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Oscillatory integrals with variable Calderón-Zygmund kernel on generalized weighted Morrey spaces

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Abstract. *In this paper, we investigate the boundedness of the oscillatory singular integrals with variable Calderón-Zygmund kernel on the generalized weighted Morrey spaces $M^{p,\varphi}(w)$. When w the weights are in the Muckenhoupt class A_p, $1 < p < \infty$ and $(\varphi_1, \varphi_2, w)$ satisfies some conditions, we show that the oscillatory singular integral operators T_λ and T_λ^* are bounded from $M^{p,\varphi_1}(w)$ to $M^{p,\varphi_2}(w)$. Meanwhile, the corresponding result for the oscillatory singular integrals with standard Calderón-Zygmund kernel are established.*

Keywords. Generalized weighted Morrey space, oscillatory integral, variable Calderón-Zygmund kernels.

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1 Introduction and main results

The classical Morrey spaces were introduced by Morrey [21] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [4,20,22] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [5,6,24]); Komori and Shirai [18] defined weighted Morrey spaces $L^{p,\kappa}(w)$; Guliyev [8] gave a concept of the generalized weighted Morrey spaces $M^{p,\varphi}(w)$ which could be viewed as extension of both $M^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,\kappa}(w)$. In [8], the boundedness of the classical operators and their commutators in spaces $M_w^{p,\varphi}$ was also studied, see also [1,3,10–12,14,16,17].

The spaces $M^{p,\varphi}(w)$ defined by the norm

$$\|f\|_{M_w^{p,\varphi}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-1/p}\, \|f\|_{L^p(B(x,r),w)},$$

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where the function φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w is a non-negative measurable function on $\mathbb{R}^n$. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $w(B(x,r)) = \int_{B(x,r)} w(y)dy$.

Suppose that K is the standard Calderón-Zygmund kernel. That is, $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $-n$, and $\int_{S^{n-1}} K(x)d\sigma(x) = 0$, where $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. The oscillatory integral operator T_λ is defined by

$$T_\lambda f(x) = p.v. \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} K(x-y)\varphi(x,y)f(y)dy, \tag{1.1}$$

where $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the space of infinitely differentiable functions on $\mathbb{R}^n \times \mathbb{R}^n$ with compact supports, and Φ is a real-analytic function or a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0,y_0) \in \operatorname{supp}\varphi$, there exists $(j_0,k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2\Phi(x_0,y_0)/\partial x_{j_0}\partial y_{k_0}$ does not vanish up to infinite order. These operators have arisen in the study of singular integrals supported on lower dimensional varieties, and the singular Radon transform. In [23], Y. B. Pan proved that T_λ are uniformly in λ bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Let $K(x,y)$ be a variable Calderón-Zygmund kernel. That means, for a. e. $x \in \mathbb{R}^n$, $K(x,\cdot)$ is a standard Calderón-Zygmund kernel and

$$\max_{|j| \le 2n, j \in \mathbb{N}_0^n} \left\| \frac{\partial^{|j|} k}{\partial y^j} \right\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} = A < \infty. \tag{1.2}$$

Define the oscillatory integral operator with variable Calderón-Zygmund kernel T_λ^* by

$$T_\lambda^* f(x) = p.v \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} K(x,x-y)\varphi(x,y)f(y)dy, \tag{1.3}$$

where λ, φ and Φ satisfy the same assumptions as those in the operator defined by (1.1).

S. Z. Lu and D. C. Yang etc. [19] investigated the L^p boundedness about this class of oscillatory integral operators. The boundedness of some operators on these spaces can be see ([2,4,6,7,20–22,25,26]). Recently, A. Eroglu [15] obtained the boundedness of a class of oscillatory integral with Calderón-Zygmund kernel and polynomial phase on generalized Morrey spaces. In [13] Guliyev etc. proved the generalized Morrey spaces $M^{p,\varphi}$ boundedness of T_λ defined by (1.1).

The purpose of this paper is to generalize the results above to the case with real - $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ or analytic phase functions. Our main results in this paper are formulated as follows.

Theorem 1.1 *Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0,y_0) \in \operatorname{supp}\varphi$, there exists $(j_0,k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2\Phi(x_0,y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume K is a standard Calderón- Zygmund kernel and T_λ is defined as in* (1.1)*. Then for any $1 \le p < \infty$, and (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \le C\,\varphi_2(x,r), \tag{1.4}$$

where C does not depend on x and r, the operator T_λ is bounded from $M^{p,\varphi_1}(w)$ to $M^{p,\varphi_2}(w)$ for $p > 1$ and from $M^{p,\varphi_1}(w)$ to $WM^{p,\varphi_2}(w)$ for $p \ge 1$.

