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Maximal function and fractional integral associated with Laplace-Bessel differential operators on generalized Morrey spaces

Ali Akbulut, Monsef Dziri, Ismail Ekincioglu*

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Abstract. *In this work we study the boundedness of the maximal and fractional integral operators associated with Laplace-Bessel differential operators on generalized Morrey spaces.*

Keywords. Maximal function, fractional integral, Laplace-Bessel differential operator, generalized Morrey space.

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1 Introduction

The study of the g-Littlewood-Paley theory enjoys a naturel motivation and arises a great interest. Many works and topic have been studied. For Euclidian analysis it is investigated by Stein. In his study of these operators, Stein uses two approaches. The first is the theory of singular integrals in the context of Hilbert space-valued functions, and the second in the theory of harmonics functions. Next, these operators play an important role in questions related to multipliers, Sobelev spaces and Hardy spaces.

The classical Morrey spaces were originally introduced by Morrey in [24] to study the local behavior of solutions of second-order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [1,24]. Guliyev, Mizuhara and Nakai [8,25,26] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see also [9,10,28]).

Over the past 20 years considerable effort has been made to extend the classical operators of harmonic analysis on the Bessel-Kingman hypergroups, the Laguerre hypergroups, the Chebli-Trimeche hypergroups, and complete Riemannian manifolds.

* Corresponding author

A. Akbulut
Ahi Evran University, Department of Mathematics, Kirsehir, Turkey
E-mail: aakbulut@ahievran.edu.tr

M. Dziri
Department of Mathematics, Faculty of Sciences of Bizerte, Tunisia
E-mail: Moncef.Dziri@iscae.rnu.tn

I. Ekincioglu
Istanbul Medeniyet University, Faculty of Engineering and Natural Sciences, Istanbul, Turkey
Dumlupinar University, Department of Mathematics, Kutahya, Turkey
E-mail: ismail.ekincioglu@dpu.edu.tr

In this paper we consider harmonic analysis associated with the following system of partial differential operators

$$\begin{cases} D_j &= \frac{\partial}{\partial x_j}, \ 1 \le j \le n, \\ \Delta_{n,\alpha} &= \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + \sum_{j=1}^n \frac{\partial^2}{\partial {x_j}^2}, \ (r,x) \in]0,\infty[\times\mathbb{R}^n. \end{cases} \tag{1.1}$$

Some problems of harmonic analysis that are associated with Laplace-Bessel operator $\Delta_{n,\alpha}$ are investigated [5,7,3,12–14,19]. In [7] was proved the (L^p, L^q)-boundedness of the B-potentials and in [13] the author proved the (L^p, L^q)-boundedness of the B-fractional maximal operators, and proved Sobolev theorem in a limit case. The maximal operator, fractional integral operator and related topics associated with the Laplace-Bessel differential operator $\Delta_{n,\alpha}$ have been investigated by many researchers, see B. Muckenhoupt and E. Stein [23], I. Kipriyanov [21], K. Trimeche [31], L. Lyakhov [19], K. Stempak [30], A.D. Gadjiev and I.A. Aliev [3,7], I.A. Aliev and S. Bayrakci [2], V.S. Guliyev [12,13], V.S. Guliyev and J.J. Hasanov [15], V.S. Guliyev, A. Serbetci and I. Ekincioglu [16], A. Serbetci and I. Ekincioglu [27] and others.

In this paper we consider the generalized shift operator, generated by the LaplaceBessel differential operator $\Delta_{n,\alpha}$ in terms of which the B-maximal operator and the B-Riesz potential are investigated in the generalized B-Morrey space. We obtain sufficient conditions for the operator I_α^β to be bounded from B-Morrey space $M_{p,\varphi,\alpha}$ to $M_{q,\varphi,\alpha}$ and from B-Morrey space $M_{1,\varphi,\alpha}$ to weak B-Morrey space $WM_{q,\varphi,\alpha}$. Also, using the boundedness of the B-maximal operator on generalized B-Morrey spaces and the same techniques as [13] we have defined and studty the boundedness of the g-function on generalized B-Morrey spaces.

The article is organized as follows: In section 2 we include definitions and auxiliary results of harmonic analysis associated with the Laplace-Bessel differential operator. In section 3 we define the generalized B-Morrey spaces and the B-maximal function we give also some results linked with B-Morrey spaces. In section 4 the boundedness of the B-maximal operator on B-Morrey spaces $M_{p,\varphi,\alpha}$ is proved. The main result of the paper is the inequality of Sobolev-Morrey type for the B-Riesz potentials, established in Section 5. Section 6 deals with the boundedness of the g-function in generalized B-Morrey spaces. Throughout the paper C denotes a positive constant whose value may vary from line to line.

2 Harmonic analysis related with $\Delta_{n,\alpha}$

In this section we recall first basic definition and some facts. We consider the system of partial differential operators

$$\begin{cases} D_j &= \frac{\partial}{\partial x_j}, \ 1 \le j \le n, \\ \Delta_{n,\alpha} &= l_\alpha + \Delta. \end{cases} \tag{2.1}$$

Where l_α is the Bessel operator with respect to the first variable r given by

$$l_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}$$

and Δ is the Laplacian operator on $\mathbb{R}^n$, $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial {x_j}^2}$. On the other hand if $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n$ and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, we put $< \lambda, x >= \sum_{i=1}^n \lambda_i x_i, \ \|\lambda\| = \sqrt{\langle\lambda,\lambda\rangle}$.

