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On a Rotar generalized condition and the central limit theorem

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Abstract. *In this article, new numerical characteristic is introduced, which does not contain any additional parameters. It is proved that this characteristic represents a general record of Rotar type characteristics. It is proved also that the tendency to zero of this characteristic is equivalent to the fulfilment of the central limit theorem for the sequence of series of independent random variables.*

Keywords. Central limit theorem, a generalized version of the Rotar characteristic, characteristic functions, conditions for uniform infinite smallness of variance.

Mathematics Subject Classification (2010): 2010 Mathematics Subject Classification: 60B12, 60F05

1 Introduction. Preliminary information

By the well-known property of normal distribution, the sum of independent normally distributed random variables also has a normal distribution. This means that if the distributions are independent close to the normal distribution law, then the distribution of the sum will be so close to the normal law, i.e. the central limit theorem (CLT) holds for the corresponding sequence of independent random variables.

Unlike other numerical characteristics used in limit theorems for sums of independent random variables, Rotar's numerical characteristic, introduced in [13, Ch. 5, 15, 261-273], [14], takes into account the reduced property of the normal distribution.

In the studies of S.V. Nagaev and his students [7], [8], [9], [10], [11], a new versions of the proof of the CLT for sequences of independent random variables connected in a homogeneous Markov chain are presented and exact estimates are found for the absolute constant in the Berri–Essen theorem.

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The references [15], [16], [17], [18], [6], [19], [1], [2], [20], [21], [22], [23], [24] are devoted to the study of the convergence rate in the CLT and the establishment of analogues of classical estimates of the Berry–Esseen type when moment conditions exist .

In recent articles [3], [12], [5], [4], modified versions of Rotar's numerical characteristics and their application in the CLT are investigated.

Arbitrary series

$$X_{1n}, \ldots, X_{nn}, \ \ldots, \ n = 1,\ 2,\ ... \tag{1.1}$$

of independent (in each series) random variables (r.v.) are considered, where the distribution of r.v. $X_{k,n}$ may depend on n.

Denote $S_n = \sum_{k=1}^{n} X_{k,n}$. Let there exist

$$\sigma_{k,n}^2 = EX_{k,n}^2 < \infty,\ k = 1, 2, ..., n,\ n \geq 1$$

From the point of view of subsequent results, it can be assumed, without loss of generality, that

$$EX_{k,n} = 0,\ k = 1,\ 2,\ \ldots, n,\ \sum_{k=1}^{n} \sigma_{k,n}^2 = 1. \tag{1.2}$$

We bear in mind that the sequence of r.v. (1.1) obeys the CLT, if the following asymptotic relationship is true

$$\sup_x |F_n(x) - \Phi(x)| \to 0,$$

where

$$F_n(x) = P(S_n < x),\ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

is the normal distribution with parameter $(0, 1)$.

In problems of checking the validity of the CLT for a sequence of independent r.v. the following characteristic plays an important role [13, Ch. 5, 18.3, p.305]

$$L_n(\varepsilon) = \sum_{k=1}^{n} \int_{|\,x\,|>\varepsilon} x^2\, d\, F_{k,n}(x),\ \varepsilon > 0,$$

where $F_{k,n}(x) = P(X_{k,n} < x)$ is the distribution function (d.f.) of r.v. $X_{k,n}$.

Condition

$$L_n(\varepsilon) \to 0,\ \ n \to \infty,\ \ \forall \varepsilon > 0 \tag{1.3}$$

is called the Lindeberg condition. In particular, it follows from the fulfilment of the Lindeberg condition that the r.v. (1.1) in this case have the property of uniform infinite smallness of variances:

$$\max_{1 \leq k \leq n} \sigma_{k,n}^2 \to 0,\ \ n \to \infty. \tag{1.4}$$

Condition (1.4) is called the Feller condition and from it, in turn, follows the so-called condition of the limiting (asymptotic) smallness of the terms, which is

$$\max_{1 \leq k \leq n} P(|X_{k,\,n}| > \varepsilon) \to 0,\ \ \forall \varepsilon > 0,\ \ n \to \infty. \tag{1.5}$$

It is well known that Lindeberg's theorem is valid if the Lindeberg condition (1.3) holds, then for a sequence of independent r.v. (1.1) the CLT is true.