Theorem 1.2 *Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp}\varphi$, there exists $(j_0, k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2\Phi(x_0, y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume K is a variable Calderón- Zygmund kernel and T_λ^* is defined as in* (1.3). *Then for any $1 \le p < \infty$, (φ_1, φ_2) satisfies the condition* (1.4), *the operator T_λ^* is bounded from $M^{p,\varphi_1}(w)$ to $M^{p,\varphi_2}(w)$ for $p > 1$ and from $M^{p,\varphi_1}(w)$ to $WM^{p,\varphi_2}(w)$ for $p \ge 1$.*

Note that for $\varphi_1(x, r) = \varphi_1(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, from Theorems 1.1 and 1.2 we get the following results, which proved in [23].

Corollary 1.1 *Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp}\varphi$, there exists $(j_0, k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2\Phi(x_0, y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume K is a standard Calderón- Zygmund kernel and T_λ is defined as in* (1.1). *Then for any $1 \le p < \infty$, and $0 < \kappa < 1$, the operator T_λ is bounded on $L^{p,\kappa}(w)$ for $p > 1$ and from $L^{p,\kappa}(w)$ to $WL^{p,\kappa}(w)$ for $p \ge 1$.*

Corollary 1.2 *Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp}\varphi$, there exists $(j_0, k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2\Phi(x_0, y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume K is a standard Calderón- Zygmund kernel and T_λ^* is defined as in* (1.3). *Then for any $1 \le p < \infty$, and $0 < \kappa < 1$, the operator T_λ^* is bounded on $L^{p,\kappa}(w)$ for $p > 1$ and from $L^{p,\kappa}(w)$ to $WL^{p,\kappa}(w)$ for $p \ge 1$.*

2 Notations and preliminary Lemmas

Let $B = B(x_0, r)$ be the ball with the center x_0 and radius r. Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B.

If w is a weight function, we denote by $L^p(w) \equiv L^p(\mathbb{R}^n, w)$ the weighted Lebesgue space defined by the norm

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty, \quad \text{when } 1 \le p < \infty$$

and by $\|f\|_{L^\infty(w)} = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} |f(x)|w(x)$ when $p = \infty$.

We recall that a weight function w is in the Muckenhoupt class A_p, $1 < p < \infty$, if

$$\begin{aligned}[w]_{A_p} &:= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} < \infty,\end{aligned}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all ball B by Hölder's inequality

$$[w]_{A_p(B)}^{1/p} = |B|^{-1}\|w\|_{L^1(B)}^{1/p}\|w^{-1/p}\|_{L^{p'}(B)} \ge 1. \tag{2.1}$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \le Cw(x)$ with $[w]_{A_1} = \sup\limits_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$ we define $A_\infty = \bigcup_{1\le p<\infty} A_p$.

Our argument based heavily on the following results.

Lemma 2.1 *[19] Assume T_λ is defined as in* (1.1). *Then for any $1 < p < \infty$ and $w \in A_p$, we have*

$$\|T_\lambda f\|_{L^p(w)} \le C(n, p, \Phi, \varphi, C_{p,w})\, C_1\, \|f\|_{L^p(w)},$$

where $C(n, p, \Phi, \varphi, C_{p,w})$ is independent of λ, K, f and $C_1 = \|k\|_{C^1(S^{n-1})}$.

Lemma 2.2 *[19] Assume T_λ^* is defined as in* (1.3). *Then for any $1 < p < \infty$ and $w \in A_p$, we have*

$$\|T_\lambda^* f\|_{L^p(w)} \le C(n, p, \Phi, \varphi, C_{p,w})\, A\, \|f\|_{L^p(w)},$$

where $C(n, p, \Phi, \varphi, C_{p,w})$ is independent of λ, K, f and A is defined in (1.2).

The generalized weighed Morrey spaces introduced by Guliyev in [8] are defined as follows.