By [13, 14] we have

Proposition 2.1 *For $(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n$, the following system of equations*

$$D_j v(r,x) = -i\lambda_j v(r,x),$$

$$\Delta_{n,\alpha} v(r,x) = -(\mu^2+\lambda^2)v(r,x),$$

$$v(0,0)=1;\ \frac{\partial v}{\partial r}(0,x)=0. \tag{2.2}$$

has a unique infinitely differentiable solution on $\mathbb{R}\times\mathbb{R}^n$ even with respect to the first variable given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha(r\mu)e^{-i<\lambda,x>}, \tag{2.3}$$

where

$$j_\alpha(s) = \frac{2^\alpha\Gamma(\alpha+1)J_\alpha(s)}{s^\alpha},\ \ \text{if}\ \ s\neq 0, 1,\ \ \text{if}\ \ s=0$$

with J_α is the Bessel function of first kind and order α.

We have for all $(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n$,

$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n}|\varphi_{\mu,\lambda}(r,x)| = 1.$$

The shift operator $\mathcal{T}_{(r,x)}$ associated with Laplace Bessel operator $\Delta_{n,\alpha}$ is defined on the space of continuous functions even with respect to the first variable by

$$\mathcal{T}_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta}, x+y)\sin^{2\alpha}\theta d\theta. \tag{2.4}$$

Denote by
- $d\nu_\alpha(r,x)$ the measure defined on $[0,\infty[\times\mathbb{R}^n$ by

$$d\nu_\alpha(r,x) = r^{2\alpha+1}dr\otimes dx. \tag{2.5}$$

- $L_{p,\alpha}(\mathbb{R}^{n+1}_+)$, $1\le p\le\infty$, the space of measurable functions f on $[0,\infty[\times\mathbb{R}^n$ satisfying

$$\|f\|_{p,\alpha} = \Big(\int_{\mathbb{R}^n}\int_0^\infty |f(r,x)|^p d\nu_\alpha(r,x)\Big)^{\frac{1}{p}} < \infty,\ \ \text{for } 1\le p<\infty$$

and

$$\|f\|_{\infty,\alpha} = \|f\|_\infty = esssup_{(r,x)\in[0,\infty[\times\mathbb{R}^n}|f(r,x)| < \infty \text{ for } p=\infty.$$

It is naturel to define the convolution product generated by the shift operator.

Definition 2.1 *The convolution product associated with $\Delta_{n,\alpha}$ of f,g in $L_{1,\alpha}(\mathbb{R}^{n+1}_+)$ is defined by the following $\forall(r,x)\in[0,+\infty[\times\mathbb{R}^n$,*

$$\big(f*_\alpha g\big)(r,x) = \int_{\mathbb{R}^n}\int_0^\infty \mathcal{T}_{(r,x)}f(s,y)g(s,y)d\nu_\alpha(s,y)$$

Note that, the following properties is valid:
i) For all $(r,x),(s,y)\in[0,\infty[\times\mathbb{R}^n,\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n$, we have

$$\varphi_{(\mu,\lambda)}(r,x)\varphi_{(\mu,\lambda)}(s,y)=\mathcal{T}_{(r,x)}\varphi_{\mu,\lambda}(s,y).$$

ii) Let f be in $L_{1,\alpha}(\mathbb{R}^{n+1}_+)$, then for all $(s,y)\in[0,\infty[\times\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n}\int_0^\infty \mathcal{T}_{(s,y)}f(r,x)d\nu_\alpha(r,x)=\int_{\mathbb{R}^n}\int_0^\infty f(r,x)d\nu_\alpha(r,x).$$

iii) If $f\in L_{p,\alpha}(\mathbb{R}^{n+1}_+),\ 1\le p\le\infty$, then for all $(s,y)\in[0,\infty[\times\mathbb{R}^n$, the function $\mathcal{T}_{(s,y)}f$ belongs to $L_{p,\alpha}(\mathbb{R}^{n+1}_+)$ and we have

$$\|\mathcal{T}_{(s,y)}f\|_{L_{p,\alpha}}\le\|f\|_{L_{p,\alpha}}.$$

iv) $\lim_{(r,x)\to(0,0}\|\mathcal{T}_{(r,x)}f-f\|_{p,\alpha}=0.$
v) For $f\in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$ and $g\in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$, Then $f*_\alpha g$ belongs to $L_\infty(\mathbb{R}^{n+1}_+)$ and the convolution product is commutative and associative
vi) For $p,q,r\in[0,\infty]$ such that $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$, the map

$$(f,g)\to f*_\alpha g$$

extends to a continuous map from $L_{p,\alpha}(\mathbb{R}^{n+1}_+)\times L_{q,\alpha}(\mathbb{R}^{n+1}_+)$ to $L_{r,\alpha}(\mathbb{R}^{n+1}_+)$ and we have,

$$\|f*_\alpha g\|_{L_{r,\alpha}}\le\|f\|_{L_{p,\alpha}}\|g\|_{L_{q,\alpha}}. \tag{2.7}$$

Definition 2.2 *The Fourier transform associated with the partial differential operators D_j and $L_{n,\alpha}$ is defined on $L_{1,\alpha}(\mathbb{R}^{n+1}_+)$ by the following , for all $(\mu,\lambda)\in[0,\infty[\times\mathbb{R}^n$,*

$$\mathcal{F}_\alpha(f)(\mu,\lambda)=\int_{\mathbb{R}^n}\int_0^\infty f(r,x)\varphi_{(\mu,\lambda)}(r,x)d\nu_\alpha(r,x).$$

