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On the Neumann problem for a second order elliptic partial operator-differential equation in Hilbert space

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Abstract. *In the paper we study well-posedness of the Neumann problem for a second order partial operator-differential equation in Hilbert space. For solving this problem we use the Fourier transform method. At first we prove that the operator corresponding to the principal part of the equation is an isomorphism. Then we prove that the operator coefficients of the part that of the equation may be chosen so that the theorem on the solvability of a boundary value problem for a complete equation to hold. This time sufficient conditions for the solvability are expressed by means of the coefficient of this equation. This factor strongly distinguishes our study from previous works where the solvability conditions involve arbitrarily smallness of the perturbed part of the equation or they are expressed by means of restrictions on the growth of the resolvent of the appropriate operator pencil.*

Keywords. Hilbert space · operator · Neumann conditions · boundary value problem · Fourier transformation · isomorphism · well-posed solvability · Banach theorem · inversible operator · spectrum

Mathematics Subject Classification (2010): 35J25, 47A10, 58J40

1 Introduction

Study of solvability of the Cauchy problem or boundary value problems with operator coefficients allows to consider the solvability of similar problems for a system of differential and integral equations, partial differential equations, integro-differential equations from a single point of view. The bases of this theory was laid by the works of the known mathematicians E. Hille, K. Iosida, T. Kato, S. Agmon, P. Lax, Z.I. Khalilov and others. The monographs of S.G. Krein [8], A.A. Dezin [3], V.I. Gorbachuk and M.L. Gorbachuk [7], S.Ya. Yakubov [17] and other mathematicians were devoted to theory of solvability of operator-differential equations. In this direction we can note the papers of M.G. Gasymov [5], V.G. Mazya and B.A. Plamenevsky [9], Yu.A. Dubinsky [4], S.S. Mirzoev [10], A.A. Shkalikov [14], N.I. Yurchuk [18], A.R. Aliyev [1], etc.

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Compared to ordinary operator-differential equations very little papers were devoted to the solvability of partial operator-differential equations in Hilbert spaces. The solvability of some classes of degenerate partial operator-differential equations was considered in the papers of V.B. Shakhmurov [15], V.B. Shakhmurov and Azad A. Babayev [16], G.I. Aslanov [2], S.S. Mirzoyev, M.F. Ismayilova [11], S.S. Mirzoyev and Dzh.I. Jafarov [12], S.S. Mirzoyev and N.M. Suleymanov [13], R.F. Gatamova [6] and others.

Let H be a separable Hilbert space, C be a positive definite self-adjoint operator in H with domain of definition $D(C)$. The domain of definition of the operator $C^p\,(p \geq 0)$ becomes a Hilbert space H_p with respect to the scalar product $(x,y)_p = (C^p x, C^p y)$, $x, y \in D(C^p)$, $p \geq 0$. For $p = 0$ we assume that $H_0 = H$.

Let $\mathbb{R}^n_+ = \left\{x : x = (x_1, x_2, \ldots, x_{n-1}, x_n), x_i \in (-\infty, \infty), i = \overline{1, n-1}, x_n \in (0, \infty)\right\}$.

By $D\left(\mathbb{R}^n_+; H_2\right)$ we denote a linear set of infinitely differentiable vector-functions in $\mathbb{R}^n_+$ having compact supports in $\mathbb{R}^n_+$ with the values in H_2.

Let $L_2\left(\mathbb{R}^n_+; H\right)$ be a Hilbert space of all vector-functions $f(x) = f(x_1, \ldots, x_{n-1}, x_n)$ determined almost everywhere in $\mathbb{R}^n_+$ with the values in $H$with the norm

$$\|f\|_{L_2\left(\mathbb{R}^n_+;H\right)} = \left(\int_{\mathbb{R}^n_+} \|f(x_1, \ldots, x_n)\|_H^2 \, dx_1 \ldots dx_n\right)^{1/2} < \infty.$$