As the following simple example of a sequence of independent r.v. shows, condition (1.3) is not necessary for the validity of the CLT. Let $X_1,\ X_2,\ \ldots$ be a sequence of independent normally distributed r.v. for which

$$EX_n = 0,\ DX_1 = 1,\ DX_k = 2^{k-2},\ k \geq 2.$$

Let us assume that $S_n = X_{1,n} + \ \ldots\ + X_{n,n}$, where

$$X_{k,n} = \frac{X_k}{B_n},\ k = 1, 2, ..., n,$$

$$B_n^2 = DS_n = \sum_{k=1}^{n} DX_k = 1 + 1 + \sum_{k=3}^{n} DX_k$$

$$= 2\left[1 + \Big(1 + \sum_{k=4}^{n} 2^{k-2}\Big)\right] = 2\big(1 + 2^{n-2} - 1\big) = 2^{n-1}.$$

It is easy to check that for the given sequence of r.v. $X_1,\ X_2,\ \ldots,\ X_n, ...$ neither the Lindeberg condition nor the condition of uniform smallness of variances (1.4) is satisfied, although the CLT is satisfied automatically, since the sums S_n are normally distributed with the parameters $ES_n = 0,\ \ DS_n = 1$.

In [14], V.I. Rotar introduced the following numerical characteristic for the sequence of r.v. (1.1):

$$R_n(\varepsilon) = \sum_{k=1}^{n} \int_{|x|>\varepsilon} |x|\, |F_{k,n}(x) - \varPhi_{k,n}(x)|\, dx,$$

where $0 < \varepsilon < \infty,\ F_{k,n}(x)$ is the d.f. of r.v. $X_{k,n}$, $\varPhi_{k,n}(x)$ is normally distributed r.v. with variance $\sigma_{k,n}^2 (i.e. \varPhi_{k,n}(x) = \varPhi\Big(\frac{x}{\sigma_{k,n}}\Big))$.

Condition

$$R_n(\varepsilon) \to 0,\ \ \forall\varepsilon > 0,\ \ \ n \to \infty \tag{1.6}$$

is called the Rotar condition. It is weaker than condition (1.3) and it is equivalent to the Lindeberg condition under the Feller condition (1.4) [13, Ch. 5, 18.7, p.310-311].

It follows from the definition of Rotar's characteristic $R_n(\varepsilon)$ that condition (1.6) is in no way connected with the condition of infinite smallness of a sequence (1.5) of independent r.v. (1.1).

In [12], an analog of the Rotar characteristic is introduced, which does not depend on any $\varepsilon > 0$. This analog is defined by the following equation

$$R_n^{(\alpha)} = \sum_{k=1}^{n} \int_{-\infty}^{\infty} \min\Big(|x|^{\alpha+1}\,,\, |x|\Big)\ \ |F_{k,n}(x) - \varPhi_{k,n}(x)|\, dx,\ \ \alpha > 0$$

and it was proved that the condition fulfilment

$$R_n^{(\alpha)} = \sum_{k=1}^{n} \int_{|\,x\,|\leq 1} |\,x\,|^{1+\alpha}\, |F_{k,n}(x) - \varPhi_{k,n}(x)|\, dx + \sum_{k=1}^{n} \int_{|\,x\,|>1} |\,x\,|\, |F_{k,n}(x) - \varPhi_{k,n}(x)|\, dx \to 0$$

for some $\alpha > 0$ is equivalent to Rotar's condition (1.6). The proof of this proposition follows from Theorem 2.1, given in Section 2 below.

2 Generalized Rotar condition and CLT

Following [5], we define class $B$of bounded nonnegative functions on the straight line $\lim\limits_{x\to 0} b(x)=0,\ \ m_b\left(\delta\right)=\inf\limits_{|x|>\delta} b(x)>0$, for all $\delta>0$.