Definition 2.1 *Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M^{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^p_{\text{loc}}(w)$ with finite norm*

$$\|f\|_{M^{p,\varphi}_w} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1}\, w(B(x, r))^{-\frac{1}{p}}\, \|f\|_{L^p(B(x,r),w)},$$

where

$$\|f\|_{L^p(B(x,r),w)} = \left(\int_{B(x,r)} |f(y)|^p w(y) dy \right)^{\frac{1}{p}}.$$

Furthermore, $WM^{p,\varphi}(w)$ is the weak generalized weighted Morrey space of all functions $f \in WL^p_{\text{loc}}(w)$ for which

$$\|f\|_{WM^{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1}\, w(B(x, r))^{-\frac{1}{p}}\, \|f\|_{WL^p(B(x,r),w)} < \infty,$$

where $WL^p(B(x, r), w)$ denotes the weak $L^p(w)$-space of measurable functions f for which

$$\|f\|_{WL^p(B(x,r),w)} \equiv \|f \chi_{B(x,r)}\|_{WL^p(w)} = \sup_{t > 0} t \left(\int_{\{y \in B(x,r) :\, |f(y)| > t\}} w(y) dy \right)^{\frac{1}{p}}.$$

Remark 2.1 (1) If $w \equiv 1$, then $M^{p,\varphi}(1) = M^{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, then $M^{p,\varphi}(w) = L^{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x, r) \equiv v(B(x, r))^{\frac{\kappa}{p}} w(B(x, r))^{-\frac{1}{p}}$, then $M^{p,\varphi}(w) = L^{p,\kappa}(v, w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M^{p,\varphi}(w) = L^{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM^{p,\varphi}(w) = WL^{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x, r) \equiv w(B(x, r))^{-\frac{1}{p}}$, then $M^{p,\varphi}(w) = L^p(\mathbb{R}^n, w)$ is the weighted Lebesgue space.

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy, \quad f \in L^1_{\rm loc}(\mathbb{R}^n).$$

A distribution kernel K is called a standard Calderòn-Zygmund kernel (SCZK) if it satisfies the following hypotheses:

$$|K(x,y)| \le \frac{C}{|x-y|^n}, \forall x \ne y,$$

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \le \frac{C}{|x-y|^{n+1}}, \forall x \ne y,$$

where C does not depend on x and y. The corresponding Calderòn-Zygmund integral operator S and oscillatory integral operator R are defined by

$$Sf(x) = p.v. \int_{\mathbb{R}^n} K(x,y) f(y) dy.$$

and

$$Rf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y) dy,$$

where $P(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$.

Theorem 2.1 *[8] Let* $1 \le p < \infty$ *and* (φ_1, φ_2) *satisfy the condition* (1.4). *Then the maximal operator* M *and the singular integral operator* T *are bounded from* $M^{p,\varphi_1}(w)$ *to* $M^{p,\varphi_2}(w)$ *for* $p > 1$ *and from* $M^{p,\varphi_1}(w)$ *to* $WM^{p,\varphi_2}(w)$ *for* $p \ge 1$.

Corollary 2.1 *Let* $1 \le p < \infty$ *and* (φ_1, φ_2) *satisfy the condition* (1.4). *If* S *is of type* (L^2, L^2), *then for any real polynomial* $P(x,y)$, *the operator* R *is bounded from* $M^{p,\varphi_1}(w)$ *to* $M^{p,\varphi_2}(w)$ *for* $p > 1$ *and from* $M^{p,\varphi_1}(w)$ *to* $WM^{p,\varphi_2}(w)$ *for* $p \ge 1$.

Remark 2.2 Note that, in the case $w \equiv 1$ Corollary 2.1 were proved in [15].

Lemma 2.3 *[27] Denote by* $\mathcal{H}_m$ *the spaces of spherical harmonic functions of degree* m. *Then*

(a) $L^2(S^{n-1}) = \oplus_{m=0}^{\infty} \mathcal{H}_m$, *and* $g_m = dim \mathcal{H}_m \le C(n) m^{n-2}$ *for any* $m \in \mathbb{N}$,

(b) for any $m = 0, 1, 2, ...$, *there exists an orthogonal system* $\{Y_{jm}\}_{j=1}^{g_m}$ *of* $\mathcal{H}_m$ *such that* $\|Y_{jm}\|_{L^\infty(S^{n-1})} \le C(n) m^{n/2-1}, Y_{jm} = (-m)^{-n}(m+n-2)^{-n} \Lambda^n Y_{jm}, j = 1, ..., g_m$, *and* Λ *is the Beltrami-Laplace operator on* S^{n-1}.

In the following the letter C will denote a constant which may vary at each occurrence.

3 Proof of Theorems 1.1 and 1.2

We will use the following results on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty g(s) w(s) ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [9].

Theorem 3.1 *[9] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{3.1}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$D := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = D$ is the best constant for (3.1).

The following lemma is valid.