By $W_2^2\left(\mathbb{R}^n_+; H\right)$ we denote a completion of the linear set $D\left(\mathbb{R}^n_+; H\right)$ with respect to the norm

$$\|u\|_{W_2^2\left(\mathbb{R}^n_+;H\right)} = \left(\sum_{k=1}^{n} \left\|\frac{\partial^2 u(x)}{\partial x_k}\right\|^2_{L_2\left(\mathbb{R}^n_+;H\right)} + \left\|C^2 u(x)\right\|^2_{L_2\left(\mathbb{R}^n_+;H\right)}\right)^{1/2}.$$

In the similar way we determine the Hilbert space

$$W^2_{2,\xi}\left(\mathbb{R}^n_+; H\right)$$

$$= \left\{v(\xi, x_n) : \left(\sum_{k=1}^{n-1} |\xi_k|^2 + C^2\right) v(\xi, x_n) \in L_2\left(\mathbb{R}^n_+; H\right), \frac{\partial v(\xi, x_n)}{\partial x_n} \in L_2\left(\mathbb{R}^n_+; H\right)\right\}$$

with the norm

$$\|v\|_{W^2_{2,\xi}\left(\mathbb{R}^n_+;H\right)} =$$

$$= \left(\left\|\frac{\partial v(\xi, x_n)}{\partial x_n}\right\|^2_{L_2(\mathbb{R}_+;H)} + \sum_{k=1}^{n-1} \left\|\xi_k^2 v(\xi, x_n)\right\|^2_{L_2\left(\mathbb{R}^n_+;H\right)} + \left\|C^2 v(\xi, x_n)\right\|^2_{L_2\left(\mathbb{R}^n_+;H\right)}\right)^{1/2}.$$

By $\overset{\circ}{W}{}^2_{2,\xi}\left(\mathbb{R}^n_+; H\right)$ we denote

$$\overset{\circ}{W}{}^2_{2,\xi}\left(\mathbb{R}^n_+; H\right) = \left\{v(\xi, x_n) \in W^2_{2,\xi}\left(\mathbb{R}^n_+; H\right), \left.\frac{\partial v(\xi, x_n)}{\partial x_n}\right|_{x_n=0} = 0\right\}.$$

2 Problem statement

In Hilbert space H we consider the following boundary value problem for a second order elliptic partial operator-differential equation

$$Lu = -\sum_{k=1}^{n} a_k \frac{\partial^2 u(x)}{\partial x_k^2} + \sum_{k=1}^{n} R_k \frac{\partial u(x)}{\partial x_k} + Tu(x) + C^2 u(x) = f(x), \ x \in \mathbb{R}_+^n, \quad (2.1)$$

$$\left.\frac{\partial u(x)}{\partial x_n}\right|_{x_n=0} = 0, \quad (2.2)$$

where $(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x = (x_1, x_2, \ldots, x_{n-1}, x_n) \in \mathbb{R}_+^n$, $f(\cdot), u(\cdot)$ are vector functions determined in $\mathbb{R}_+^n$ almost everywhere, with the values in H, and the operator coefficients satisfy the conditions:

1. $a_k > 0, \ k = 1, 2, \ldots, n.$
2. The operators $Q_k = R_k C^{-1}$ $(k = 1, 2, \ldots, n)$ and $F = TC^{-2}$ are bounded in H.

Denote by

$$L_0 u = -\sum_{k=1}^{n} a_k \frac{\partial^2 u(x)}{\partial x_k^2} + C^2 u, \ \ u(\cdot) \in \overset{0}{W}{}_2^2 \left(\mathbb{R}_+^n; H\right),$$

$$L_1 u = \sum_{k=1}^{n} R_k \frac{\partial u(x)}{\partial x_k} + Tu, \ \ u(\cdot) \in \overset{0}{W}{}_2^2 \left(\mathbb{R}_+^n; H\right),$$

$$Lu = L_0 u + L_1 u, \ \ u(\cdot) \in \overset{0}{W}{}_2^2 \left(\mathbb{R}_+^n; H\right).$$

Definition 2.1 *If for all $f(\cdot) \in L_2\left(\mathbb{R}_+^n; H\right)$ there exists a vector-function $u(\cdot) \in W_2^2\left(\mathbb{R}_+^n; H\right)$ that satisfies equation* (2.1) *almost everywhere in $\mathbb{R}_+^n$ then the function $u(\cdot)$ is said to be a regular solution of equation* (2.1).

