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Some characterizations of BMO spaces via maximal commutators in Orlicz spaces over Carleson curves

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Abstract. *We study the maximal commutators $M_{b,\Gamma}$ and the commutators $[b, M_\Gamma]$ of the maximal operator over the Orlicz spaces $L^\Phi(\Gamma)$ defined on Carleson curves Γ. We give necessary and sufficient conditions for the boundedness of the operators $M_{b,\Gamma}$ on Orlicz spaces $L^\Phi(\Gamma)$ when b belongs to $BMO(\Gamma)$ spaces, by which some new characterizations for certain subclasses of $BMO(\Gamma)$ spaces are obtained.*

Keywords. Carleson curves, maximal operator, Orlicz space, commutator; BMO

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1 Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le l \le \infty\}$ be a rectifiable Jordan curve in the complex plane with arc-length measure $\nu(t) = s$, here $l = \nu\Gamma =$ lengths of Γ.

We denote

$$\Gamma(t, r) = \Gamma \cap B(t, r),\ t \in \Gamma,\ r > 0,$$

where $B(t, r) = \{z \in \mathbb{C} : |z - t| < r\}$.

A rectifiable Jordan curve Γ is called a Carleson curve (regular curve) if the condition

$$\nu\Gamma(t, r) \le c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on t and r.

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Let $f \in L_1^{\rm loc}(\Gamma)$. The maximal function $M_\Gamma f$ is defined by

$$M_\Gamma f(t) = \sup_{r>0} (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau),\ t \in \Gamma.$$

The maximal commutator $M_{b,\Gamma}$, generated by $b \in L_1^{loc}(\Gamma)$, is defined by

$$M_{b,\Gamma}(f)(t) = \sup_{r>0} (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} |b(t) - b(\tau)|\, |f(\tau)| d\nu(\tau),\ t \in \Gamma.$$

The commutator, generated by a function b and the operator M_Γ, is defined by

$$[b, M_\Gamma](f)(t) = b(t) M_\Gamma(f)(t) - M_\Gamma(bf)(t),\ t \in \Gamma.$$

Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas than other methods.

Although the $M_{b,\Gamma}$ and $[b, M_\Gamma]$ operators are very similar, they are fundamentally different. The maximal commutator $M_{b,\Gamma}$ plays a significant role in the study of commutators of singular integral operators with the symbol BMO, and this topic has attracted the attention of many mathematicians (check it out, for exaample, [1–5, 10–16, 18, 24]). The nonlinear commutator $[b, M]$ of maximal operator, can be used in studying the product of a function in H_1 and a function in BMO (see [6], for instance). In [5], Bastero et al. studied the necessary and sufficient condition for the boundedness of $[b, M]$ on L^p spaces.

Our main aim is to characterize the commutator functions b, involved in the boundedness on Orlicz spaces of the maximal commutator $M_{b,\Gamma}$ (Theorems 4.1).

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 On Young Functions and Orlicz Spaces

Orlicz space was first introduced by Orlicz as a generalizations of Lebesgue spaces L^p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when L^1 space does not work.

First, we recall the definition of Young functions.

Definition 2.1 *A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.*

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} , & r \in [0,\infty) \\ \infty , & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \qquad r \ge 0$$

for some $C > 1$.

Lemma 2.1 *[20] Let Φ be a Young function with canonical representation*

$$\Phi(r) = \int_0^r \varphi(s)ds, r > 0.$$

(1) *Assume that $\Phi \in \Delta_2$. More precisely $\Phi(2r) \le A\Phi(r)$ for some $A \ge 2$. Set $\beta = \log_2 A$. If $p > \beta + 1$, then the following inequality is valid:*

$$\int_r^\infty \frac{\varphi(s)}{s^p}ds \lesssim \frac{\Phi(r)}{r^p},\ t > 0.$$

(2) *Assume that $\Phi \in \nabla_2$. Then the following inequality is valid:*

$$\int_0^r \frac{\varphi(s)}{s}ds \lesssim \frac{\Phi(r)}{r},\ r > 0.$$

Definition 2.2 *(Orlicz Space). For a Young function Φ, the set*

$$L^{\Phi}(\Gamma) = \left\{ f \in L^1_{\rm loc}(\Gamma) : \int_\Gamma \Phi(k|f(\tau)|)d\nu(\tau) < \infty \, for\ some\ k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^{\Phi}(\Gamma) = L^p(\Gamma)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^{\Phi}(\Gamma) = L^\infty(\Gamma)$. The space $L^{\Phi}_{\rm loc}(\Gamma)$ is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in L^{\Phi}(\Gamma)$ for all balls $\mathcal{E} \subset \Gamma$.

$L^{\Phi}(\Gamma)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\Gamma)} = \inf\left\{ \lambda > 0 : \int_\Gamma \Phi\Big(\frac{|f(\tau)|}{\lambda}\Big)d\nu(\tau) \le 1 \right\}.$$

For a measurable set $\Gamma_0 \subset \Gamma$, a measurable function f and $r > 0$, let $m(\Gamma_0,\ f,\ r) = |\{t \in \Gamma_0 : |f(t)| > r\}|$. In the case $\Gamma_0 = \Gamma$, we shortly denote it by $m(f,\ r)$.

