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The Jost solutions to the Schrödinger equation with an additional complex potential

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Abstract. *We consider the differential equation* $-y'' + xe^{ix}y + q(x)y = k^2y$. *Using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. Estimates for the kernels of the transformation operators are obtained.*

Keywords. Schrödinger equation · non-self-adjoint differential operator · the space $L_2(-\infty, +\infty)$ · transformation operator · the Jost solution.

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1 Introduction and main results

In many aspects of the theory of inverse problems of spectral analysis, an important role is played by so-called transformation operators. The latter first appeared in the theory of generalized translation operators of J. Delsarte [1] and B.M. Levitan [5]. For arbitrary Sturm-Liouville equations, transformation operators were constructed by A.Ya. Povzner [9]. V.A. Marchenko [6] used transformation operators for studying inverse spectral problems and the asymptotic behavior of the spectral function of the singular Sturm-Liouville operator. It should be remarked that in the effective solution of various inverse problems of scattering theory, an important role is played by the transformation operators with a conditional which were discovered by B.Ya. Levin [4] Similar problems for the Schrödinger equation with unbounded potentials were considered in [3, 8, 10].

We consider the differential equation

$$-y'' + xe^{ix}y + q(x)y = k^2y, \tag{1.1}$$

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where $q(x)$ is a continuously differentiable function with bounded support and is a complex parameter. If $q(x) = 0$, then from [2], the equation (1.1) has unique solution $f_0(x, k)$, which can be given as a series

$$f_0(x,k) = e^{ikx} + \sum_{n=1}^{\infty}\sum_{s=0}^{n} p_{ns}(k)\, x^s e^{i(n+k)x}. \tag{1.2}$$

Here $p_{ns}(k)$ is a regular rational function with poles at the points $k = -\frac{j}{2}, j = 1, 2, ..., n$ and multiplicities at most $j + 1$, while the series (1.2) admits term-by-term differentiation with respect to x any number of times for $k \neq -\frac{n}{2}, n = 1, 2, ...$ It was proved in [2] (see also [7]), for any k with $Imk > 0$, the function $f_0(x, k)$ belongs to $L_2(0, +\infty)$ and the function belongs to $L_2(-\infty, 0)$. Moreover, the functions $f_0(x, k)$ and $f_0(x, -k)$ form the fundamental system of solutions of equation (1.1) for $k \neq 0$ when $q(x) = 0$.

This paper is devoted to the study of the solutions of (1.1) with asymptotic conditions

$$f_{\pm}(x,k) = f_0(x, \pm k) + o(1),\, x \to \pm\infty.$$

We shall derive the integral representation, which is usually called the Jost translation representation between $f_{\pm}(x, k)$ and $f_0(x, \pm k)$. The obtained results can be used to study the spectral properties of the non-self-adjoint differential operator L, generated by the differential expression $l(y) = -y'' + xe^{ix}y + q(x)y$ in the space $L_2(-\infty, +\infty)$.

The main result of the present paper is as follows.

Theorem 1.1 *For any $k \neq -\frac{n}{2}, n = 1, 2, ...$ from the complex plane, equation (1.1) has solutions $f_+(x, k)$ and $f_-(x, k)$, which can be represented in the form*

$$f_+(x,k) = f_0(x,k) + \int_x^{+\infty} K(x,t)\, f_0(t,k)\, dt \tag{1.3}$$

and

$$f_-(x,k) = f_0(x,-k) + \int_{-\infty}^{x} A(x,t)\, f_0(t,-k)\, dt. \tag{1.4}$$

Moreover,

$$K(x,x) = \frac{1}{2}\int_x^{+\infty} q(t)\, dt, \tag{1.5}$$

$$A(x,x) = \frac{1}{2}\int_{-\infty}^{x} q(t)\, dt. \tag{1.6}$$

2 Proof of the theorem

Without loss of generality, we consider the case " + " and assume that $x \geq 0$. We shall use the following notation

$$p(x) = xe^{ix}, \sigma(x) = \frac{1}{2}\int_x^{+\infty} |q(t)|\, dt.$$

We first consider the following lemmas before turning to the proof of the theorem.

Lemma 2.1 *If $q(x)$ is a continuously differentiable function with bounded support, then the integral equation*

$$U(\xi_0, \eta_0) = \frac{1}{2}\int_{\xi_0}^{+\infty} q(\xi)\, d\xi$$

$$+\int_0^{\eta_0}\int_{\xi_0}^{+\infty} \left[p(\xi-\eta) - p(\xi+\eta) + q(\xi-\eta)\right] U(\xi,\eta)\, d\xi d\eta \tag{2.1}$$

has one and only one solution $U(\xi_0, \eta_0)$. Furthermore, if $q(x) = 0$ when $x > a$, then

$$U(\xi_0, \eta_0) = 0 \text{ when } \xi_0 \geq a. \tag{2.2}$$

Proof. Using the method of successive approximation, let

$$U_0(\xi_0, \eta_0) = \frac{1}{2}\int_{\xi_0}^{+\infty} q(\xi)\, d\xi, \tag{2.3}$$

$$U_n(\xi_0, \eta_0) = \int_0^{\eta_0}\int_{\xi_0}^{+\infty} \left[p(\xi-\eta) - p(\xi+\eta) + q(\xi-\eta)\right] U_{n-1}(\xi,\eta)\, d\xi d\eta. \tag{2.4}$$