Definition 2.2 *If for any $f(\cdot) \in L_2\left(\mathbb{R}_+^n; H\right)$ there exists a regular solution of equation* (2.1) *that satisfies the boundary condition*

$$\left.\frac{\partial u(x)}{\partial x_n}\right|_{x_n=0} = 0$$

and the estimation

$$\|u\|_{W_2^2\left(\mathbb{R}_+^n; H\right)} \le const \cdot \|f\|_{L_2\left(\mathbb{R}_+^n; H\right)}$$

holds, then the problem (2.1), (2.2) *is said to be a correctly solvable problem.*

In this paper we find conditions on the coefficients of equation (2.1) that provide coercive solvability of problem (2.1), (2.2).

We represent the boundary value problem in the following form

$$Lu = \left(-a_n \frac{\partial^2 u(x)}{\partial x_n^2} + R_n \frac{\partial u(x)}{\partial x_n} + C^2 u(x)\right)$$

$$+ \left(-\sum_{k=1}^{n-1} a_k \frac{\partial^2 u(x)}{\partial x_k^2} + \sum_{k=1}^{n} R_k \frac{\partial u(x)}{\partial x_k} + Tu(x)\right) = f(x), \quad (2.3)$$

$$\left.\frac{\partial u(x)}{\partial x_n}\right|_{x_n=0} = 0. \tag{2.4}$$

Denote by $\hat{f}(\xi_1, \xi_2, \ldots \xi_{n-1}, x_n)$ the Fourier transform of the vector-function $f(x)$ with respect to the variables $x_1, \ldots x_{n-1}$, i.e.

$$\hat{f}(\xi_1, \xi_2, \ldots \xi_{n-1}, x_n)$$
$$= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} f(x_1, \ldots x_{n-1}, x_n)\, e^{-i(x_1\xi_1 + \ldots + x_{n-1}\xi_{n-1})}\, dx_1 \ldots dx_{n-1}.$$

Then after applying the Fourier transform, from problem (2.3), (2.4) we get the following boundary value problem

$$\hat{L}\hat{u}(\xi, x_n) = -a_n \frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2} + \left(R_n \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n} + C^2 \hat{u}(\xi, x_n) \right.$$
$$\left. + \sum_{k=1}^{n-1} a_k^2 \xi_k^2 \hat{u}(\xi, x_n) + \sum_{k=1}^{n-1} i R_k \hat{u}(\xi, x_n) + T\hat{u}(\xi, x_n) \right) = \hat{f}(\xi, x_n), \tag{2.5}$$

$$\left.\frac{\partial \hat{u}(\xi_1, \ldots, \xi_{n-1}, x_n)}{\partial x_n}\right|_{x_n=0} = 0, \tag{2.6}$$

where $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}_+ = (0, \infty)$, $\hat{u}(\xi, x_n)$ is a Fourier transform $u(x)$ with respect to the variables $x_1, x_2, \ldots x_{n-1}$.

Denote by $B(\xi) = \sum_{k=1}^{n-1} a_k \xi_k^2 E + C^2$, where E is a unit operator in the space H.

For any $\xi \in \mathbb{R}^{n-1}$ the operator-function $B(\xi)$ is a self-adjoint, positive-definite operator, and $B(\xi) \geq \mu_0 E$, where μ_0 is a lower bound of the spectrum of the operator C. Therefore, there exists the operator $B^{1/2}(\xi)$, where $D\left(B^{1/2}(\xi)\right) = D\left(C^2\right)$.