Definition 2.3 *The weak Orlicz space*

$$WL^{\Phi}(\Gamma) = \{f \in L^1_{\rm loc}(\Gamma) : \|f\|_{WL^{\Phi}} < \infty\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi}(\Gamma)} = \inf\left\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(r) m\Big(\frac{f}{\lambda},\ r\Big) \ \le 1 \right\}.$$

We note that $\|f\|_{WL^\Phi} \le \|f\|_{L^\Phi}$,

$$\sup_{r>0} \Phi(r) m(\Gamma_0,\ f,\ r) = \sup_{r>0} r\, m(\Gamma_0,\ f,\ \Phi^{-1}(r)) = \sup_{r>0} r\, m(\Gamma_0,\ \Phi(|f|),\ r)$$

and

$$\int_{\Gamma_0} \Phi\Big(\frac{|f(\tau)|}{\|f\|_{L^\Phi(\Gamma_0)}}\Big) d\nu(\tau) \le 1, \qquad \sup_{r>0} \Phi(r) m\Big(\Gamma_0,\ \frac{f}{\|f\|_{WL^\Phi(\Gamma_0)}},\ r\Big) \le 1, \tag{2.2}$$

where $\|f\|_{L^\Phi(\Gamma_0)} = \|f\chi_{\Gamma_0}\|_{L^\Phi(\Gamma)}$ and $\|f\|_{WL^\Phi(\Gamma_0)} = \|f\chi_{\Gamma_0}\|_{WL^\Phi(\Gamma)}$.

The following analogue of the Hölder's inequality is well known (see, for example, [26]).

Theorem 2.1 *Let $\Gamma_0 \subset \Gamma$ be a measurable set and functions f and g measurable on Γ_0. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_{\Gamma_0} |f(\tau) g(\tau)|\, d\nu(\tau) \le 2\, \|f\|_{L^\Phi(\Gamma_0)}\, \|g\|_{L^{\tilde{\Phi}}(\Gamma_0)}.$$

By elementary calculations we have the following property.

Lemma 2.2 *Let Φ be a Young function and $\Gamma(t,r)$ be a balls in Γ. Then*

$$\|\chi_{\Gamma(t,r)}\|_{L^\Phi(\Gamma)} = \|\chi_{\Gamma(t,r)}\|_{WL^\Phi(\Gamma)} = \frac{1}{\Phi^{-1}\left(\nu(\Gamma(t,r))^{-1}\right)}.$$

By Theorem 2.1, Lemma 2.2 and (2.1) we get the following estimate.

Lemma 2.3 *For a Young function Φ and for the balls $\Gamma(t,r)$ the following inequality is valid:*

$$\int_{\Gamma(t,r)} |f(\tau)|\, d\nu(\tau) \le 2\, \nu(\Gamma(t,r))\, \Phi^{-1}\left(\nu(\Gamma(t,r))^{-1}\right)\, \|f\|_{L^\Phi(\Gamma(t,r))}.$$

3 Maximal operator defined on Carleson curves in the Orlicz spaces $L^\Phi(\Gamma)$

In this section the boundedness of the maximal operator M_Γ defined on Carleson curves Γ in Orlicz spaces $L^\Phi(\Gamma)$ have been obtained.

Theorem 3.1 *Let Φ any Young function and Γ be a Carleson curve. Then the maximal operator M_Γ is bounded from $L^\Phi(\Gamma)$ to $WL^\Phi(\Gamma)$ and for $\Phi \in \nabla_2$ bounded in $L^\Phi(\Gamma)$.*

Proof. At first proved that the maximal operator M_Γ is bounded from $L^\Phi(\Gamma)$ to $WL^\Phi(\Gamma)$.

We take $f \in L^\Phi(\Gamma)$ satisfying $\|f\|_{L^\Phi(\Gamma)} = 1$ so that the modular

$$\rho_\Phi(f) := \int_\Gamma \Phi(|f(\tau)|) d\nu(\tau) \le 1.$$

We know that by Jensen inequality

$$\Phi\left(\frac{1}{\nu(\Gamma(t,r))} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau)\right) \le \frac{1}{\nu(\Gamma(t,r))} \int_{\Gamma(t,r)} \Phi(|f(\tau)|) d\nu(\tau) \tag{3.1}$$

for all balls $\Gamma(t,r)$. Using (3.1) and definition of maximal operator we have

$$\Phi(M_\Gamma f(t)) \le M_\Gamma[(\Phi \circ f)(t)]. \tag{3.2}$$

Using (3.2) and weak (1,1) boundedness of maximal operator (see [7]) we get

$$\begin{aligned}\nu\big(\{t\in\Gamma : M_\Gamma f(t)>r\}\big) &= \nu\big(\{t\in\Gamma : \Phi(M_\Gamma f(t))>\Phi(r)\}\big)\\ &\le \nu\big(\{t\in\Gamma : M_\Gamma(\Phi\circ f)(t)>\Phi(r)\}\big)\\ &\le \frac{C}{\Phi(r)}\int_\Gamma \Phi(|f(t)|)d\nu(t)\\ &\le \frac{C}{\Phi(r)} \le \frac{1}{\Phi(\frac{r}{C\|f\|_{L^\Phi(\Gamma)}})},\end{aligned}$$

since $\|f\|_{L^\Phi(\Gamma)}=1$ and $\frac{1}{C}\Phi(r)\ge\Phi\left(\frac{r}{C}\right)$, if $C\ge1$.

Since $\|\cdot\|_{L^\Phi(\Gamma)}$ norm is homogeneous the inequality

$$\nu(\{t\in\Gamma : M_\Gamma f(t)>r\}) \le \frac{1}{\Phi(\frac{r}{C\|f\|_{L^\Phi(\Gamma)}})}$$

is true for every $f\in L^\Phi(\Gamma)$.