Because the function $q(x)$ with bounded support, there exists an $a > 0$ such that $q(x) = 0$ for $x > a$. By induction with respect to n , we have

$$U_n(\xi_0, \eta_0) \text{ for } \xi_0 > 2a, n = 0, 1, 2, ... \tag{2.5}$$

For any $R > 0$, suppose that $0 < \eta_0 < R$, $0 < \xi_0 < +\infty$. By (2.3), we have

$$|U(\xi_0, \eta_0)| \leq \sigma(\xi_0).$$

Taking the notation

$$M = \max_{\substack{0\leq\xi\leq 2a \\ 0\leq\eta\leq R}} |p(\xi-\eta) - p(\xi+\eta) + q(\xi-\eta)|$$

into account, we obtain

$$|U_1(\xi_0, \eta_0)| \leq \sigma(\xi_0)(M\eta_0).$$

Using induction, by (2.4) we next prove that

$$|U_n(\xi_0, \eta_0)| \leq \sigma(\xi_0)\frac{1}{n!}(M\eta_0)^n. \tag{2.6}$$

Hence the series

$$U(\xi_0, \eta_0) = \sum_{n=0}^{\infty} U_n(\xi_0, \eta_0) \tag{2.7}$$

is uniformly and absolutely convergent, so $U(\xi_0, \eta_0)$ is the solution of the integral equation (2.1). From (2.6) and (2.7), it follows that

$$|U(\xi_0, \eta_0)| \leq \sigma(\xi_0)\exp(M\eta_0). \tag{2.8}$$

This implies obviously the uniqueness of the solution to the equation (2.1). The assertion (2.2) is justified by (2.5) and (2.7).

Lemma 2.2 *Suppose $q(x)$ is a continuously differentiable function with bounded support. Then the solution $U(\xi_0, \eta_0)$ of the integral equation (2.1) satisfies the following differential equation*

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial \xi_0 \partial \eta_0} + [p(\xi-\eta) - p(\xi+\eta) + q(\xi-\eta)]\, U(\xi_0,\eta_0) = 0 \tag{2.9}$$

and

$$U(\xi_0, 0) = \frac{1}{2}\int_{\xi_0}^{+\infty} q(\xi)\, d\xi. \tag{2.10}$$

Proof. From (2.1) the differentiability of $U(\xi_0,\eta_0)$ is evident. Differentiating equation (2.1) directly, we get the equation (2.9). Putting $\xi_0 = 0$ in (2.1), we get the result (2.10). We now let $\xi_0 = \frac{t+x}{2}, \eta_0 = \frac{t-x}{2}$ and express the function $K(x,t) = U(\xi_0,\eta_0)$ as a function of x, t . Then the function $K(x,t)$ is twice continuously differentiable. Moreover, from the two preceding lemmas we get the following lemma.

Lemma 2.3 *Suppose $q(x)$ is a continuously differentiable function with bounded support. Then the function $K(x,t) = U\left(\frac{t+x}{2}, \frac{t-x}{2}\right)$ satisfies both the differential equation*

$$\frac{\partial^2 K(x,t)}{\partial x^2} - [p(x) + q(x)]\, K(x,t) = \frac{\partial^2 K(x,t)}{\partial t^2} - p(t)\, K(x,t) \tag{2.11}$$

and the condition

$$K(x,x) = \frac{1}{2}\int_x^{+\infty} q(t)\, dt.$$

Furthermore, if $q(x) = 0$ when $x > a$, then $K(x,t) = 0$ when $x + t > 2a$.

Now the theorem can be proved. By differentiation from (1.3), we have

$$f_+'(x,k) = f_0'(x,k) - K'(x,x)\, f_0(x,k) + \int_x^{+\infty} K_x'(x,t)\, f_0(t,k)\, dt \tag{2.12}$$

$$\begin{aligned} f_+''(x,k) = f_0''(x,k) - \frac{dK(x,x)}{dx} f_0(x,k) - K(x,x)\, f_0'(x,k) \\ -K_x'(x,t)\, f_0(x,k) + \int_x^{+\infty} K_{xx}''(x,t)\, f_0(t,k)\, dt. \end{aligned} \tag{2.13}$$

From Lemma 2.3, it is easily seen that when t sufficiently large, $K(x,t) = 0$, so the last terms of (1.3), (2.12), (2.13) are integrable. From

$$-f_0''(x,k) + p(x)\, f_0(x,k) = k^2 f_0(x,k) \tag{2.14}$$

and (1.3), we have

$$\begin{aligned} k^2 f_+(x,k) = k^2 f_0(x,k) \\ + \int_x^{+\infty} K(x,t)\, p(t)\, f_0(t,k)\, dt - \int_x^{+\infty} K(x,t)\, f_0''(t,k)\, dt. \end{aligned} \tag{2.15}$$

Hence, integrating by parts, we obtain

$$\int_x^{+\infty} K(x,t)\, f_0''(t,k)\, dt = -K(x,x)\, f_0'(x,k) - \int_x^{+\infty} K_t'(x,t)\, f_0'(t,k)\, dt$$

$$= -K(x,x)\, f_0'(x,k) + K_t'(x,x)\, f_0(x,k) + \int_x^{+\infty} K_{tt}''(x,t)\, f_0(t,k)\, dt. \qquad (2.16)$$

By virtue of (1.3) and (2.13)-(2.16), we have

$$-f_+''(x,k) + p(x)\, f_+(x,k) - k^2 f_+(x,k)$$

$$= \int_x^{+\infty} \left[K_{tt}''(x,t) - K_{xx}''(x,t) + K(x,t)\,(p(x) + q(x) - p(t))\right] f_0(t,k)\, dt$$

$$+ \left[2\frac{dK(x,x)}{dx} + q(x)\right] f_0(x,k).$$

From the lemma 2.3 and the last relation, $f_+(x,k)$ satisfies equation (1.1). Furthermore, by virtue of (2.8)-(2.14), it follows that $f_+(x,k) = f_0(x,k)$ when x sufficiently large. Hence, the $f_+(x,k)$ is a Jost solution. Thus, the proof of the theorem is complete.

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