We assume that

$$R_n^b=\sum_{k=1}^{n}\int_{-\infty}^{\infty}|x|\,b(x)\ |F_{k,n}(x)-\Phi_{k,n}(x)|\,dx.$$

Theorem 2.1 *The following three conditions are equivalent:*
1) Rotar's condition is satisfied;
2) $\lim\limits_{n\to\infty} R_n^b=0$ *for some* $b(\cdot)\in B$,
3) $\lim\limits_{n\to\infty} R_n^b=0$ *for all* $b(\cdot)\in B$.

Proof. Obviously, 3) implies 2). Let us prove that 2) implies 1). In fact, for any $\varepsilon>0$ there holds

$$R_n(\varepsilon)=\sum_{k=1}^{n}\int_{|x|>\varepsilon}|x|\,|F_{k,n}(x)-\Phi_{k,n}(x)|\,dx$$

$$\le\frac{1}{m_b(\varepsilon)}\sum_{k=1}^{n}\int_{|x|>\varepsilon}b(x)\,|x|\ \ |F_{k,n}(x)-\Phi_{k,n}(x)|\,dx\le\frac{1}{m_b(\varepsilon)}R_n^b\to 0. \tag{2.1}$$

Hence, the implication 2) $\Rightarrow$ 1) is proved. Now, let Rotar's condition hold. Then we have

$$R_n^b=\sum_{k=1}^{n}\int_{-\infty}^{\infty}|x|\,b(x)\ |F_{k,n}(x)-\Phi_{k,n}(x)|\,dx$$

$$\le\sup_x b(x)\sum_{k=1}^{n}\int_{|x|>\varepsilon}|x|\ |F_{k,n}(x)-\Phi_{k,n}(x)|\,dx+\sum_{k=1}^{n}\int_{|x|\le\varepsilon}b(x)\,|x|\ |F_{k,n}(x)-\Phi_{k,n}(x)|\,dx$$

$$\le\sup_x b(x)\cdot R_n(\varepsilon)+\sup_{0<|x|\le\varepsilon}b(x)\sum_{k=1}^{n}\int_{-\infty}^{\infty}|x|\ |F_{k,n}(x)-\Phi_{k,n}(x)|\,dx. \tag{2.2}$$

Furthermore, applying integration by parts, we can be sure that for any d.f. $F(x)$ with finite invariance the following equation holds

$$\int_{-\infty}^{0}|x|\ F(x)dx+\int_{0}^{\infty}x(1-F(x))dx=\frac{1}{2}\int_{-\infty}^{\infty}x^2dF(x). \tag{2.3}$$

Now using the formula

$$E\,|X|^n=\int_{-\infty}^{\infty}|x|^n\,dF(x)=n\int_{0}^{\infty}x^{n-1}(1-F(x)+F(-x))dx$$

and equality (2.3), we obtain

$$\sum_{k=1}^{n}\int_{-\infty}^{\infty}|x|\ |F_{k,n}(x)-\Phi_{k,n}(x)|\,dx\le\sum_{k=1}^{n}\sigma_{k,n}^2\le 1. \tag{2.4}$$

Thus, it follows from relation (2.2)–(2.4) that

$$\lim_{n\to\infty}\sup R_n^b\le\sup_{|x|\le\varepsilon}b(x),\quad\forall\varepsilon>0,$$

and therefore

$$\lim_{n\to\infty} R_n^b = 0.$$

The proof of the Theorem 2.1 is complete.

Expression R_n^b is called the generalized Rotar characteristic, and condition $\left(R^b\right)$ $R_n^b \to 0$, for each $b(\bullet) \in R$ is called the generalized Rotar condition. The Rotar characteristic $R_n(\varepsilon)$ is obtained from the form R_n^b, when

$$b(\bullet) = b_0(x) = \begin{cases} 0, & \text{for } |x| \le \varepsilon, \\ 1, & \text{for } |x| > \varepsilon. \end{cases}$$

According to the result of Theorem 2.1, the proposition $R_n^b \to 0$, for any function $b \in B$ is equivalent to the fact that the Rotar condition is met $\{R_n(\varepsilon) \to 0,\ \forall\varepsilon > 0\}$. Taking into account the latter, the main theorem in V. I. Rotar's article [14] can be formulated as follows.