Lemma 3.1 *Let $1 \le p < \infty$, $w \in A_p$ and T_λ is defined as in* (1.1).
Then, for $1<p<\infty$ the inequality

$$\|T_\lambda f\|_{L^p(B,w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p} \frac{dt}{t}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.
Moreover, for $p=1$ the inequality

$$\|T_\lambda f\|_{WL^1(B,w)} \lesssim w(B) \int_{2r}^\infty \|f\|_{L^1(B(x_0,t),w)}\, w(B(x_0,t))^{-1} \frac{dt}{t} \tag{3.2}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L^1_{\rm loc}(\mathbb{R}^n,w)$.

Proof. Let $p \in (1,\infty)$ and $w \in A_p$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y)$$

and have

$$\|T_\lambda f\|_{L^p(B,w)} \le \|T_\lambda f_1\|_{L^p(B,w)} + \|T_\lambda f_2\|_{L^p(B,w)}.$$

It is known that (see Lemma 2.1) the operator T_λ is bounded on $L^p(w)$. Since $f_1 \in L^p(w)$, $T_\lambda f_1 \in L^p(w)$ and boundedness of T_λ in $L^p(w)$ (see [19]) it follows that

$$\|T_\lambda f_1\|_{L^p(B,w)} \le \|T_\lambda f_1\|_{L^p(w)} \le C\|f_1\|_{L^p(w)} = C\|f\|_{L^p(2B,w)},$$

where the constant $C>0$ is independent of f.

We now estimate $T_\lambda f_2$. We can write

$$\Big|T_\lambda f_2(x)\Big| = \Big|\int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K(x-y)\varphi(x,y) f(y) dy\Big|.$$

Now by an argument similar to the proof of Lemma 6 in [19], we choose $\phi_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\phi_1(x) \equiv 1$ when $|x| \le 1$, and $\phi_1(x) \equiv 0$ when $|x| > 2$. Let $\phi_2 = 1 - \phi_1$ and $N \in \mathbb{N}$ which is large enough and will be determined later. Write

$$K(x) = K_\lambda^1(x) + K_\lambda^2(x),$$

where

$$K_\lambda^j(x) = K(x)\phi_j(\lambda^{1/N} x), \; j = 1,2.$$

Then

$$T_\lambda f_2(x) = p.v. \int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K_\lambda^1(x-y)\varphi(x,y)f(y)dy$$
$$+ p.v. \int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K_\lambda^2(x-y)\varphi(x,y)f(y)dy := T_\lambda^1 f_2(x) + T_\lambda^2 f_2(x).$$

Let us first estimate $T_\lambda^1 f_2(x)$. To do so, using Taylor's expansion and the compactness of $\operatorname{supp}\varphi$, we write

$$\Phi(x,y) = \Phi(x,x) + P(x,y) + r_N(x,y)$$

for $(x,y) \in \operatorname{supp}\varphi$, where $P(x,y)$ is a polynomial with deg $P < N$ and $|r_N(x,y)| \le C|x-y|^N$ with C in dependent of x and y. Define

$$Rf(x) = p.v. \int_{\complement_{(2B)}} e^{i\lambda P(x,y)} K_\lambda^1(x-y)\varphi(x,y)f(y)dy.$$

Therefore

$$e^{-i\lambda\Phi(x,x)} T_\lambda^1 f_2(x) - Rf(x)$$
$$= \int_{B(x,2\lambda^{-1/N})} e^{i\lambda P(x,y)}[e^{i\lambda r_N(x,y)} - 1] K_\lambda^1(x-y)\varphi(x,y)f(y)dy$$
$$= \sum_{j=0}^{\infty} \int_{B(x,2^{-j+1}\lambda^{-1/N})\setminus B(x,2^{-j}\lambda^{-1/N})} e^{i\lambda P(x,y)}[e^{i\lambda r_N(x,y)} - 1] K_\lambda^1(x-y)\varphi(x,y)f(y)dy$$
$$\equiv \sum_{j=0}^{\infty} T_{\lambda j}^1 f_2(x).$$

On $T_{\lambda j}^1 f_2(x)$, by the properties of r_N and k, we have

$$|T_{\lambda j}^1 f_2(x)| \le C 2^{-jN} Mf(x).$$

So we have

$$|T_\lambda^1 f_2(x)| \le C \sum_{j=0}^{\infty} 2^{-jN} Mf(x) + C|Rf(x)| \le CMf(x) + C|Rf(x)|.$$

By Theorem 3.1 in [8], we have

$$\|T_\lambda^1 f_2\|_{L^p(B(x_0,r),w)} \lesssim \|Mf\|_{L^p(B(x_0,r),w)} + \|Rf\|_{L^p(B(x_0,r),w)}$$
$$\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p} \frac{dt}{t}.$$

Now, let us turn to estimate $T_\lambda^2 f_2(x)$. We consider the following two cases.