We have the following properties.
i) Let $f\in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$. Then for all $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, we have, $\forall(\mu,\lambda)\in[0,\infty[\times\mathbb{R}^n$;

$$\mathcal{F}_\alpha(\mathcal{T}_{(r,x)}(f))(\mu,\lambda)=\varphi_{(\mu,\lambda)}(r,x)\mathcal{F}_\alpha(f)(\mu,\lambda).$$

ii) For $f,g\in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$

$$\mathcal{F}_\alpha(f*_\alpha g)(\mu,\lambda)=\mathcal{F}_\alpha(f)(\mu,\lambda)\cdot\mathcal{F}_\alpha(g)(\mu,\lambda).$$

Proposition 2.2 *Let $f\in L_{p,\alpha}(\mathbb{R}^{n+1}_+)$ with $p\in[1,2]$. Then $\mathcal{F}_\alpha(f)$ belongs to $L_{p',\alpha}(\mathbb{R}^{n+1}_+)$ with $\frac{1}{p}+\frac{1}{p'}=1$ and we have*

$$\|\mathcal{F}_\alpha(f)\|_{L_{p',\alpha}}\le\|f\|_{L_{p,\alpha}}.$$

Proposition 2.3 *The fourier transform $\mathcal{F}_\alpha$ is a topological isomorphism from $\mathrm{S}_*(\mathbb{R}\times\mathbb{R}^n)$ (the space of infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, rapidly decreasing together with all their derivatives) onto itself. The inverse mapping is given by*

$$\mathcal{F}_\alpha^{-1}(f)(r,x)=\int_{\mathbb{R}^n}\int_0^\infty f(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}}d\gamma_\alpha(\mu,\lambda),$$

where

$$d\gamma(\mu,\lambda)=\frac{\mu^{2\alpha+1}}{(2\pi)^n2^{2\alpha}(\Gamma(\alpha+1))^2}d\mu d\lambda.$$

Lemma 2.1 *The following equality is valid*

$$\int_{\mathbb{R}_+^{n+1}}g(s,y)s^{2\alpha+1}dsdy=\frac{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}\int_{\mathbb{R}_+^{n+2}}g\left(\sqrt{r^2+\overline{r}^2},x\right)\overline{r}^{2\alpha+1}d\overline{r}drdx.$$

Lemma 2.2 *The following equality is valid*

$$\int_{B((r,x),t)}g(s,y)s^{2\alpha+1}dsdy=\frac{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}\int_{B(((r,x),0),t)}g\left(\sqrt{\tau^2+\overline{\tau}^2},z\right)\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz,$$

where

$$B(((r,x),0),t)=\left\{(\overline{\tau},(\tau,z))\in\mathbb{R}_+^{n+2}:|r-\sqrt{\tau^2+\overline{\tau}^2}|^2+|x-z|^2<t^2\right\}.$$

Lemma 2.3 *For all $(r,x)\in\mathbb{R}_+^{n+1}$ the following equality is valid*

$$\int_{B((0,0),t)}\mathcal{T}_{(r,x)}g(s,y)s^{2\alpha+1}dsdy=\int_{B(((r,x),0),t)}g\left(\sqrt{\tau^2+\overline{\tau}^2},z\right)\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz,$$

where $B(((r,x),0),t)=\{(\overline{\tau},\tau,z)\in\mathbb{R}_+^{n+2}:\ |(\overline{\tau},(r-\tau,x-z))|<t\}$.

Lemma 2.4 *For all $(r,x)\in\mathbb{R}_+^{n+1}$ the following equality is valid*

$$\begin{aligned}&\int_{B((0,0),t)}\mathcal{T}_{(r,x)}g(s,y)M_\alpha\chi_{B((0,0),t)}(y)s^{2\alpha+1}dsdy\\&\qquad=\int_{B(((r,x),0),t)}g\left(\sqrt{\tau^2+\overline{\tau}^2,z}\right)M_\nu\chi_{B((r,x),0),t)}(\overline{\tau},(\tau,z))\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz,\end{aligned}$$

where $B((r,x,0),t)=\{(\overline{\tau},\tau,z)\in\mathbb{R}_+^{n+2}\ :\ |(\overline{\tau},r-\tau,x-z)|<t\}$.

Lemmas 2.3, 2.4 is straightforward via the following substitutions

$$\begin{gathered}z=y,\tau=s\cos\theta,\ \ \overline{\tau}=s\sin\theta,\ \ 0\le\alpha<\pi,\\(s,y)\in\mathbb{R}_+^{n+1},\ (\overline{\tau},(\tau,x))\in\mathbb{R}_+^{n+2}.\end{gathered}$$

3 Definitions, notation and preliminaries

Definition 3.1 *[12,13] Let $1 \leq p < \infty$, $0 \leq \lambda \leq 2\alpha+2$. We denote by $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R}^{n+1}_+)$ Morrey space ($\equiv B$-Morrey space), associated with the Laplace-Bessel differential operator as the set of locally integrable functions $f(r,x)$, $(r,x) \in \mathbb{R}^{n+1}_+$, with the finite norm*

$$\|f\|_{\mathcal{M}_{p,\lambda,\alpha}} = \sup_{t>0,\,(r,x)\in\mathbb{R}^{n+1}_+} \left(t^{-\lambda} \int_{B((0,0),t)} \left(\mathcal{T}_{(r,x)}|f(s,y)|\,\right)^p s^{2\alpha+1} ds dy \right)^{1/p}.$$