Now proved that for $\Phi\in\nabla_2$ the maximal operator M_Γ is bounded in $L^\Phi(\Gamma)$.

Let $\Lambda>0$ and $f\in L^\Phi(\Gamma)\backslash\{0\}$. Then we have

$$\begin{aligned}\int_\Gamma \Phi\left(\frac{M_\Gamma f(t)}{\Lambda}\right)d\nu(t) &= \int_\Gamma\int_0^{\frac{M_\Gamma f(t)}{\Lambda}} \varphi(s)ds d\nu(t)\\ &= \int_\Gamma\int_0^\infty \chi_{\{s\in[0,\infty):\frac{M_\Gamma f(t)}{\Lambda}>s\}}\varphi(s)dsd\nu(t)\\ &= \int_0^\infty \varphi(s)\int_\Gamma \chi_{\{t\in\Gamma:M_\Gamma f(t)>\Lambda s\}}d\nu(t)ds\\ &= \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{\lambda}{\Lambda}\right)\nu\big(\{t\in\Gamma : M_\Gamma f(t)>\lambda\}\big)\,d\lambda\\ &= \frac{2}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\nu\big(\{t\in\Gamma : M_\Gamma f(t)>2\lambda\}\big)\,d\lambda.\end{aligned}$$

From the weak $(1,1)$ maximal inequality

$$\nu\big(\{t\in\Gamma : M_\Gamma f(t)>2\lambda\}\big) \lesssim \frac{1}{\lambda}\int_{\{t\in\Gamma:|f(t)|>\lambda\}} |f(t)|d\nu(t)$$

and change the order of integration

$$\begin{aligned}\int_\Gamma \Phi\left(\frac{M_\Gamma f(t)}{\Lambda}\right)d\nu(t) &\lesssim \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\left(\int_{\{t\in\Gamma:|f(t)|>\lambda\}}|f(t)|d\nu(t)\right)\frac{d\lambda}{\lambda}\\ &\lesssim \frac{1}{\Lambda}\int_\Gamma |f(t)|\left(\int_0^{|f(t)|}\varphi\left(\frac{2\lambda}{\Lambda}\right)\frac{d\lambda}{\lambda}\right)d\nu(t)\\ &\lesssim \frac{1}{\Lambda}\int_\Gamma |f(t)|\left(\int_0^{2\Lambda^{-1}|f(t)|}\varphi(\lambda)\frac{d\lambda}{\lambda}\right)d\nu(t).\end{aligned}$$

Now we use Lemma 2.1 which yields

$$\left(\int_0^{2\Lambda^{-1}|f(t)|}\varphi(\lambda)\frac{d\lambda}{\lambda}\right) \lesssim |f(t)|^{-1}\Lambda\Phi\left(\frac{2|f(t)|}{\Lambda}\right),$$

if $f(t) \neq 0$. Recall that $k\Phi(r) \leq \Phi(kr)$ for $k \geq 1$ and $r > 0$, assuming Φ convex. Therefore, it follows that

$$\int_\Gamma \Phi\left(\frac{M_\Gamma f(t)}{\Lambda}\right) d\nu(t) \leq c_0 \int_\Gamma \Phi\left(\frac{2|f(t)|}{\Lambda}\right) d\nu(t) \leq \int_\Gamma \Phi\left(\frac{c_0|f(t)|}{\Lambda}\right) d\nu(t).$$

Here c_0 is a constant we would like to shed light on. Choosing $\Lambda = c_0\|f\|_{L^\Phi(\Gamma)}$, we obtain

$$\int_\Gamma \Phi\left(\frac{M_\Gamma f(t)}{\Lambda}\right) d\nu(t) \leq 1.$$

This means

$$\|M_\Gamma f\|_{L^\Phi(\Gamma)} \leq \Lambda = c_0\|f\|_{L^\Phi(\Gamma)}$$

from the definition of the norm.

4 Maximal commutator in the Orlicz spaces $L^\Phi(\Gamma)$

In this section we find necessary and sufficient conditions for the boundedness of the maximal commutator $M_{b,\Gamma}$ on Orlicz spaces. To study the boundedness of the commutators of some integral operators, we need the bounded mean oscillation space first introduced by John and Nirenberg [19].

Suppose that $b \in L^1_{\rm loc}(\Gamma)$. Then b is said to be in $BMO(\Gamma)$ if the seminorm given by

$$\|b\|_* = \sup_{r>0} (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} |b(\tau) - b_{\Gamma(t,r)}| d\nu(\tau)$$

is finite, where the supremum is taken over all balls $\Gamma(t,r) \subset \Gamma$ and

$$b_{\Gamma(t,r)} = (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} b(\tau) d\nu(\tau).$$

For any measurable set $\Gamma_0 \subset \Gamma$ with $\nu(\Gamma_0) < \infty$ and any suitable function f, the norm $\|f\|_{L(\log L),\Gamma_0}$ is defined by

$$|f\|_{L(\log L),\Gamma_0} = \inf\left\{\lambda > 0 : \frac{1}{\nu(\Gamma_0)} \int_{\Gamma_0} \frac{|f(t)|}{\lambda} \log\Big(2 + \frac{|f(t)|}{\lambda}\Big) d\nu(t) \leq 1\right\}.$$

The norm $\|f\|_{\exp L,\Gamma_0}$ is defined by

$$|f\|_{\exp L,\Gamma_0} = \inf\left\{\lambda > 0 : \frac{1}{\nu(\Gamma_0)} \int_{\Gamma_0} \exp\Big(\frac{|f(t)|}{\lambda}\Big) d\nu(t) \leq 2\right\}.$$

Then for any suitable functions f and g the generalized Hölders inequality holds (see [26])

$$\frac{1}{\nu(\Gamma_0)} \int_{\Gamma_0} |f(t)||g(t)| d\nu(t) \lesssim \|f\|_{\exp L,\Gamma_0} \|g\|_{L(\log L),\Gamma_0}. \tag{4.1}$$

The following John-Nirenberg inequalities on spaces of homogeneous type come from [21, Propositions 6, 7].