Theorem 2.2 *Let* $\{X_{n,1}, \ldots X_{n,n},\ n = 1, 2, \ldots\}$ *be a sequence of series of independent random variables. For this sequence to satisfy the CLT, it is necessary and sufficient to satisfy the generalized Rotar condition*

$$R_n^b = \sum_{k=1}^{n} \int_{-\infty}^{\infty} |x|\, b(x)\, |F_{k,n}(x) - \Phi_{k,n}|\, d(x) \to 0 \tag{2.5}$$

for any function $b(\bullet) \in B$.

In section 3 it will be provide a self-standing and independent of ([14], a shorter version) proof of Theorem 2.2.

3 Proof of Theorem 2.2

Let the characteristic function (ch.f) of r.v. $X_{k,n}$

$$f_{k,n}(t) = E\, e^{i\,t\,X_{k,n}}$$

and let

$$g_{k,n}(t) = e^{-\frac{\sigma_{k,n}^2 t^2}{2}}$$

be ch.f. of the normally distributed r.v. with parameter $\left(0, \sigma_{k,n}^2\right)$ $(k = 1, 2, \ldots, n)$. Then

$$\overline{f_n(t)} = E\, e^{i\,t\,S_n} = \prod_{k=1}^{n} f_{k,n}(t)$$

is the ch.f. of the standard normally distributed r.v. with parameter $(0, 1)$:

$$g_n(t) = g(t) = \prod_{k=1}^{n} e^{-\ \sigma_{k,n}^2 t^2/2} = e^{-t^2/2}.$$

To prove the sufficiency of the condition for the validity of the CLT according to the continuous accordance theorem, it is suffice to show that under condition (2.5) the following convergence takes place

$$\overline{f_n(t)} \to g(t), \quad t \in R. \tag{3.1}$$

In turn, the limit relationship (3.1) is proved using the following auxiliary assertions.

Lemma 3.1 . *Let*

$$D_n = \sum_{k=1}^{n} \int_{-\infty}^{\infty} \min\left(|x|\,,\, x^2\right) |F_{k,n}(x) - \Phi_{k,n}(x)|\; d\,x = \sum_{k=1}^{n} \int_{|\,x\,|\,\leq 1} x^2\, |F_{k,n}(x) - \Phi_{k,n}(x)|$$

$$+ \sum_{k=1}^{n} \int_{|\,x\,|>1} |\,x\,|\,|\,F_{k,n}(x) - \Phi_{k,n}(x)|\; d\,x = K_n + R_n(1) = K_n + R_n.$$

Then the condition

$$D_n \to 0, \quad n \to \infty \tag{3.2}$$

is equivalent to Rotar's condition (1.6).

Proof. Let condition (1.6) be true. Then we have

$$K_n = \sum_{k=1}^{n} \int_{|x|\leq\varepsilon} x^2\, |F_{k,n}(x) - \Phi_{k,n}(x)|\, d\,x$$

$$+ \sum_{k=1}^{n} \int_{\varepsilon<|x|\leq 1} x^2\, |F_{k,n}(x) - \Phi_{k,n}(x)|\, dx = K_n^{(1)} + K_n^{(2)}, \;\; 0 < \varepsilon < 1.$$

In view of inequality (2.4), we obtain

$$K_n^1 \leq \varepsilon \int_{|x|\leq\varepsilon} |x|\, |F_{k,n}(x) - \Phi_{k,n}(x)|\, dx \leq \varepsilon \sum_{k=1}^{n} \sigma_{k,n}^2 \leq \varepsilon. \tag{3.3}$$