We will make use the Laplace-Bessel maximal function associated with the differential operators $\Delta_{n,\alpha}$. The Laplace-Bessel maximal function was introdused by Guliyev in [11], see also [12,13]

$$M_\alpha f(r,x) = \sup_{\varepsilon>0} |B((0,0),\varepsilon)|_\alpha^{-1} \int_{B((0,0),\varepsilon)} \mathcal{T}_{(s,y)}(|f(r,x)|) d\nu_\alpha(s,y),$$

where

$$B((0,0),\varepsilon) = \{(s,y) \in \mathbb{R}^{n+1}_+ : s^2 + |y|^2 \leq \varepsilon^2\}$$

and

$$|B((0,0),\varepsilon)|_\alpha = \frac{\pi^{1/2}}{2^n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)} \varepsilon^{n+2\alpha+2}.$$

Also consider the fractional maximal function

$$M^\beta_\alpha f(r,x) = \sup_{\varepsilon>0} |B((0,0),\varepsilon)|_\alpha^{\frac{\beta}{2\alpha+n+2}-1} \int_{B((0,0),\varepsilon)} \mathcal{T}_{(s,y)}(|f(r,x)|) d\nu_\alpha(s,y),$$

with

$$0 \leq \beta < 2\alpha + n + 2.$$

Note that for $\beta = 0$ we have $\mathcal{M}^0_\alpha f(r,x) = M_\alpha f(r,x)$.

We consider the B-Riesz potential

$$I^\beta_\alpha f(r,x) = \int_{\mathbb{R}^{n+1}_+} \mathcal{T}_{(r,x)}|f(s,y)|\,|(s,y)|^{\beta-2\alpha-2} s^{2\alpha+1} ds dy, \quad 0 < \beta < 2\alpha+2.$$

Theorem 3.1 *[15] 1) If $f \in \mathcal{M}_{1,\lambda,\alpha}(\mathbb{R}^{n+1}_+)$, $0 \leq \lambda < 2\alpha+2+n$, then $M_\alpha f \in W\mathcal{M}_{1,\lambda,\alpha}(\mathbb{R}^{n+1}_+)$ and*

$$\|M_\alpha f\|_{W\mathcal{M}_{1,\lambda,\alpha}} \leq C_{1,\lambda,\alpha}\|f\|_{\mathcal{M}_{1,\lambda,\alpha}},$$

where $C_{1,\lambda,\alpha}$ depends only on λ, α and n.

2) If $f \in \mathcal{M}_{p,\lambda,\alpha}(\mathbb{R}^{n+1}_+)$, $1 < p < \infty$, $0 \leq \lambda < 2\alpha+2+n$, then $M_\alpha f \in \mathcal{M}_{p,\lambda,\alpha}(\mathbb{R}^{n+1}_+)$ and

$$\|M_\alpha f\|_{\mathcal{M}_{p,\lambda,\alpha}} \leq C_{p,\lambda,\alpha}\|f\|_{\mathcal{M}_{p,\lambda,\alpha}},$$

where $C_{p,\lambda,\alpha}$ depends only on p, λ, α and n.

Theorem 3.2 *[15] Let $0 < \alpha < Q$, $0 \leq \lambda < 2\alpha+2+n-\beta$ and $1 \leq p < \frac{2\alpha+2+n-\lambda}{\beta}$.*

1) If $1 < p < \frac{2\alpha+2-\lambda}{\beta}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2+n-\lambda}$ is necessary and sufficient for the boundedness I^β_α from $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R}^{n+1}_+)$ to $\mathcal{M}_{q,\lambda,\alpha}(\mathbb{R}^{n+1}_+)$.

2) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2+n-\lambda}$ is necessary and sufficient for the boundedness I^β_α from $\mathcal{M}_{1,\lambda,\alpha}(\mathbb{R}^{n+1}_+)$ to $W\mathcal{M}_{q,\lambda,\alpha}(\mathbb{R}^{n+1}_+)$.

Definition 3.2 *Let $\omega(r,x)$ positive measurable weight function on $\mathbb{R}_+^{n+1}$. We denote by $\mathcal{M}_{p,\omega,\alpha}(\mathbb{R}_+^{n+1})$ the generalized Morrey spaces, the spaces of all functions $f \in L_p^{loc}(\mathbb{R}_+^{n+1})$ with finite quasinorm*

$$\|f\|_{\mathcal{M}_{p,\omega,\alpha}} = \sup_{(r,x)\in\mathbb{R}_+^{n+1},\, t>0} \frac{t^{-\frac{2\alpha+n+2}{p}}}{\omega((r,x),t)} \|\mathcal{T}_{(r,x)}|f(\cdot)|\|_{L_{p,\alpha}(B((0,0),t))}.$$

If $\omega((r,x),t) \equiv t^{-\frac{2\alpha+2+n}{p}}$, then $\mathcal{M}_{p,\omega,\alpha}(\mathbb{R}_+^{n+1}) \equiv L_{p,\alpha}(\mathbb{R}_+^{n+1})$, if $\omega((r,x),t) \equiv t^{\frac{\lambda-2\alpha+2}{p}}$, $0 \le \lambda < 2\alpha+n+2$, then $\mathcal{M}_{p,\omega,\alpha}(\mathbb{R}_+^{n+1}) \equiv \mathcal{M}_{p,\lambda,\alpha}(\mathbb{R}_+^{n+1})$.

4 Maximal function associated with Laplace-Bessel differential operators on generalized B-Morrey spaces

In this section we study the boundedness of the maximal operator associated with Laplace-Bessel differential operators on generalized B-Morrey spaces.