Lemma 4.1 *Let $b \in BMO(\Gamma)$. Then there exist constants $C_1, C_2 > 0$ such that for every ball $B \subset \Gamma$ and every $\alpha > 0$, we have*

$$\nu\Big(\big\{\tau \in \Gamma(t,r) : |b(\tau) - b_{\Gamma(t,r)}| > \alpha\big\}\Big) \leq C_1\, \nu(\Gamma(t,r)) \exp\Big\{-\frac{C_2}{\|b\|_*}\, \alpha\Big\}.$$

By the generalized Hölder's inequality in Orlicz spaces (see [26, page 58]) and John-Nirenberg's inequality, we get (see also [22, (2.14)]).

$$\frac{1}{\nu(\Gamma(t,r))}\int_{\Gamma(t,r)}\big|b(\tau)-b_{\Gamma(t,r)}\big||g(\tau)|d\nu(\tau)\lesssim\|b\|_*\,\|g\|_{L(\log L),\Gamma(t,r)}. \tag{4.2}$$

We refer for instance to [19] and [23] for details on this space and properties.

For a function b defined on Γ, we denote

$$b^-(t):=\begin{cases}0\,, & \text{if } b(t)\ge 0,\\ |b(t)|, & \text{if } b(t)<0\end{cases}$$

and $b^+(t):=|b(t)|-b^-(t)$. Obviously, $b^+(t)-b^-(t)=b(t)$.

Before proving our theorems, we need the following lemmas and theorem.

Lemma 4.2 *[17] Let $b\in BMO(\Gamma)$ and Φ be a Young function with $\Phi\in\Delta_2$, then*

$$\|b\|_*\approx\sup_{\Gamma(t,r)}\Phi^{-1}\big((\nu\Gamma(t,r))^{-1}\big)\,\Big\|\big(b-b_{\Gamma(t,r)}\big)\chi_{\Gamma(t,r)}\Big\|_{L^\Phi(\Gamma)}. \tag{4.3}$$

For proving our main results, we need the following estimate.

Lemma 4.3 *[8] Let $b\in L^1_{\rm loc}(\Gamma)$, $t_0\in\Gamma$ and $r_0>0$. Then*

$$|b(t)-b_{\Gamma(t_0,r_0)}|\le CM_{b,\Gamma}\big(\chi_{\Gamma(t_0,r_0)}\big)(t)\ \textit{ for every }\ t\in\Gamma(t_0,r_0).$$

Lemma 4.4 *Let $f\in L^1_{\rm loc}(\Gamma)$ and $t\in\Gamma$. Then*

$$M_\Gamma\big(M_\Gamma f\big)(\tau)\approx\sup_{r>0}\|f\|_{L(1+\log^+L),\Gamma(t,r)}. \tag{4.4}$$

Proof. Let $\Gamma(t,r)$ be a ball in Γ. We are going to use weak type estimates (see [25], for instance): there exist positive constants $c>1$ such that for every $f\in L^1_{\rm loc}(\Gamma)$ and for every $t>\big(1/\nu(\Gamma(t,r))\big)\int_{\Gamma(t,r)}|f(\tau)|d\nu(\tau)$ we have

$$\begin{aligned}\frac{1}{cr}\int_{\{\tau\in\Gamma(t,r):|f(\tau)|>r\}}|f(\tau)|d\nu(\tau)&\le\nu(\{\tau\in\Gamma(t,r):M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)>r\})\\&\le\frac{c}{r}\int_{\{\tau\in\Gamma(t,r):|f(\tau)|>r/2\}}|f(\tau)|d\nu(\tau).\end{aligned}$$

Then

$$\int_{\Gamma(t,r)} M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)d\nu(\tau) = \int_0^\infty \nu(\{\tau\in\Gamma(t,r): M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)>\lambda\})d\lambda$$
$$= \int_0^{|f|_{\Gamma(t,r)}} \nu(\{\tau\in\Gamma(t,r): M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)>\lambda\})d\lambda$$
$$+ \int_{|f|_{\Gamma(t,r)}}^\infty \nu(\{\tau\in\Gamma(t,r): M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)>\lambda\})d\lambda$$
$$= \nu(\Gamma(t,r))\,|f|_{\Gamma(t,r)} + \int_{|f|_{\Gamma(t,r)}}^\infty \nu(\{\tau\in\Gamma(t,r): M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)>\lambda\})d\lambda$$
$$\geq \nu(\Gamma(t,r))\,|f|_{\Gamma(t,r)} + \frac{1}{c}\int_{|f|_{\Gamma(t,r)}}^\infty \Big(\int_{\{\tau\in\Gamma(t,r):|f(\tau)|>\lambda\}} |f(\tau)|d\nu(\tau)\Big)\frac{d\lambda}{\lambda}$$
$$= \nu(\Gamma(t,r))\,|f|_{\Gamma(t,r)} + \frac{1}{c}\int_{\{\tau\in\Gamma(t,r):|f(\tau)|>|f|_{\Gamma(t,r)}\}} \Big(\int_{|f|_{\Gamma(t,r)}}^{|f(\tau)|}\frac{d\lambda}{\lambda}\Big)|f(\tau)|d\nu(\tau)$$
$$= \nu(\Gamma(t,r))\,|f|_{\Gamma(t,r)} + \frac{1}{c}\int_{\{\tau\in\Gamma(t,r):|f(\tau)|>|f|_{\Gamma(t,r)}\}} |f(\tau)|\log\frac{|f(\tau)|}{|f|_{\Gamma(t,r)}}\,d\nu(t)$$
$$\geq \frac{1}{c}\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\frac{|f(\tau)|}{|f|_{\Gamma(t,r)}}\Big)d\nu(\tau).$$