Thus, we can rewrite the boundary value problem (2.5), (2.6) in the following form:

$$\hat{L}\hat{u}(\xi, x_n) = \left(-a_n \frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2} + B(\xi)\hat{u}(\xi, x_n) \right) + \left(R_n \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n} \right.$$
$$\left. + \sum_{k=1}^{n-1} i\xi_k R_k \hat{u}(\xi, x_n) + T\hat{u}(\xi, x_n) \right) = f(\xi, x_n), \tag{2.7}$$

$$\left.\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right|_{x_n=0} = 0. \tag{2.8}$$

Denote by

$$\hat{L}_0 \hat{u}(\xi, x_n) = -a_n \frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2} + B(\xi)\hat{u}(\xi, x_n), \quad \hat{u}(\xi, x_n) \in W_{2,\xi}^2\left(\mathbb{R}_+^n; H\right), \tag{2.9}$$

$$\hat{L}_1 \hat{u}(\xi, x_n) = R_n \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n} + \sum_{k=1}^{n-1} i\xi_k R_k \hat{u}(\xi, x_n) + T\hat{u}(\xi, x_n) = f(\xi, x_n), \tag{2.10}$$

and

$$\hat{L}\hat{u}(\xi, x_n) = \hat{L}_0 \hat{u}(\xi, x_n) + \hat{L}_1 \hat{u}(\xi, x_n), \hat{u}(\xi, x_n) \in \overset{0}{W}{}_2^2\left(\mathbb{R}_+^n; H\right).$$

In the paper [6] it is proved that the operator $\hat{L}_0$ isomorphically maps the space $W_{2,\xi}^2\left(\mathbb{R}_+^n; H_2\right)$ onto the space $L_2\left(\mathbb{R}_+^n; H\right)$. Now we prove the following auxiliary lemma.

3 Proof of some auxiliary theorems

We now prove some auxiliary suppositions.

Lemma 3.1. For any $\hat{u}\left(\xi,x_n\right)\in \overset{0}{W}{}^{2}_{2,\xi}\left(\mathbb{R}^n_+;H\right)$ we have the inequality

$$\left\|-a_n\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2}+B(\xi)\hat{u}(\xi,x_n)\right\|^2_{L_2\left(\mathbb{R}^n_+;H\right)}=a_n^2\left\|\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2}\right\|^2_{L_2\left(\mathbb{R}^n_+;H\right)}$$

$$+\left\|B(\xi)\hat{u}(\xi,x_n)\right\|^2_{L_2\left(\mathbb{R}^n_+;H\right)}+2\left\|B^{1/2}(\xi)\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}\right\|^2_{L_2\left(\mathbb{R}^n_+;H\right)}. \tag{3.1}$$

Proof. Let $\xi\in\mathbb{R}^{n-1}$, $x_n\in\mathbb{R}_+$. Then bearing in mind the condition $\frac{\partial\hat{u}(\xi,0)}{\partial x_n}=0$ we have:

$$\left\|-a_n\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2}+B(\xi)\hat{u}(\xi,x_n)\right\|^2_{L_2(\mathbb{R}_+;H)}=a_n^2\left\|\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2}\right\|^2_{L_2(\mathbb{R}_+;H)}$$

$$+\left\|B(\xi)\hat{u}(\xi,x_n)\right\|^2_{L_2(\mathbb{R}_+;H)}-2Re\left(\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2},B(\xi)\hat{u}(\xi,x_n)\right)_{L_2(\mathbb{R}_+;H)}. \tag{3.2}$$

On the other hand,

$$\left(\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2},B(\xi)\hat{u}(\xi,x_n)\right)_{L_2(\mathbb{R}_+;H)}=\int_0^\infty\left(\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2},B(\xi)\hat{u}(\xi,x_n)\right)_H dx_n$$

$$=\left(B^{\frac{3}{4}}(\xi)\hat{u}(\xi,x_n),B^{\frac{1}{4}}(\xi)\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}\right)\Bigg|_0^\infty$$

$$-\int_0^\infty\left(B^{\frac{1}{2}}(\xi)\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n},B^{1/2}(\xi)\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}\right)_H dx_n$$

$$=-\int_0^\infty\left\|B^{\frac{1}{2}}(\xi)\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n}\right\|^2_H dx_n=-\left\|B^{\frac{1}{2}}(\xi)\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}\right\|^2_{L_2(\mathbb{R}_+;H)}. \tag{3.3}$$