Theorem 4.1 *Let $1 \le p < \infty$ and the $\omega(r,x)$ positive measurable weight function on $\mathbb{R}_+^{n+1} \times (0,\infty)$ satisfying the condition*

$$\int_t^\infty \omega((r,x),\tau)\frac{d\tau}{\tau} \le C\omega((r,x),t), \tag{4.1}$$

where C does not depend on (r,x) and t.

Then for $p>1$ the maximal operator M_α is bounded from $\mathcal{M}_{p,\omega,\alpha}(\mathbb{R}_+^{n+1})$ to $\mathcal{M}_{p,\omega,\alpha}(\mathbb{R}_+^{n+1})$ and for $p=1$ the maximal operator M_α is bounded from $\mathcal{M}_{1,\omega,\alpha}(\mathbb{R}_+^{n+1})$ to $W\mathcal{M}_{1,\omega,\alpha}(\mathbb{R}_+^{n+1})$.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y,d,σ). By this we mean a topological space $Y=\mathbb{R}_+^{n+1}\times(0,\infty)$ equipped with a continuous pseudometric d and a positive measure σ satisfying

$$\sigma(B((\overline{r},r,x),2t)) \le C_1\sigma(B((\overline{r},r,x),t)) \tag{4.2}$$

with a constant C_1 independent of $(\overline{r},r,x)$ and $r>0$. Here $B((\overline{r},r,x),t) = \{(\overline{s},s,y)\in Y : d(((\overline{r},r,x),(\overline{s},s,y)) < r\}$, $d\sigma(\overline{s},s,y) = (\overline{s})^{2\alpha+1}d\overline{s}\, ds\, dy$, $d((\overline{r},r,x),(\overline{s},s,y)) = |(\overline{r},r,x)-(\overline{s},s,y)| \equiv ((\overline{r}-\overline{s})^2+|r-s|^2+|x-y|^2)^{\frac{1}{2}}$.

Let (Y,d,σ) be a space of homogeneous type. Define

$$M_\sigma\overline{f}(\overline{r},r,x) = \sup_{r>0}\sigma(B((\overline{r},r,x),t))^{-1}\int_{B((\overline{r},r,x),t)} \left|\overline{f}(\overline{s},s,y)\right| d\sigma(\overline{s},s,y),$$

where $\overline{f}(\overline{r},r,x) = f\left(\sqrt{\overline{r}^2+r^2},x\right)$.

It is well known that the maximal operator M_σ is of weak type $(1,1)$ and is bounded on $L_p(Y,d\sigma)$ for $1<p<\infty$ (see [6]). Here we are concerned with the maximal operator defined by $d\sigma(\overline{s},s,y) = (\overline{s})^{2\alpha+1}d\overline{s}\, ds\, dy$. It is clear that this measure satisfies the doubling condition (4.2).

It can be proved that

$$M_\alpha f\left(\sqrt{\overline{\tau}^2+\tau^2},z\right) = M_\sigma\overline{f}\left(\sqrt{\overline{\tau}^2+\tau^2},z,0\right) \tag{4.3}$$

and

$$M_\alpha f(r,x) = M_\sigma \overline{f}(r,x,0). \tag{4.4}$$

Indeed, from Lemma 2.3

$$\int_{B((0,0),t)} \mathcal{T}_{(s,y)} \left| f\left(\sqrt{\overline{\tau}^2+\tau^2}, z\right)\right| \overline{\tau}^{2\alpha+1} d\overline{\tau} d\tau dz$$
$$= \int_{B\left(\left(\sqrt{\overline{\tau}^2+\tau^2},z,0\right),t\right)} \left|\overline{f}(\overline{s},s,y)\right| \, d\sigma(\overline{s},s,y)$$

and

$$|B((0,0),t)|_\alpha = \sigma B\left(\left(\sqrt{\overline{\tau}^2+\tau^2},z,0\right),t\right)$$

imply (4.3). Furthermore, taking $\overline{\tau}=0$ in (4.3) we get (4.4).

Using Lemma 2.3 and equality (4.3) we have

$$\left(\int_{B((0,0),t)} [\mathcal{T}_{(r,x)}\left(M_\alpha f(y)\right)]^p d\nu(s,y)\right)^{1/p}$$
$$= \left(\int_{\mathbb{R}^{n+1}_+} [\mathcal{T}_{(r,x)}\left(M_\alpha f(y)\right)]^p \chi_{B((0,0),t)}(s,y) d\nu(s,y)\right)^{1/p}$$
$$= \left(\int_{\mathbb{R}^{n+1}_+\times(0,\infty)} \left(M_\alpha f\left(\sqrt{\overline{\tau}^2+\tau^2}, z\right)\right)^p \chi_{E(((r,x),0),t)}(\overline{\tau},\tau,z)\, d\sigma(\overline{\tau},\tau,z)\right)^{1/p}$$
$$= \left(\int_{B(((r,x),0),t)} \left(M_\sigma \overline{f}\left(\sqrt{\overline{\tau}^2+\tau^2}, z, 0\right)\right)^p \, d\sigma(\overline{\tau},\tau,z)\right)^{1/p}.$$

In [17] there was proved that the analogue of the Fefferman-Stein theorem for the maximal operator defined on a space of homogeneous type is valid, if condition (4.2) is satisfied. Therefore

$$\int_Y \left(M_\sigma \varphi(\overline{s},s,y)\right)^p \psi(\overline{s},s,y) d\sigma(\overline{s},s,y)$$
$$\le C_2 \int_Y |\varphi(\overline{s},s,y)|^p M_\sigma \psi(\overline{s},s,y)\, d\sigma(\overline{s},s,y). \tag{4.5}$$