Case 1. $\lambda \le 1$. Similar to that estimate of T_λ^2 in Lemma 6 in [19], we have

$$|T_\lambda^2 f_2(x)| \le CMf(x),$$

where the constant $C > 0$ is independent of f. By Theorem 3.1 in [8] we have

$$\|T_\lambda^2 f_2\|_{L^p(B(x_0,r),w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p} \frac{dt}{t}.$$

Case 2. $\lambda > 1$. We choose $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp} \varphi_0 \subseteq \{x \in \mathbb{R}^n : 1 < |x| \le 2\},$$

and

$$\phi_2(x) = \sum_{j=0}^{\infty} \varphi_0(2^{-j}x).$$

Let

$$K_{\lambda,j}^2(x) = K(x)\, \varphi_0(2^{-j}\lambda^{1/N}x).$$

Then

$$\begin{aligned} T_\lambda^2 f_2(x) &= \int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K_\lambda^2(x-y)\varphi(x,y)f(y)dy \\ &= \sum_{j=0}^{\infty} \int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K_{\lambda,j}^2(x-y)\varphi(x,y)f(y)dy \\ &\equiv \sum_{j=0}^{\infty} T_{\lambda,j}^2 f_2(x). \end{aligned}$$

For $T_{\lambda,j}^2$, by the definition of it, we can get

$$|T_\lambda^2 f_2(x)| \le C \int_{B(x,2^{-j+1}\lambda^{-1/N})\setminus B(x,2^{-j}\lambda^{-1/N})} \frac{|f(y)|}{|x-y|^n}\, dy \le CMf(x). \tag{3.3}$$

The inequality (3.3) also can be see in [19], we omit the detail here.
By Theorem 3.1 in [8], we have

$$\|T_\lambda^2 f_2\|_{L^p(B(x_0,r),w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p}\, \frac{dt}{t}.$$

Therefore

$$\begin{aligned} \|T_\lambda f_2\|_{L^p(B(x_0,r),w)} &\le \|T_\lambda^1 f_2\|_{L^p(B(x_0,r),w)} + \|T_\lambda^2 f_2\|_{L^p(B(x_0,r),w)} \\ &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p}\, \frac{dt}{t}. \end{aligned}$$

This finishes the proof of Lemma 3.1.

Proof of Theorem 1.1.

By Lemma 3.1 and Theorem 3.1 we get

$$\begin{aligned} \|T_\lambda f\|_{M^{p,\varphi_2}(w)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L^p(B(x,t),w)}\, w(B(x,t))^{-1/p}\, \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-1/p}\, \|f\|_{L^p(B(x,r),w)} = \|f\|_{M^{p,\varphi_1}(w)}. \end{aligned}$$

This finishes the proof of Theorem 1.1.

The following lemma is valid.

Lemma 3.2 *Let $1 \le p < \infty$ and T^*_λ is defined as in* (1.3).
Then, for $1 < p < \infty$ the inequality

$$\|T^*_\lambda f\|_{L^p(B,w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.
Moreover, for $p = 1$ the inequality

$$\|T^*_\lambda f\|_{WL^1(B,w)} \lesssim w(B) \int_{2r}^{\infty} \|f\|_{L^1(B(x_0,t),w)}\, w(B(x_0,t))^{-1} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, w)$.

Proof. Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and radius r, $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y)$$

and have

$$\|T^*_\lambda f\|_{L^p(B,w)} \le \|T^*_\lambda f_1\|_{L^p(B,w)} + \|T^*_\lambda f_2\|_{L^p(B,w)}.$$

It is known that (see Lemma 2.2) the operator T^*_λ is bounded on $L^p(w)$. Since $f_1 \in L^p(w)$, $T^*_\lambda f_1 \in L^p(w)$ and boundedness of T^*_λ in $L^p(w)$ (see [19]) it follows that

$$\|T^*_\lambda f_1\|_{L^p(B,w)} \le \|T^*_\lambda f_1\|_{L^p(w)} \le C\|f_1\|_{L^p(w)} = C\|f\|_{L^p(2B,w)},$$

where the constant $C > 0$ is independent of f.