Then from (3.2) we get

$$\left\|-a_n\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2}+B(\xi)\hat{u}(\xi,x_n)\right\|^2_{L_2(\mathbb{R}_+;H)}=a_n^2\left\|\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2}\right\|_{L_2(\mathbb{R}_+;H)}$$

$$+2\left\|B^{1/2}(\xi)\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}\right\|^2_{L_2(\mathbb{R}_+;H)}+\left(B(\xi)\hat{u}(\xi,x_n)\right)^2_{L_2(\mathbb{R}_+;H)}.$$

Then, integrating this inequality with respect to $\xi\in\mathbb{R}^{n-1}$, we get the statement of the lemma.

We have the following theorem.

Theorem 3.1 *For any* $\hat{u}(\xi, x_n) \in \overset{\circ}{W}{}^{2}_{2,\xi}\left(\mathbb{R}^n_+; H\right)$ *we have the following inequalities*

$$\left\|B(\xi)\,\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)} \le \left\|\hat{L}_0\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}, \tag{3.4}$$

$$\left\|B^{1/2}(\xi)\,\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right\|_{L_2\left(\mathbb{R}^n_+;H\right)} \le \frac{1}{2\sqrt{a_n}}\left\|L_0\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}, \tag{3.5}$$

$$\left\|\xi_j C\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)} \le \frac{1}{2\sqrt{a_j}}\left\|L_0\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}, \tag{3.6}$$

$$\left\|C^2\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)} \le \left\|L_0\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}. \tag{3.7}$$

Proof. Inequality (3.4) follows from lemma 3.1. We prove inequality (3.5). Let $\hat{u}(\xi, x_n) \in \overset{\circ}{W}{}^{2}_{2,\xi}\left(\mathbb{R}^n_+; H\right)$. Then integrating by parts, we get:

$$a_n\int_0^\infty \left\|B^{1/2}(\xi)\frac{\partial u(\xi, x_n)}{\partial x_n}\right\|^2 dx_n = a_n\int_0^\infty \left(B^{\frac{1}{2}}(\xi)\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}, B^{1/2}(\xi)\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right) dx_n$$

$$= a_n\left(B^{\frac{3}{4}}(\xi)\hat{u}(\xi, x_n), B^{\frac{1}{4}}(\xi)\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right)\Bigg|_0^\infty - a_n\int_0^\infty \left(B(\xi)\hat{u}(\xi, x_n), \frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2}\right) dx_n$$

$$= -a_n\left(B(\xi)\hat{u}(\xi, x_n), \frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2}\right)_{L_2(\mathbb{R}_+;H)}$$

$$\le a_n\left\|B(\xi)\hat{u}(\xi, x_n)\right\|_{L_2(\mathbb{R}_+;H)} \cdot \left\|\frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2}\right\|_{L_2(\mathbb{R}_+;H)}.$$

Then for any $\varepsilon > 0$ we have

$$a_n\left\|B(\xi)\hat{u}(\xi, x_n)\right\|_{L_2(\mathbb{R}_+;H)} \cdot \left\|\frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2}\right\|_{L_2(\mathbb{R}_+;H)}$$

$$\le a_n\left(\frac{\varepsilon}{2}\left\|B(\xi)\hat{u}(\xi, x_n)\right\|^2_{L_2(\mathbb{R}_+;H)} + \frac{1}{2\varepsilon}\left\|\frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2}\right\|^2_{L_2(\mathbb{R}_+;H)}\right). \tag{3.8}$$