Then taking $\varphi(\overline{s},s,y) = \overline{f}\left(\sqrt{\overline{s}^2+s^2}, y, 0\right)$ and $\psi(\overline{s},s,y) \equiv \chi_{B((\overline{r},r,x),t)}(\overline{s},s,y)$ we obtain from inequality (4.5) and Lemma 2.3 that

$$\left(\int_{B((0,0),r)} [\mathcal{T}_{(r,x)}\left(M_\alpha f(s,y)\right)]^p d\nu(s,y)\right)^{1/p}$$

$$= \left(\int_Y \left(M_\sigma \overline{f}\left(\sqrt{\overline{s}^2+s^2}, y, 0\right)\right)^p \chi_{B((x,0),t)}(\overline{s},s,y)\, d\sigma(\overline{s},s,y)\right)^{1/p}$$

$$\le C_2 \left(\int_Y \left|\overline{f}\left(\sqrt{\overline{s}^2+s^2}, y, 0\right)\right|^p M_\sigma \chi_{B((x,0),t)}(\overline{s},s,y)\, d\sigma(\overline{s},s,y)\right)^{1/p}$$

$$= C_2 \left(\int_Y \left| f\left(\sqrt{\overline{s}^2 + s^2}, y\right)\right|^p M_\sigma \chi_{B((x,0),t)}(\overline{s}, s, y)\, d\sigma(\overline{s}, s, y) \right)^{1/p}$$

$$= C_2 \left(\int_{\mathbb{R}_+^{n+1}} [\mathcal{T}_{(r,x)}|f(s,y)|\,]^p M_\alpha \chi_{B((0,0),r)}(s,y) d\nu(s,y) \right)^{1/p}$$

$$\leq C_2 \Big(\int_{B((0,0),r)} [\mathcal{T}_{(r,x)}|f(s,y)|\,]^p d\nu(s,y)$$

$$+ C_2 \sum_{j=1}^{\infty} \int_{B_{2^{j+1}r} \setminus B_{2^j r}} [\mathcal{T}_{(r,x)}|f(s,y)|\,]^p M_\alpha \chi_{B((0,0),r)}(s,y) d\nu(s,y) \Big)^{1/p}$$

$$\leq C_2 \left(\int_{B((0,0),r)} [\mathcal{T}_{(r,x)}|f(s,y)|\,]^p d\nu(s,y) \right)^{1/p}$$

$$+ C_2 \left(\sum_{j=1}^{\infty} \int_{B_{2^{j+1}r} \setminus B_{2^j r}} [\mathcal{T}_{(r,x)}|f(s,y)|\,]^p \frac{r^Q}{(|y|+r)^Q} d\nu(s,y) \right)^{1/p}$$

$$\leq C_3 \, \|f\|_{\mathcal{M}_{p,\omega,\alpha}} r^{\frac{2\alpha+2+n}{p}} \left(\omega(x,r) + \sum_{j=1}^{\infty} \frac{1}{(2^j+1)^Q} (2^{j+1}r)^{\frac{2\alpha+2}{p}} \omega(x, 2^{j+1}r) \right)$$

$$\leq C_3 \, \|f\|_{\mathcal{M}_{p,\omega,\alpha}} r^{\frac{2\alpha+2+n}{p}} \left(\omega((r,x),t) + C \int_t^{\infty} \omega((r,x),\tau) \frac{d\tau}{\tau} \right)$$

$$\leq C_4 \, r^{\frac{2\alpha+2+n}{p}} \omega((r,x),t) \|f\|_{\mathcal{M}_{p,\omega,\alpha}}.$$

5 Fractional integral associated with Laplace-Bessel differential operators on generalized B-Morrey spaces

In this section we study the boundedness of the fractional integral operator associated with Laplace-Bessel differential operators on generalized B-Morrey spaces.

Theorem 5.1 *[3, 13] Let $0 < \beta < 2\alpha + 2 + n$ and $1 \leq p < \frac{2\alpha+2+n}{\beta}$.*

1) If $1 < p < \frac{2\alpha+2+n}{\beta}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2+n}$ is necessary and sufficient for the boundedness of I_α^β from $L_{p,\alpha}(\mathbb{R}_+^{n+1})$ to $L_{q,\alpha}(\mathbb{R}_+^{n+1})$.

2) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2+n}$ is necessary and sufficient for the boundedness of I_α^β from $L_{1,\alpha}(\mathbb{R}_+^{n+1})$ to $WL_{q,\alpha}(\mathbb{R}_+^{n+1})$.

Theorem 5.2 *Let $0 < \beta < 2\alpha + n + 2$, $1 \leq p < \frac{2\alpha+2+n}{\beta}$ and the $\omega(r,x)$ positive measurable weight function on $\mathbb{R}_+^{n+1} \times (0,\infty)$ satisfying the condition* (4.1) *and*

$$\int_t^{\infty} \omega((r,x),\tau) \frac{d\tau}{\tau^{1-\beta}} \leq C\, t^\beta \, \omega((r,x),t), \tag{5.1}$$

where C does not depend on (r,x) and t.

1) If $1 < p < \frac{2\alpha+2+n}{\beta}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2+n}$, then I_α^β is bounded from $\mathcal{M}_{p,\omega,\alpha}(\mathbb{R}_+^{n+1})$ to $\mathcal{M}_{q,\omega,\alpha}(\mathbb{R}_+^{n+1})$.