Furthermore, it is obvious that

$$K_n^2 \leq \sum_{k=1}^{n} \int_{\varepsilon<|\,x\,|\leq 1} |\,x\,|\,|F_{k,n}(x) - \Phi_{k,n}(x)|\, dx$$

$$\leq \sum_{k=1}^{n} \int_{|\,x\,|>\varepsilon} |x|\, |F_{k,n}(x) - \Phi_{k,n}(x)|\, dx = R_n(\varepsilon)\,. \tag{3.4}$$

It follows from (3.3) and (3.4) that $K_n \leq \varepsilon + R_n(\varepsilon)$, and therefore $\lim_{n\to\infty} \sup K_n \leq \varepsilon$. Thus, due to the arbitrariness of $0 < \varepsilon < 1$

$$K_n \to 0, \quad n \to \infty \tag{3.5}$$

it is obvious that

$$R_n = R_n(1) \leq R_n(\varepsilon) \to 0, \quad \forall\varepsilon > 0, \quad n \to \infty. \tag{3.6}$$

Now, by virtue of (3.5), (3.6), we obtain under condition (1.6)

$$D_n \to 0, \quad n \to \infty. \tag{3.7}$$

Now let the condition (3.2) be satisfied. Then we have

$$K_n \to 0, \quad R_n \to 0, \quad n \to \infty. \tag{3.8}$$

Furthermore, we have ($0 < \varepsilon < 1$)

$$R_n(\varepsilon) = \sum_{k=1}^{n} \int_{|x|>\varepsilon} |x|\, |F_{k,n}(x) - \Phi_{k,n}(x)|\, dx = \sum_{k=1}^{n} \int_{\varepsilon<|x|\leq 1} |x|\, |F_{k,n}(x) - \Phi_{k,n}(x)|\, dx$$

$$+ \sum_{k=1}^{n} \int_{|\,x\,|>1} |x|\, |F_{k,n}(x) - \Phi_{k,n}(x)|\, d\,x \leq \frac{1}{\varepsilon} K_n + R_n \to 0, \quad n \to \infty. \tag{3.9}$$

The proof of Lemma 3.1 follows from relationships (3.7) and (3.9).

Lemma 3.2 *If condition (3.2) is satisfied, then*

$$\left|\sum_{k=1}^{n}\left(f_{k,n}\left(t\right)-g_{k,n}\left(t\right)\right)\right|\leq h\left(t\right)D_{n}\to 0,$$

where $h\left(t\right)=\max\left(\frac{t^{4}}{6},\ \frac{|t|^{3}}{2},\ 2t^{2}\right)$.

Proof. Indeed, by virtue of the definition of the ch.f. of distributions. We can write the following system of equalities and inequalities:

$$\left|\sum_{k=1}^{n}\left(f_{k,n}\left(t\right)-g_{k,n}\left(t\right)\right)\right|=\left|\sum_{k=1}^{n}\int_{-\infty}^{\infty}e^{itx}d\left(F_{k,n}\left(x\right)-\varPhi_{k,n}\left(x\right)\right)\right|$$

$$=\left|\sum_{k=1}^{n}\int_{-\infty}^{\infty}\left(e^{i\,t\,x}-1-i\,t\,x-\frac{\left(itx\right)^{2}}{2}\right)d\left(F_{k,n}-\varPhi_{k,n}\right)\right|$$

$$=\left|\sum_{k=1}^{n}\left(it\right)\int_{-\infty}^{\infty}\left(e^{itx}-1-itx\right)\left(F_{k,n}\left(x\right)-\varPhi_{k,n}\left(x\right)\right)d\,x\right|$$

$$\leq\left|\sum_{k=1}^{n}\left(it\right)\int_{|\,x\,|\leq 1}\left(e^{itx}-1-itx-\frac{\left(itx\right)^{2}}{2}\right)\left(F_{k,n}\left(x\right)-\varPhi_{k,n}\left(x\right)\right)d\,x\right|$$

$$+\frac{|t|^{3}}{2}\sum_{k=1}^{n}\int_{|\,x\,|\leq 1}x^{2}\left|\left(F_{k,n}\left(x\right)-\varPhi_{k,n}\left(x\right)\right)\right|d\,x$$

$$+\left|\,t\,\right|\sum_{k=1}^{n}\left|\int_{|\,x\,|>1}\left(e^{itx}-1-i\,t\,x\right)\left(F_{k,n}\left(x\right)-\varPhi_{k,n}\left(x\right)\right)d\,x\right|.$$