Assuming $\varepsilon = a^{-1}$, we get

$$a_n\left\|B^{1/2}(\xi)\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right\|^2_{L_2(\mathbb{R}_+;H)}$$

$$\le \frac{1}{2}\left(a_n^2\left\|\frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2}\right\|^2_{L_2(\mathbb{R}_+;H)} + \left\|B(\xi)\hat{u}(\xi, x_n)\right\|^2_{L_2(\mathbb{R}_+;H)}\right). \tag{3.9}$$

Integrating the last inequality with respect to $\xi \in \mathbb{R}^{n-1}$, we get:

$$a_n\left\|B(\xi)\frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2}\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}$$

$$\leq \frac{1}{2}\left(a_n^2\left\|\frac{\partial^2\hat{u}(\xi,x_n)}{\partial x_n^2}\right\|_{L_2\left(\mathbb{R}_+^n;H\right)}+\|B(\xi)\hat{u}(\xi,x_n)\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}\right).$$

By Lemma 3.1 we have :

$$a_n^2\left\|\frac{\partial^2\hat{u}(\xi,x_n)}{\partial x_n^2}\right\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}+\|B(\xi)\hat{u}(\xi,x_n)\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}$$

$$=\left\|\hat{L}_0u(\xi,x_n)\right\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}-2a_n\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}. \quad (3.10)$$

From (3.9) and (3.10) it follows

$$a_n\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}$$

$$\leq\frac{1}{2}\|L_0\hat{u}(\xi,x_n)\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}-a_n\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2\left(\mathbb{R}_+^n;H\right)},$$

i.e.

$$2a_n\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}\leq\frac{1}{2}\left\|\hat{L}_0\hat{u}(\xi,x_n)\right\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}.$$

Hence

$$\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}\leq\frac{1}{4a_n}\left\|\hat{L}\hat{u}(\xi,x_n)\right\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}.$$

We finally get:

$$\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|_{L_2\left(\mathbb{R}_+^n;H\right)}\leq\frac{1}{2\sqrt{a_n}}\left\|\hat{L}\hat{u}(\xi,x_n)\right\|_{L_2\left(\mathbb{R}_+^n;H\right)}.$$

Thus, we get the validity of inequality (3.5).
We now prove inequality (3.6). Let $j=1,2,\ldots n-1$.
Then we have:

$$\|\xi_jC\hat{u}(\xi,x_n)\|_{L_2\left(\mathbb{R}_+^n;H\right)}=\left\|\xi_jCB^{-1}(\xi)\,B(\xi)\,\hat{u}(\xi,x_n)\right\|_{L_2\left(\mathbb{R}_+^n;H\right)}$$

$$\leq\sup_{\xi}\left\|\xi_jCB^{-1}(\xi)\right\|\cdot\|B(\xi)\,\hat{u}(\xi,x_n)\|^2_{L_2\left(\mathbb{R}_+^n;H\right)}. \quad (3.11)$$

By inequality (3.4)

$$\|B(\xi)\,\hat{u}(\xi,x_n)\|_{L_2\left(\mathbb{R}_+^n;H\right)}\leq\left\|\hat{L}_0\hat{u}(\xi,x_n)\right\|_{L_2\left(\mathbb{R}_+^n;H\right)},$$

while

$$\left\|\xi_jCB^{-1}(\xi)\right\|=\sup_{\mu\in\sigma(c)}\left|\xi_j\mu\left(\sum_{k=1}^{n-1}a_k\xi_k^2+\mu^2\right)^{-1}\right|$$

$$\leq\sup_{\mu\in\sigma(c)}\left|\xi_j\mu\left(a_j\xi_j+\mu^2\right)^{-1}\right|=\sup_{\mu\in\sigma(c)}\frac{1}{a_j^{1/2}}\cdot\left|\left(a_j^{\frac{1}{2}}\xi_j\right)\mu\left(a_j\xi_j^2+\mu^2\right)^{-1}\right|$$

$$\leq \sup_{\mu\in\sigma(c)} \frac{1}{a_j^{1/2}} \cdot \left| \frac{1}{2} \left(a_j\xi_j^2 + \mu^2\right) \left(a_j\xi_j^2 + \mu^2\right)^{-1} \right| = \frac{1}{2\sqrt{a_j}}.$$

