_\infty(\Omega) \cap W_p^1(\Omega)$,which allows us to rewrite problem (1.1) as the following abstract Caushy problem in X:

$$\begin{cases} V'(t) + B_{\gamma,u}\left(V(t)\right) = (f,\, g)\,, \quad t \in (0,\, T)\,; \\ \qquad V\left(0\right) = \left(\gamma\left(u\left(0\right)\right),\, u\left(0\right)|_{\partial\Omega}\right). \end{cases} \tag{2.5}$$

The operator $B_{\gamma,u}$ is T-accretive in X and on these condition problem (2.5) there exists a unique solution.

The existence part of theorem 2.2 is shown by prowing that the solution of problem (2.5) is a weak solution of problem (1.1) whenever the assumptions of theorem 2.2 are fulfiled. Before giving the proof we need to prove some technical Lemmas.

The following lemma in the proof of the existence part and in the proof of the Contraction principle is using. Denote by $(\cdot\,,\,\cdot)$ the pairing between $\left(W_p^1(\Omega)\right)'$ and $W_p^1(\Omega)$.

Lemma 2.3 *Let* $(\gamma(u),\, u\,|_{\partial\Omega}) \in C\left([0,T];\, L_1(\Omega)\times L_1\left(\partial\Omega\right)\right)$ *and* $F \in L_{p'}(0,T;(W_p^1(\Omega)'))$ *such that*

$$\int\limits_0^T\int\limits_{\Omega} \gamma\big(u(x,t)\big)\psi_t dxdt + \int\limits_0^T\int\limits_{\partial\Omega} u\big|_{\partial\Omega}\cdot\psi_t dxdt = \int\limits_0^T \left(F(t), \psi(t)\right)dt, \tag{2.6}$$

for any $\psi \in W_1^1\left(0,\, T;\, W_1^1(\Omega)\cap L_\infty(\Omega)\right) \cap L_p\left(0,\, T;\, W_p^1(\Omega)\right)$ *with* $\psi\left(0\right) = \psi(t) = 0.$ *Then*

$$\int\limits_0^T\int\limits_{\Omega}\left(\int\limits_0^{\gamma(u(x,t))} H((x,t),(\gamma^{-1})(s))ds\right)\psi_t dxdt + \int\limits_0^T\int\limits_{\partial\Omega}\left(\int\limits_0^{u|_{\partial\Omega}} H((x,t),(u^{-1})(s))ds\right)\psi_t dsdt$$

$$= \int\limits_0^T \left(F(t),\, H\left((x,t),u(t)\right)\right)\psi(t)dt,$$

for any $u \in L_p\left(0,\, T;\, W_p^1(\Omega)\right)$ *with* $\gamma(u)$ *in* Q_T*,* u *in* S_T*, for any* ψ*, and for Caratheodory function* $H\,:\,\Omega\times[0,T) \to \mathbb{R}$ *such that* $H\left(x,r\right)$ *is nondecreasing in* r*,* $H\left(\cdot,u\right) \in L_p\left(0,\, T;\, W_p^1(\Omega)\right)$*,*

$$\int\limits_0^{\gamma(u)} H\left((x,t),\left(\gamma^{-1}(u)(s)\right)\right)ds \in L_1\left(Q_T\right) \quad \text{and} \quad \int\limits_0^{u|_{\partial\Omega}} H\left((x,t),\left(u^{-1}\right)(s)\right)ds \in L_1\left(S_T\right).$$

Proof: Before this result is proved for dirichlet boundary condition. Our proof is similar with correspondingly changes.