When writing the last chain of equalities and in subsequent inequalities, identical transformations and elementary estimates

$$\left|e^{i\,t\,x}-1-i\,t\,x-\frac{\left(i\,t\,x\right)^{2}}{2}\right|\leq\frac{\left|\,t\,x\,\right|^{3}}{6},\ \left|e^{i\,t\,x}-1-i\,t\,x\right|\leq\frac{\left(t\,x\right)^{2}}{2}$$

are valid for any t and x. Taking into account the last remarks, we can conclude that

$$\left|\sum_{k=1}^{n}\left(f_{k,n}\left(t\right)-g_{k,n}\left(t\right)\right)\right|\leq I_{n\,1}\left(t\right)+I_{n\,2}\left(t\right)+I_{n\,3}\left(t\right),\tag{3.10}$$

where

$$\left|I_{n\,1}\left(t\right)\right|\leq\frac{t^{4}}{6}\left[\sum_{k=1}^{n}\int_{|\,x\,|\leq 1}x^{2}\,|\,F_{k,n}\left(x\right)-\varPhi_{k,n}\left(x\right)|\,dx\right],$$

$$\left|\,I_{n\,2}\left(t\right)\,\right|\leq\frac{\left|\,t\,\right|^{3}}{2}\left[\sum_{k=1}^{n}\int_{|\,x\,|\leq 1}x^{2}\left|F_{k,n}\left(x\right)-\varPhi_{k,n}\left(x\right)\right|\,d\,x\right],$$

$$\left|I_{n\,3}\left(t\right)\right|\leq 2\,t^{2}\left[\sum_{k=1}^{n}\int_{|\,x\,|>1}|\,x\,|\,|\,F_{k,n}\left(x\right)-\varPhi_{k,n}\left(x\right)|\,d\,x\right].$$

Therefore, for any t, the following estimates hold:

$$\max\left(I_{n1}\left(t\right),\,I_{n2}\left(t\right)\right)\leq\max\left(\frac{t^{4}}{6},\,\frac{\left|\,t\,\right|^{3}}{2}\right)K_{n}\to 0,\tag{3.11}$$

$$\left|I_{n\,3}\left(t\right)\right|\leq 2\,t^{\,2}R_{n}\to 0.\tag{3.12}$$

The assertion of Lemma 3.2 follows from relationships (3.10)–(3.12).

Furthermore, when applying induction, it is easy to verify that for any

$$\max_{1\leq k\leq n}\left(\left|a_{\,k}\right|,\,\left|b_{\,k}\right|\right)\leq 1$$

the following inequality holds

$$\left|\prod_{k=1}^{n}a_{\,k}-\prod_{k=1}^{n}b_{\,k}\right|\leq\sum_{k=1}^{n}\left|a_{k}-b_{k}\right|.\tag{3.13}$$

By virtue of (3.13) and Lemma 3.2, for any t we have

$$\left|\overline{f_{n}\left(t\right)}-g\left(t\right)\right|=\left|\prod_{k=1}^{n}f_{n\,k}\left(t\right)-\prod_{k=1}^{n}g_{n\,k}\left(t\right)\right|\leq\sum_{k=1}^{n}\left|\,f_{n\,k}\left(t\right)-g_{n\,k}\left(t\right)\right|\to 0,\quad n\to\infty.$$

So, the limit relationship (3.1) is proved, and the sufficiency of condition (2.5) for the validity of the CLT is proved as well.

The necessity of the generalized Rotar condition is proved as follows. In [2], V.I. Rotar has proved that the Rotar condition (1.6) is necessary for CLT to hold. However, according to Theorem 2.1, the condition (2.5) is equivalent to Rotar's condition (1.6). Consequently,the generalized Rotar condition (2.5) is a necessary condition for the validity of the CLT.

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