Therefore, it follows from inequality (3.11) that

$$\|\xi_j C\hat{u}(\xi, x_n)\|_{L_2\left(\mathbb{R}_+^n;H\right)} \leq \frac{1}{2\sqrt{a_j}} \left\|\hat{L}_0\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}_+^n;H\right)},$$

$j = 1, 2, \ldots n-1$. Thus, inequality (3.6) also is proved. At last we prove inequality (3.7). Using the above given inequality (3.4), we get:

$$\left\|C^2\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}_+^n;H\right)} \leq \left\|C^2 B^{-1}(\xi) B(\xi) \hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}_+^n;H\right)}$$

$$\leq \sup_{\xi} \left\|C^2 B^{-1}(\xi)\right\| \cdot \|B(\xi)\hat{u}(\xi, x_n)\|_{L_2\left(\mathbb{R}_+^n;H\right)}$$

$$\leq \sup_{\xi} \left\|C^2 \cdot B^{-1}(\xi)\right\| \cdot \|L_0\hat{u}(\xi, x_n)\|_{L_2\left(\mathbb{R}_+^n;H\right)}.$$

Since for any $\xi \in \mathbb{R}^{n-1}$

$$\left\|C^2 B^{-1}(\xi)\right\| \leq \sup_{\mu\in\tau(c)} \left| \mu^2 \left(\sum_{k=1}^{n-1} a_k\xi_k^2 + \mu^2 \right)^{-1} \right| \leq 1,$$

we have

$$\left\|C^2\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}_+^n;H\right)} \leq \left\|\hat{L}_0\hat{u}(\xi, x_n)\right\|_{L_2\left(\mathbb{R}_+^n;H\right)}.$$

This complete the proof of theorem 3.1.

4 Main results

We now prove the main theorem of the present paper.

Theorem 4.1 *If the coefficients of differential equation* (2.1) *satisfy conditions 1)-2) and it hold the inequality*

$$q = \frac{1}{2} \sum_{k=1}^{n} \|Q_k\| + \|F\| < 1,$$

where $Q_k = R_k C^{-1}, F = T \cdot C^{-2}$*, then the operator* $\hat{L}$ *isomorphically maps the space* $\overset{\circ}{W}{}^2_{2,\xi}\left(\mathbb{R}_+^n; H\right)$ *onto the space* $L_2\left(\mathbb{R}_+^n; H\right)$.

Proof. Above we obtained that the operator $\hat{L}$ is represented in the form $\hat{L} = \hat{L}_0 + \hat{L}_1$. Since the operator $\hat{L}_0$ isomorphically maps the space $\overset{\circ}{W}{}^2_{2,\xi}\left(\mathbb{R}_+^n; H\right)$ onto $L_2\left(\mathbb{R}_+^n; H\right)$, then after change of $\hat{L}_0\hat{u}(\xi, x_n) = \hat{\omega}(\xi, x_n)$ we get the following equation in $L_2\left(\mathbb{R}_+^n; H\right)$:

$$\hat{\omega}(\xi, x_n) + \hat{L}_1 \cdot \hat{L}_0^{-1}\hat{\omega}(\xi, x_n) = \hat{f}(\xi, x_n),$$

i.e.

$$\left(E + \hat{L}_1 \cdot \hat{L}_0^{-1}\right) \hat{\omega}(\xi, x_n) = \hat{f}(\xi, x_n),$$

where E is a unit operator in the space $L_2\left(\mathbb{R}_+^n; H\right)$.