Let $\psi \geq 0$ and for $H_\tau = T_{1/\tau}H,\ \tau > 0$, let

$$\eta_\tau(t) = \frac{1}{\tau}\int\limits_t^{t+\tau} H_\tau\left((x,t),u(s)\right)\psi(s)ds.$$

Then η_τ can be used as a test function in (2.6), after calculations and letting $\tau \to 0+$ we get

$$\int_0^T \left(F(t),\, H\left((x,t),\, u(t)\right)\right)\psi(t)dt \le \int_0^T\int_\Omega \left(\int_0^{\gamma(u(t))} H\left((x,t),\left(\gamma^{-1}\right)^\circ(s)\right)ds\right)\psi_t dxdt$$

$$+\int_0^T\int_{\partial\Omega} \left(\int_0^{u|_{\partial\Omega}} H\left((x,t),\left(u^{-1}\right)^\circ(s)\right)ds\right)\psi_t dsdt.$$

Takin now $\hat{\eta}_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau} H_\tau\left(x, u\left(s-\tau\right)\right)\psi(s)ds$, and arguing as above we get the required inequality.□

To prove the existence of weak solutions we shall

Lemma 2.4 *Let $\{u_n\}_{n\in\mathbb{N}} \subset W_p^1(\Omega)$, $\{\gamma\left(u_n\right)\}_{n\in\mathbb{N}} \subset L_1(\Omega)$, $\{u_n\left|_{\partial\Omega}\right.\}_{n\in\mathbb{N}} \subset L_1\left(\partial\Omega\right)$ be such that, for every $n \in \mathbb{N}$. Supppose that 1) if $R^+_{\gamma,u} = +\infty$, there exists $M > 0$ such that*

$$\int_\Omega \gamma\big(u_n\big)^+ + \int_{\partial\Omega} u_n\big|^+_{\partial\Omega} < M, \quad \forall n\in\mathbb{N},$$

2) if $R^+_{\gamma,u} < +\infty$, there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_\Omega \gamma\big(u_n\big)^+ + \int_{\partial\Omega} u_n\big|^+_{\partial\Omega} < M < R^+_{\gamma,u}, \quad \forall n\in\mathbb{N}$$

and

$$\max\left\{\int_{\left\{x\in\Omega:\gamma\left(u_n(x)\right)<-h\right\}} \big|\gamma\big(u_n\big)\big|,\ \int_{\left\{x\in\partial\Omega:u_n\big|_{\partial\Omega}<-h\right\}} \big|u_n\big|_{\partial\Omega}\big|\right\} < \frac{R^+_{\gamma,u}-M}{8}, \quad \forall n\in\mathbb{N}.$$

Then there exist a constant $C(M)$ in case 1), and $C(M,h)$ in case 2), such that

$$\left\|u_n^+\right\|_{L_p(\Omega)} \le C\left(\left\|Du_n^+\right\|_{L_p(\Omega)}+1\right),\ \ \forall n\in\mathbb{N}.$$

Proof: Assume first that $R^+_{\gamma,u} = +\infty$. Then $\gamma^+ = +\infty$, and by assumption, there exists $M > 0$ such that

$$\int_\Omega \gamma^+\left(u_n\right)dx < M\,,\ \ \forall n\in\mathbb{N}.$$

Let $K_n = \left\{x\in\Omega : \gamma^+\big(u_n(x)\big) < \frac{2M}{|\Omega|}\right\}$, for every $n \in \mathbb{N}$. Then

$$0 \le \int_{K_n}\gamma^+\left(u_n\right) = \int_\Omega \gamma^+\left(u_n\right) - \int_{\Omega\backslash K_n}\gamma^+\left(u_n\right) \le M - \big(|\Omega| - \big|K_n\big|\big)\frac{2M}{|\Omega|} = \big|K_n\big|\frac{2M}{|\Omega|} - M.$$

Therefore, $\left|K_n\right| \geq \frac{|\Omega|}{2}$, and

$$\left\|u_n^+\right\|_{L_p\left(K_n\right)} \leq \left|K_n\right|^{1/p} \sup \gamma^{-1}(u_n) \frac{2M}{|\Omega|}.$$