2) If $p = 1$ and $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2+n}$, then I_α^β is bounded from $\mathcal{M}_{1,\omega,\alpha}(\mathbb{R}_+^{n+1})$ to $W\mathcal{M}_{q,\omega,\alpha}(\mathbb{R}_+^{n+1})$.

Proof. Let $1 < p < \frac{2\alpha+2+n}{\beta}$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2+n}$ and $f \in \mathcal{M}_{p,\omega,\alpha}(\mathbb{R}^{n+1}_+)$. Then

$$I^\beta_\alpha f(r,x) = \left(\int_{B((0,0),t)} + \int_{\mathbb{R}^{n+1}_+ \setminus B((0,0),t)}\right) \mathcal{T}_{(r,x)} f(s,y) |(s,y)|^{\beta-2\alpha-2} d\nu(s,y)$$
$$= A_1(r,x,t) + A_2(r,x,t). \tag{5.2}$$

Let $I_{\alpha,\nu} f$ be the fractional integral operator on the space of homogeneous type (Y,d,σ):

$$I^\beta_\sigma f(\overline{r},r,x) = \int_Y f(s,y) d((\overline{r},r,x),(\overline{s},s,y))^{\beta-1}\, d\sigma(\overline{s},s,y).$$

Also, in the work [20], [22] it was proved:

Proposition 1. *Let $0 < \alpha < 1$, $1 \le p < \frac{1}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \alpha$. Then the following two conditions are equivalent:*

1) There is a constant $C > 0$ such that for any $f \in L_{p,\varphi}(Y)$ the inequality

$$\|I^\beta_\sigma (f\varphi^\alpha)\|_{L_{q,\varphi}} \le C \|f\|_{L_{p,\varphi}}$$

holds.

2) $\varphi \in A_{1+\frac{q}{p'}}(Y)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

By the Proposition 1 and $\varphi(\overline{s},s,y) = \left(M\chi_{E((0,r,x),t)}(\overline{s},s,y)\right)^\theta \in A_p(Y)$, $0 < \theta < 1$, we have

$$\left(\int_{B((0,0),t)} \mathcal{T}_{(r,x)} |F_1(s,y)|^q \, d\nu(s,y)\right)^{1/q}$$
$$\le \left(\int_{\mathbb{R}^{n+1}_+} \mathcal{T}_{(r,x)} |F_1(s,y)|^q \, (M_\alpha \chi_{B((0,0),t)}(s,y))^\theta d\nu(s,y)\right)^{1/q}$$
$$= \left(\int_Y \left| I^\beta_\sigma (f\varphi^\alpha)\left(\sqrt{\overline{s}^2+s^2}, y, 0\right)\right|^q \varphi(\overline{s},s,y)\, d\sigma(\overline{s},s,y)\right)^{1/q}$$
$$\le C_2 \left(\int_Y \left|\overline{f}\left(\sqrt{\overline{s}^2+s^2}, y, 0\right)\right|^p \left(M_\sigma \chi_{B(((r,x),0),t)}(\overline{s},s,y)\right)^\theta \, d\sigma(\overline{s},s,y)\right)^{1/p}$$
$$= C_2 \left(\int_Y \left|f\left(\sqrt{\overline{s}^2+s^2}, y\right)\right|^p \left(M_\sigma \chi_{B(((r,x),0),t)}(\overline{s},s,y)\right)^\theta \, d\sigma(\overline{s},s,y)\right)^{1/p}$$
$$= C_2 \left(\int_{\mathbb{R}^{n+1}_+} \mathcal{T}_{(r,x)} |f(s,y)|^p (M_\alpha \chi_{B((0,0),t)}(s,y))^\theta d\nu(s,y)\right)^{1/p}$$
$$\le C_2 \left(\int_{B((0,0),t)} \mathcal{T}_{(r,x)} |f(s,y)|^p d\nu(s,y)\right)^{1/p}$$
$$+ C_2 \left(\sum_{j=1}^\infty \int_{B((0,0),2^{j+1}t)\setminus B((0,0),2^j t)} \mathcal{T}_{(r,x)} |f(s,y)|^p (M_\alpha \chi_{E_r}(s,y))^\theta d\nu(s,y)\right)^{1/p}$$

$$\leq C_2 \left(\int_{B((0,0),t)} \mathcal{T}_{(r,x)} |f(s,y)|^p d\nu(s,y) \right)^{1/p}$$

$$+C_2 \left(\sum_{j=1}^{\infty} \int_{B((0,0),2^{j+1}t)\setminus B((0,0),2^j t)} \mathcal{T}_{(r,x)} |f(s,y)|^p \frac{t^{(2\alpha+2+n)\theta}}{(|(s,y)|+r)^{(2\alpha+2+n)\theta}} d\nu(s,y) \right)^{1/p}$$

$$\leq C_3 \, \|f\|_{\mathcal{M}_{p,\omega,\alpha}} \Big(r^{\frac{2\alpha+2+n}{p}} \omega((r,x),t)$$

$$+ \sum_{j=1}^{\infty} \frac{1}{(2^j+1)^{(2\alpha+2+n)\theta}} (2^{j+1}t)^{\frac{2\alpha+2+n}{p}} \omega((r,x),2^{j+1}t) \Big)^{1/p}$$

$$\leq C_3 \, \|f\|_{\mathcal{M}_{p,\omega,\alpha}} t^{\frac{2\alpha+2+n}{p}} \left(\omega((r,x),t) + C \int_t^{\infty} \omega((r,x),\tau) \frac{d\tau}{\tau} \right)$$

$$\leq C_4 \, t^{\frac{2\alpha+2+n}{p}} \omega((r,x),t) \|f\|_{\mathcal{M}_{p,\omega,\alpha}}.$$