On the other hand,

$$\int_{\Gamma(t,r)} M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)d\nu(\tau) = \int_0^\infty \nu(\{\tau\in\Gamma(t,r): M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)>\lambda\})d\lambda$$
$$\approx \int_0^\infty \nu(\{\tau\in\Gamma(t,r): M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)>2\lambda\})d\lambda$$
$$= \int_0^{|f|_{\Gamma(t,r)}} \nu(\{\tau\in\Gamma(t,r): M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)>2\lambda\})d\lambda$$
$$+ \int_{|f|_{\Gamma(t,r)}}^\infty \nu(\{\tau\in\Gamma(t,r): M_\Gamma(f\,\chi_{\Gamma(t,r)})(\tau)>2\lambda\})d\lambda$$
$$\leq \nu(\Gamma(t,r))\,|f|_{\Gamma(t,r)} + c\int_{|f|_{\Gamma(t,r)}}^\infty \Big(\int_{\{\tau\in\Gamma(t,r):|f(\tau)|>\lambda\}} |f(\tau)|d\nu(t)\Big)\frac{d\lambda}{\lambda}$$
$$= \nu(\Gamma(t,r))\,|f|_{\Gamma(t,r)} + c\int_{\{\tau\in\Gamma(t,r):|f(\tau)|>|f|_{\Gamma(t,r)}\}} |f(\tau)|\log\frac{|f(\tau)|}{|f|_{\Gamma(t,r)}}\,d\nu(\tau)$$
$$\leq c\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\frac{|f(\tau)|}{|f|_{\Gamma(t,r)}}\Big)d\nu(\tau).$$

Therefore, for all $f\in L^1_{\rm loc}(\Gamma)$ we get

$$M_\Gamma\big(M_\Gamma f\big)(t) \approx \sup_{r>0}\nu(\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\frac{|f(\tau)|}{|f|_{\Gamma(t,r)}}\Big)d\nu(\tau). \tag{4.5}$$

Since

$$1 \leq (\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\frac{|f(\tau)|}{|f|_{\Gamma(t,r)}}\Big)d\nu(\tau),$$

then

$$|f|_{\Gamma(t,r)} \le \|f\|_{L(1+\log^+ L),\Gamma(t,r)}.$$

Using the inequality $\log^+(ab) \le \log^+ a + \log^+ b$ with $a, b > 0$, we get

$$(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\frac{|f(\tau)|}{|f|_{\Gamma(t,r)}}\Big)d\nu(\tau)$$
$$= (\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\Big(\frac{|f(\tau)|}{\|f\|_{L(1+\log^+ L),\Gamma(t,r)}}\frac{\|f\|_{L(1+\log^+ L),\Gamma(t,r)}}{|f|_{\Gamma(t,r)}}\Big)\Big)d\nu(\tau)$$
$$= (\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\frac{|f(t)|}{\|f\|_{L(1+\log^+ L),\Gamma(t,r)}}\Big)d\nu(\tau)$$
$$+ (\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} |f(\tau)|\log^+\frac{\|f\|_{L(1+\log^+ L),\Gamma(t,r)}}{|f|_{\Gamma(t,r)}}d\nu(\tau)$$
$$\le \|f\|_{L(1+\log^+ L),\Gamma(t,r)} + |f|_{\Gamma(t,r)}\log^+\frac{\|f\|_{L(1+\log^+ L),\Gamma(t,r)}}{|f|_{\Gamma(t,r)}}.$$

Since $\frac{\|f\|_{L(1+\log^+ L),\Gamma(t,r)}}{|f|_{\Gamma(t,r)}} \ge 1$ and $\log r \le r$ when $r \ge 1$, we get

$$(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\frac{|f(\tau)|}{|f|_{\Gamma(t,r)}}\Big)d\nu(\tau) \le 2\|f\|_{L(1+\log^+ L),\Gamma(t,r)}. \quad (4.6)$$

On the other hand, since

$$\|f\|_{L(1+\log^+ L),\Gamma(t,r)} = (\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\frac{|f(\tau)|}{\|f\|_{L(1+\log^+ L),\Gamma(t,r)}}\Big)d\nu(\tau),$$

on using

$$|f|_{\Gamma(t,r)} \le \|f\|_{L(1+\log^+ L),\Gamma(t,r)},$$

we get that

$$\|f\|_{L(1+\log^+ L),\Gamma(t,r)} \lesssim (\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} |f(\tau)|\Big(1+\log^+\frac{|f(\tau)|}{|f|_{\Gamma(t,r)}}\Big)d\nu(\tau). \quad (4.7)$$

Therefore, from (4.5), (4.6) and (4.7) we have (4.4).