We now estimate $T^*_\lambda f_2$. For each $m \in \mathbb{N}$ and $j = 1, \ldots, g_m$, we get

$$a_{jm}(x) = \int_{S^{n-1}} \Omega(x,z) Y_{jm}(z) d\sigma_z,$$

where $\Omega(x,z) = |z|^n K(x,z)$. Then for $a.e. x \in \mathbb{R}^n$,

$$\Omega(x,z) = \sum_{m=1}^{\infty} \sum_{j=1}^{g_m} a_{jm}(x) Y_{jm}(z'), \tag{3.4}$$

where $z' = z/|z|$ for any $z \in \mathbb{R}^n \backslash \{0\}$. By Lemma 2.3, we have that for any $x \in \mathbb{R}^n$,

$$\begin{aligned} |a_{jm}(x)| &= m^{-n}(m+n-2)^{-n} \left| \int_{S^{n-1}} \Omega(x,z) \Lambda^n Y_{jm}(z) d\sigma_z \right| \\ &= m^{-n}(m+n-2)^{-n} \left| \int_{S^{n-1}} \Lambda^n \Omega(x,z) Y_{jm}(z) d\sigma_z \right| \\ &\le C(n) A m^{-2n}. \end{aligned} \tag{3.5}$$

By Lemma 2.3 again, we can verify that for any $\epsilon > 0, N \in \mathbb{N}$, and a.e. $x \in \mathbb{R}^n$, if $|y - x| \ge \epsilon$, then

$$\left| \sum_{m=1}^{N} \sum_{j=1}^{g_m} e^{i\lambda \Phi(x,y)} \frac{a_{jm}(x) Y_{jm}((x-y)')}{|x-y|^n} \varphi(x,y) f_2(y) \right| \le C(\epsilon) A |f_2(y)|. \tag{3.6}$$

Therefore, from (3.4), (3.6) and the Lebesgue dominated convergence theorem, it follows that

$$
\begin{aligned}
T_\lambda^* f_2(x) &= \lim_{\epsilon\to 0}\int_{\complement_{B(x,\varepsilon)}} e^{i\lambda\Phi(x,y)}K(x,x-y)\varphi(x,y)f_2(y)dy\\
&= \lim_{\epsilon\to 0}\sum_{m=1}^{\infty}\sum_{j=1}^{g_m}\int_{\complement_{B(x,\varepsilon)}} e^{i\lambda\Phi(x,y)}\frac{a_{jm}(x)Y_{jm}((x-y)')}{|x-y|^n}\varphi(x,y)f_2(y)dy\\
&= \lim_{\epsilon\to 0}\sum_{m=1}^{\infty}\sum_{j=1}^{g_m}a_{jm}(x)\int_{\complement_{B(x,\varepsilon)}} e^{i\lambda\Phi(x,y)}\frac{Y_{jm}((x-y)')}{|x-y|^n}\varphi(x,y)f_2(y)dy.
\end{aligned}
$$

We write

$$
R_{jm}f_2(x) = \int_{\complement_{B(x,\varepsilon)}} e^{i\lambda\Phi(x,y)}\frac{Y_{jm}((x-y)')}{|x-y|^n}\varphi(x,y)f_2(y)dy.
$$

It is easy to see that $R_{jm}f_2(x)$ is the oscillatory integral operator defined by (1.1). By Theorem 1.1 we have R_{jm} bounded from $M^{p,\varphi_1}(w)$ to $M^{p,\varphi_2}(w)$. Therefore, by (3.5) and the above discussion we have

$$
\|T_\lambda^* f_2\|_{L^p(B(x_0,r),w)} \lesssim w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L^p(B(x_0,t),w)}\,w(B(x_0,t))^{-1/p}\,\frac{dt}{t}.
$$

This finishes the Lemma 3.2.

Proof of Theorem 1.2.

By Lemma 3.2 and Theorem 3.1 we get

$$
\begin{aligned}
\|T_\lambda^* f\|_{M^{p,\varphi_2}(w)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0}\varphi_2(x,r)^{-1}\int_r^{\infty}\|f\|_{L^p(B(x,t),w)}\,w(B(x,t))^{-1/p}\,\frac{dt}{t}\\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0}\varphi_1(x,r)^{-1}\,w(B(x,r))^{-1/p}\,\|f\|_{L^p(B(x,r),w)} = \|f\|_{M^{p,\varphi_1}(w)}.
\end{aligned}
$$

This finishes the proof of Theorem 1.2.

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