Since $\hat{L}_0 : \overset{\circ}{W}{}^2_{2,\xi}\left(\mathbb{R}^n_+;H\right) \to L_2\left(\mathbb{R}^n_+;H\right)$ is an isomorphism, then for any $\hat{u}\left(\xi,x_n\right) \in L_2\left(\mathbb{R}^n_+;H\right)$ we have:

$$\left\|\hat{L}_1\hat{L}_0^{-1}\hat{\omega}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}$$
$$=\left\|\hat{L}_1\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)} \le \left\|R_n\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}$$
$$+\sum_{k=1}^{n-1}\left\|\xi_kR_k\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}+\left\|Tu\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}$$
$$\le\left\|R_nB^{-1/2}\left(\xi\right)B^{1/2}\left(\xi\right)\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}$$
$$+\sum_{k=1}^{n-1}\left\|\xi_kR_kC^{-1}C\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}+\left\|TC^{-2}\cdot C^2\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}.$$

Using theorem 4.1, we get:

$$\left\|\hat{L}_1\hat{L}_0^{-1}\hat{\omega}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}$$
$$\le\sup_\xi\left\|R_nC^{-1}CB^{-\frac{1}{2}}\left(\xi\right)\right\|\cdot\left\|B^{\frac{1}{2}}\left(\xi\right)\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}$$
$$+\sum_{k=1}^{n-1}\left\|R_kC^{-1}\right\|\cdot\left\|C\xi_k\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}+\left\|F\right\|\cdot\left\|C^2\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}$$
$$\le\left\|Q_n\right\|\cdot\left\|CB^{-\frac{1}{2}}\left(\xi\right)\right\|\cdot\frac{1}{2\sqrt{a_n}}\left\|\hat{L}_0\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}$$
$$+\sum_{k=1}^{n-1}\left\|Q_k\right\|\frac{1}{2\sqrt{a_k}}\left\|\hat{L}_0\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}+\left\|F\right\|\ \left\|L_0\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(R^n;H\right)}$$
$$\le\left(\frac{1}{2}\sum_{k=1}^{n}\frac{\left\|Q_k\right\|}{\sqrt{a_k}}+\left\|F\right\|\right)\left\|\hat{L}_0\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}=q\ \left\|\hat{\omega}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}.$$

Here we used the inequality

$$\sup_\xi\left\|CB^{-1/2}\left(\xi\right)\right\|=\sup_\xi\sup_{\mu\in\sigma(c)}\left|\mu\left(\sum_{k=1}^{n-1}a_k\xi_k^2+\mu^2\right)^{-1}\right|\le 1.$$

By the theorem condition $0<q<1$. Therefore, there exists an inverse operator $\left(E+\hat{L}_1\hat{L}_0^{-1}\right)^{-1}$ and $\hat{\omega}\left(\xi,x_n\right)=\left(E+\hat{L}_1\hat{L}_0^{-1}\right)^{-1}f\left(\xi,x_n\right)$, i.e.

$$\hat{u}\left(\xi,x_n\right)=\hat{L}_0^{-1}\left(E+\hat{L}_1\hat{L}_0^{-1}\right)^{-1}\hat{f}\left(\xi,x_n\right).$$

Hence it follows

$$\left\|\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}\le const\left\|\hat{f}\left(\xi,x_n\right)\right\|_{L_2\left(\mathbb{R}^n_+;H\right)}. \tag{4.1}$$

Theorem 4.1 is proved.

From this theorem it follows

Theorem 4.2 *Let all the conditions of Theorem 4.1 be fulfilled. Then problem* (2.1),(2.2) *is correctly solvable.*

Proof. From Theorem 4.1 it follows that the equation $\hat{L}_0 u(\xi, x_n) = f(\xi, x_n)$ is solvable for all $\hat{f}(\xi, x_n) \in L_2(\mathbb{R}^n_+; H)$. Since the Fourier transform is a unitary operator, then the solution of the equation $Lu(x) = f(x)$ can be found by means of the inverse transformation with respect to the variables $(x_1, \ldots x_{n-1})$. From inequality (4.1) it follows

$$\|u\|_{L_2(\mathbb{R}^n_+;H)} \leq const \|f\|_{L_2(\mathbb{R}^n_+;H)} .$$

This means correct solvability of problem (2.1), (2.2). Theorem 4.2 is proved.

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