Lemma 4.5 *Let Γ be a Carleson curve and $b \in BMO(\Gamma)$. Then there exists a positive constant C such that*

$$M_{b,\Gamma} f(t) \le C\|b\|_* M_\Gamma\big(M_\Gamma f\big)(t) \quad (4.8)$$

for almost every $t \in \Gamma$ and for all functions $f \in L^1_{\rm loc}(\Gamma)$.

Proof. Let $t \in \Gamma$ and fix a ball $\Gamma(t,r)$. We write f as $f = f_1 + f_2$, where $f_1(\tau) = f(\tau)\chi_{\Gamma(t,3r)}(\tau)$, $f_2(\tau) = f(\tau)\chi_{\complement_{(\Gamma(t,3r))}}(\tau)$, and $\chi_{\Gamma(t,3r)}$ denotes the characteristic function of $\Gamma(t,3r)$. Then for any $\tau \in \Gamma$

$$M_{b,\Gamma} f(\tau) = M_\Gamma\big((b-b(\tau))f\big)(\tau) = M_\Gamma\big((b-b_{\Gamma(t,3r)}+b_{\Gamma(t,3r)}-b(\tau))f\big)(\tau)$$
$$\le M_\Gamma\big((b-b_{\Gamma(t,3r)})f\big)(\tau) + M_\Gamma\big((b_{\Gamma(t,3r)}-b(\tau))f\big)(\tau)$$
$$\le M_\Gamma\big((b-b_{\Gamma(t,3r)})f_1\big)(\tau) + M_\Gamma\big((b-b_{\Gamma(t,3r)})f_2\big)(\tau) + \big|b_{\Gamma(t,3r)}-b(\tau)\big|M_\Gamma f(\tau).$$

For $0<\delta<1$ we have

$$\begin{aligned}
&\Big((\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}\big(M_{b,\Gamma}f(\tau)\big)^{\delta}d\nu(\tau)\Big)^{\frac{1}{\delta}}\\
&\le\Big((\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}\big(M_{\Gamma}\big((b-b_{\Gamma(t,3r)})f_1\big)(\tau)\big)^{\delta}d\nu(\tau)\Big)^{\frac{1}{\delta}}\\
&+\Big((\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}\big(M_{\Gamma}\big((b-b_{\Gamma(t,3r)})f_2\big)(\tau)\big)^{\delta}d\nu(\tau)\Big)^{\frac{1}{\delta}}\\
&+\Big((\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}\big|b(\tau)-b_{\Gamma(t,3r)}\big|\big(M_{\Gamma}f\big)(\tau)\big)^{\delta}d\nu(\tau)\Big)^{\frac{1}{\delta}}\\
&=I_1(t,r)+I_2(t,r)+I_3(t,r).
\end{aligned}$$

We first estimate I_1. Recall that M_Γ is weak-type $(1,1)$, (cf. [7]). We have

$$\begin{aligned}
I_1^{\delta}(t,r)&\le(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}\big|M_{\Gamma}\big((b-b_{\Gamma(t,3r)})f_1\big)(\tau)\big|^{\delta}d\nu(\tau)\\
&\le(\nu\Gamma(t,r))^{-1}\int_0^{\nu(\Gamma(t,r))}\big[\big(M_{\Gamma}\big((b-b_{\Gamma(t,3r)})f_1\big)\big)^{*}(t)\big]^{\delta}dt\\
&\le(\nu\Gamma(t,r))^{-1}\Big[\sup_{0<t<\nu(\Gamma(t,r))}t\,\big(M_{\Gamma}\big((b-b_{\Gamma(t,3r)})f_1\big)\big)^{*}(t)\Big]^{\delta}\int_0^{\nu(\Gamma(t,r))}t^{-\delta}dt\\
&\lesssim(\nu\Gamma(t,r))^{-1}\big\|(b-b_{\Gamma(t,3r)})\,f_1\big\|_{L^1(\Gamma)}^{\delta}\,\nu(\Gamma(t,r))^{-\delta+1}\\
&\lesssim\big\|(b-b_{\Gamma(t,3r)})\,f\big\|_{L^1(\Gamma(t,3r))}^{\delta}\,\nu(\Gamma(t,r))^{-\delta}.
\end{aligned}$$

Thus

$$I_1(t,r)\le(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,3r)}|b(\tau)-b_{\Gamma(t,3r)}|\,|f(\tau)|d\nu(\tau).$$

Then, by (4.1) and Lemmas 4.1 and 4.4 , we obtain

$$\begin{aligned}
I_1(t,r)&\le\|b-b_{\Gamma(t,3r)}\|_{\exp L,\Gamma(t,3r)}\,\|f\|_{L(\log L),\Gamma(t,3r)}\\
&\lesssim\|b\|_{*}\,\|f\|_{L(\log L),\Gamma(t,3r)}\\
&\le\|b\|_{*}\,M_{\Gamma}\big(M_{\Gamma}f\big)(t).
\end{aligned}$$

Let us estimate $I_2(t,r)$. Since for any points $\tau\in\Gamma(t,r)$, we have

$$M_{\Gamma}\big((b-b_{\Gamma(t,3r)})f_2\big)(\tau)\le CM_{\Gamma}\big((b-b_{\Gamma(t,3r)})f_2\big)(t)$$

with C an absolute constant (see, for example, [9, p. 160]).