Then by well known result, for all $n \in N$

$$\left\|u_n^+\right\|_{L_p(\Omega)} \leq C \left(\left(\frac{2}{|\Omega|} \right)^{1/p} \left\|Du_n^+\right\|_{L_p(\Omega)} + \sup \gamma^{-1}(u_n) \left(\frac{2M}{|\Omega|} \right) \right).$$

Now assume $R^+_{\gamma,u} < +\infty$, and let $\delta = R^+_{\gamma,u} - M$. Then by assumption

$$\int_\Omega \gamma(u_n) + \int_{\partial\Omega} u_n < R^+_{\gamma,u} - \delta.$$

Consequently, for every $n \in N$

$$\int_\Omega \gamma(u_n) < \gamma^+ |\Omega| - \frac{\delta}{2}, \quad \int_{\partial\Omega} u_n < \gamma^+ |\partial\Omega| - \frac{\delta}{2}. \tag{2.6}$$

Thus

$$\left\|u_n^+\right\|_{L_p(K_n)} \leq \left|K_n\right|^{1/p} \sup \gamma^{-1} \left(\gamma^+ - \frac{\delta}{4|\Omega|} \right).$$

Then, by well known results

$$\left\|u_n^+\right\|_{L_p(\Omega)} \leq C \left(\left(\frac{h - \delta(4|\Omega| + \gamma^+)}{\delta/8} \right)^{1/p} \left\|Du_n^+\right\|_{L_p(\Omega)} + \sup \gamma^{-1} \left(\gamma^+ - \frac{\delta}{4|\Omega|} \right) \right).$$

Lemma is proved.

Proof of the existence part of Theorem 2.2. Let $f \in L_{p'}\left(0, T; L_{p'}(\Omega)\right)$, $g \in L_{p'}\left(0, T; L_{p'}(\partial\Omega)\right)$, $u_0 \in L_{p'}(\Omega)$, and $V(t)$ solution of problem (2.5), where $V(t) = \left(\gamma(u(t)), u(t)|_{\partial\Omega}\right)$. Our aim is to prove that $\left(\gamma(u(t)), u|_{\partial\Omega}\right)$ is a weak solution of problem (1.1). For $n \in N$, let $\varepsilon = \frac{T}{n}$, and consider a subdivision $t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \varepsilon$ and $f_1, \cdots, f_n \in L_{p'}(\Omega)$ and $g_1, \cdots, g_n \in L_{p'}(\partial\Omega)$ with

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\|f(t) - f_i\|^{p'}_{L_{p'}(\Omega)} + \|g(t) - g_i\|^{p'}_{L_{p'}(\partial\Omega)} \right) dt \leq \varepsilon.$$

If we set $f_\varepsilon(t) = f_i$, $g_\varepsilon(t) = g_i$ and $u_\varepsilon(t) = u_i$ for $t \in (t_{i-1}, t_i)$, $i = 1, \cdots n$ and after some calculations, using Young's inequality, the trace theorem, from Lemma 2.4 we have there exists a constant $C > 0$ such that

$$\|u_\varepsilon(t)\|_{L_p(\Omega)} \leq C \left(\|Du_\varepsilon(t)\|_{L_p(\Omega)} + 1 \right), \ for\ all\ t \in [0, T). \tag{2.7}$$

Then we deduce that there exists $C > 0$ such that

$$\int_0^T \int_\Omega |Du_\varepsilon(x,t)|^p \, dxdt < C. \tag{2.8}$$

By this estimates $\{u_\varepsilon\}$ is bounded in $L_p\left(0, T; W^1_p(\Omega)\right)$. So, there exists a subsequence, denoted again $\{u_\varepsilon\}$, such that $u_\varepsilon \to u$ weakly in $L_p\left(0, T; W^1_p(\Omega)\right)$ as $\varepsilon \to 0+$ and $u_\varepsilon \to u$ weakly in $L_p(S_T)$ as $\varepsilon \to 0+$. This as $\varepsilon \to 0+$ we have the proof.

Theorem is proved.

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