Therefore, by (4.1) and Lemma 4.4 we obtain

$$\begin{aligned}
I_2(t,r) &\lesssim \Big((\nu\Gamma(t,r))^{-1}\int_{b_{\Gamma(t,r)}} \big(M_\Gamma\big((b-b_{\Gamma(t,3r)})f_2\big)(t)\big)^\delta d\nu(z)\Big)^{\frac{1}{\delta}} \\
&= M_\Gamma\big((b-b_{\Gamma(t,3r)})f_2\big)(t) \\
&= \sup_{r_0>0}\nu(\Gamma(t,r_0)^{-1}\int_{\Gamma(t,r_0)}|b(\tau)-b_{\Gamma(t,3r)}|\,|f_2(\tau)|d\nu(\tau) \\
&= \sup_{r_0>0}\nu(\Gamma(t,r_0))^{-1}\int_{\Gamma(t,r_0)\cap\complement(\Gamma(t,3r))}|b(\tau)-b_{\Gamma(t,3r)}|\,|f(\tau)|d\nu(\tau) \\
&= \sup_{r_0>3r}\nu(\Gamma(t,r_0))^{-1}\int_{\Gamma(t,r_0)\setminus\Gamma(t,3r)}|b(\tau)-b_{\Gamma(t,3r)}|\,|f(\tau)|d\nu(\tau) \\
&\le \sup_{r_0>3r}\nu(\Gamma(t,r_0))^{-1}\int_{\Gamma(t,r_0)\setminus\Gamma(t,3r)}|b(\tau)-b_{\Gamma(t,r_0)}|\,|f(\tau)|d\nu(\tau) \\
&+ \sup_{r_0>3r}\nu(\Gamma(t,r_0))^{-1}\,|b_{\Gamma(t,3r)}-b_{\Gamma(t,r_0)}|\int_{\Gamma(t,r_0)\setminus\Gamma(t,3r)}|f(\tau)|d\nu(\tau) \\
&\le \sup_{r_0>3r}\|b-b_{\Gamma(t,r_0)}\|_{\exp L,\Gamma(t,r_0)}\,\|f\|_{L(\log L),\Gamma(t,r_0)} \\
&+ \|b\|_*\sup_{r_0>3r}\log\frac{r_0}{r}\,\nu(\Gamma(t,r_0))^{-1}\,\|f\|_{L,\Gamma(t,r_0)\setminus\Gamma(t,3r)} \\
&\le \|b\|_*\sup_{r_0>3r}\|f\|_{L(\log L),\Gamma(t,r_0)} \\
&\lesssim \|b\|_*\,M_\Gamma\big(M_\Gamma f\big)(t).
\end{aligned}$$

Therefore we get

$$I_2(t,r)\lesssim\|b\|_*\,M_\Gamma\big(M_\Gamma f\big)(t).$$

Finally, for estimate I_3, applying Hölders inequality with exponent $a=1/\delta$, $0<\delta<1$, by Lemmas 4.2 for $\Phi(t)=t^a$, $1<a<\infty$ we get

$$\begin{aligned}
I_3(t,r) &\le \Big((\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}\big|b(\tau)-b_{\Gamma(t,3r)}\big|^a d\nu(\tau)\Big)^{\frac{1}{a}}(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}M_\Gamma f(\tau)d\nu(\tau) \\
&\lesssim \|b\|_*\,M\big(M_\Gamma f\big)(t).
\end{aligned}$$

Lemma 4.5 is proved by the estimate of $I_1(t,r)$, $I_2(t,r)$, $I_3(t,r)$ and the Lebesgue differentiation theorem.

Remark 4.1 Note that, in the case of the Carnot groups Lemma 4.5 was proved in [15].

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $M_{b,\eta}$ on $L^\Phi(\Gamma)$.

Theorem 4.1 *Let $b\in L^1_{\rm loc}(\Gamma)$, $\Phi\in\mathcal{Y}$ be a Young function and Γ be a Carleson curve.*

1. *If $\Phi\in\nabla_2$, then the condition $b\in BMO(\Gamma)$ is sufficient for the boundedness of $M_{b,\eta}$ on $L^\Phi(\Gamma)$.*
2. *If $\Phi\in\Delta_2$, then the condition $b\in BMO(\Gamma)$ is necessary for the boundedness of $M_{b,\eta}$ on $L^\Phi(\Gamma)$.*
3. *Let $\Phi\in\Delta_2\cup\nabla_2$. Then the condition $b\in BMO(\Gamma)$ is necessary and sufficient for the boundedness of $M_{b,\eta}$ on $L^\Phi(\Gamma)$.*

Proof. 1. Let $b \in BMO(\Gamma)$. Then from Lemma 4.8 we have

$$M_{b,\eta} f(t) \lesssim \|b\|_* \, M_\Gamma\big(M_\Gamma f\big)(t) \tag{4.9}$$

for almost every $t \in \Gamma$ and for all functions from $f \in L^1_{\rm loc}(\Gamma)$.

Combining Theorem 3.1 and Lemma 4.5 and from (4.9), we get

$$\begin{aligned}\|M_{b,\Gamma} f\|_{L^\Phi(\Gamma)} &\lesssim \|b\|_* \|M_\Gamma\big(M_\Gamma f\big)\|_{L^\Phi(\Gamma)} \\ &\lesssim \|b\|_* \|M_\Gamma f\|_{L^\Phi(\Gamma)} \lesssim \|b\|_* \|f\|_{L^\Phi(\Gamma)}.\end{aligned}$$

2. We shall now prove the second part. Suppose that $M_{b,\Gamma}$ is bounded on $L^\Phi(\Gamma)$. Choose any ball $\Gamma_0 = \Gamma(t,r)$ in Γ, by (2.1)

$$\begin{aligned}&\frac{1}{\nu(\Gamma_0)} \int_{\Gamma_0} |b(\tau) - b_{\Gamma_0}| d\nu(\tau) = \frac{1}{\nu(\Gamma_0)} \int_{\Gamma_0} \Big| \frac{1}{\nu(\Gamma_0)} \int_{\Gamma_0} (b(\tau) - b(z)) d\nu(z) \Big| d\nu(\tau) \\ &\le \frac{1}{\nu(\Gamma_0)^2} \int_{\Gamma_0} \int_{\Gamma_0} |b(\tau) - b(z)| d\nu(z) d\nu(\tau) \\ &= \frac{1}{\nu(\Gamma_0)} \int_{\Gamma_0} \frac{1}{\nu(\Gamma_0} \int_{\Gamma_0} |b(\tau) - b(z)| \chi_B(z) d\nu(z) d\nu(\tau) \\ &\le \frac{1}{\nu(\Gamma_0)} \int_{\Gamma_0} M_{b,\Gamma}\big(\chi_{\Gamma_0}\big)(\tau) d\nu(\tau) \le \frac{2}{\nu(\Gamma_0)} \|M_{b,\Gamma}\big(\chi_{\Gamma_0}\big)\|_{L^\Phi(\Gamma_0)} \|1\|_{L^{\tilde{\Phi}}(\Gamma_0)}\end{aligned}$$

Thus $b \in BMO(\Gamma)$.

3. The third statement of the theorem follows from the first and second parts of the theorem.

If we take $\Phi(t) = t^p$ in Theorem 4.1 we get the following corollary.

Corollary 4.1 *Let Γ be a Carleson curve, $1 < p < \infty$ and $b \in L^1_{\rm loc}(\Gamma)$. Then $M_{b,\Gamma}$ is bounded on $L^p(\Gamma)$ if and only if $b \in BMO(\Gamma)$.*

5 Commutators of fractional maximal operator in Orlicz spaces

In this section we find sufficient conditions for the boundedness of the commutator $[b, M_\Gamma]$ of the maximal operator on Orlicz spaces.

The following relations between $[b, M_\Gamma]$ and $M_{b,\Gamma}$ are valid :

Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\rm loc}(\Gamma)$ and $t \in \Gamma$ the following inequality is valid

$$\begin{aligned}&\big|[b, M_\eta] f(t)\big| = \big|b(t) M_\eta f(t) - M_\eta(bf)(t)\big| \\ &= \big|M_\eta(b(t)f)(t) - M_\eta(bf)(t)\big| \le M_\eta(|b(t) - b| f)(t) \le M_{b,\eta}(f)(t).\end{aligned}$$

If b is any locally integrable function on X, then

$$|[b, M_\Gamma] f(t)| \le M_{b,\Gamma}(f)(t) + 2b^-(t) M_\Gamma f(t), \qquad t \in \Gamma \tag{5.1}$$

holds for all $f \in L^1_{\rm loc}(\Gamma)$ (see, for example, [16,27]).

Obviously, operators $M_{b,\Gamma}$ and $[b, M_\Gamma]$ essentially differ from each other since $M_{b,\Gamma}$ is positive and sublinear and $[b, M_\Gamma]$ is neither positive nor sublinear.

From Lemma 4.5 and inequality (5.1) we get the following corollary.

Corollary 5.1 *Let $b \in BMO(\Gamma)$ such that $b^- \in L^\infty(\Gamma)$. Then there exists a positive constant C such that*

$$[b, M_\Gamma]f(t) \le C\big(\|b^+\|_* + \|b^-\|_{L^\infty}\big)M_\Gamma\big(M_\Gamma f\big)(t) \tag{5.2}$$

for almost every $t \in \Gamma$ and for all functions from $f \in L^1_{\rm loc}(\Gamma)$.

Proof. By Lemma 4.5, inequality (5.1) and the fact that $M_\Gamma f \le M_\Gamma\big(M_\Gamma f\big)$, we have

$$\begin{aligned}
|[b, M_\Gamma]f(t)| &\le M_{b,\Gamma}(f)(t) + 2b^-(t)M_\Gamma f(t)\\
&\lesssim \|b\|_* \, M\big(M_\Gamma f\big)(t) + b^-(t)M_\Gamma f(t)\\
&\lesssim \big(\|b^+\|_* + \|b^-\|_*\big)M_\Gamma\big(M_\Gamma f\big)(t) + \|b^-\|_{L^\infty(\Gamma)}M_\Gamma\big(M_\Gamma f\big)(t)\\
&\lesssim \big(\|b^+\|_* + \|b^-\|_{L^\infty(\Gamma)}\big)M_\Gamma\big(M_\Gamma f\big)(t).
\end{aligned}$$

By Corollary 5.1 and Theorem 4.1 we have

Corollary 5.2 *Let Γ be a Carleson curve, $b \in BMO(\Gamma)$, $b^- \in L^\infty(\Gamma)$ and Φ be a Young function with $\Phi \in \Delta_2 \cup \nabla_2$. Then the operator $[b, M_\Gamma]$ is bounded on $L^\Phi(\Gamma)$.*

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