Hence

$$\begin{aligned}
\|A_1\|_{\mathcal{M}_{q,\omega,\alpha}} &= \sup_{(r,x)\in\mathbb{R}^{n+1}_+,\, t>0} t^{-\frac{2\alpha+2+n}{q}} \omega^{-1}((r,x),t) \\
&\quad \times \left(\int_{B((0,0),t)} \mathcal{T}_{(r,x)} \, |A(s,y,t)|^q \, d\nu(s,y) \right)^{\frac{1}{q}} \\
&\leq C_4 \|f\|_{\mathcal{M}_{p,\omega,\alpha}}.
\end{aligned}$$

Now we estimate $|A_2(r,x,t)|$. By the Hölder's inequality we have

$$\begin{aligned}
|A_2(r,x,t)| &\leq \int_{\mathbb{R}^{n+1}_+ \setminus B((0,0),t)} |(s,y)|^{\beta-2\alpha-2} d\nu(s,y) \\
&= \sum_{j=1}^{\infty} \int_{B((0,0),2^{j+1}t)\setminus B((0,0),2^j t)} |(s,y)|^{\beta-2\alpha-2} \mathcal{T}_{(r,x)} |f(s,y)| d\nu(s,y) \\
&\leq \sum_{j=1}^{\infty} \left(\int_{B((0,0),2^{j+1}t)\setminus B((0,0),2^j t)} |(s,y)|^{(\beta-(2\alpha+2)p'} d\nu(s,y) \right)^{\frac{1}{p'}} \\
&\quad \times \left(\int_{B((0,0),2^{j+1}t)\setminus B((0,0),2^j t)} \mathcal{T}_{(r,x)} |f(s,y)|^p d\nu(s,y) \right)^{\frac{1}{p}} \\
&\leq C \, \|f\|_{\mathcal{M}_{p,\omega,\alpha}} \sum_{j=1}^{\infty} (2^j t)^{\alpha} \omega((r,x),2^j t) \\
&\leq C \, \|f\|_{\mathcal{M}_{p,\omega,\alpha}} \int_t^{\infty} \omega((r,x),\tau) \tau^{\beta-1} d\tau \\
&\leq C t^{\beta} \omega((r,x),t) \, \|f\|_{\mathcal{M}_{p,\omega,\alpha}} .
\end{aligned}$$

Hence

$$\|A_2\|_{\mathcal{M}_{q,\omega,\alpha}} = \sup_{(r,x)\in\mathbb{R}^{n+1}_+,\, t>0} t^{-\frac{2\alpha+2+n}{q}} \omega^{-1}((r,x),t) \left(\int_{B((0,0),t)} \mathcal{T}_{(r,x)} \left|F_2(s,y)\right|^q \right)^{\frac{1}{q}}$$
$$\leq C \sup_{(r,x)\in\mathbb{R}^{n+1}_+,\, t>0} \omega^{-1}((r,x),t) t^{\alpha} \left\|f\right\|_{\mathcal{M}_{p,\omega,\alpha}} \omega((r,x),t) \|\chi_{B((0,0),t)}\|_{L_{q,\alpha}}$$
$$\leq C\|f\|_{\mathcal{M}_{p,\omega,\alpha}}.$$

Therefore $I^{\beta}_{\alpha} f \in \mathcal{M}_{q,\omega^{q/p},\alpha}(\mathbb{R}^{n+1}_+)$ and

$$\left\| I^{\beta}_{\alpha} f \right\|_{\mathcal{M}_{q,\omega^{q/p},\alpha}} \leq C\|f\|_{\mathcal{M}_{p,\omega,\alpha}}.$$

2) Let $f \in \mathcal{M}_{1,\omega,\alpha}(\mathbb{R}^{n+1}_+)$. By the (5.2), we get

$$|F_1| \leq \int_{B((0,0),t)} \mathcal{T}_{(r,x)} |f(s,y)||(s,y)|^{\beta-2\alpha-2-n} d\nu(s,y)$$
$$\leq \sum_{k=-\infty}^{-1} \left(2^k t\right)^{\beta-2\alpha-2-n} \int_{E_{2^{k+1}t}\setminus E_{2^k t}} T^x \left|f(y)\right| d\nu(s,y).$$

Hence

$$|F_1(r,x)| \leq Ct^{\beta} M_{\alpha} f(r,x). \tag{5.3}$$

Then

$$\left|\left\{(s,y) \in B((0,0),t) \ : \ \mathcal{T}_{(r,x)} \left| I^{\beta}_{\alpha} f(s,y) \right| > 2\beta \right\}\right|_{\alpha}$$
$$\leq |\left\{(s,y) \in B((0,0),t) \ : \ \mathcal{T}_{(r,x)} |F_1(s,y)| > \beta \right\}|_{\alpha}$$
$$+ |\left\{(s,y) \in B((0,0),t) \ : \ \mathcal{T}_{(r,x)} |F_2(s,y)| > \beta \right\}|_{\alpha}.$$

Taking into account inequality (5.3) and Theorem 3.1 we have

$$|\left\{(s,y) \in B((0,0),t) \ : \ \mathcal{T}_{(r,x)} |F_1(s,y)| > \beta \right\}|_{\alpha}$$
$$\leq \left|\left\{(s,y) \in B((0,0),t) \ : \ \mathcal{T}_{(r,x)} (M_{\alpha} f(s,y)) > \frac{\beta}{Ct^{\beta}} \right\}\right|_{\alpha}$$
$$\leq \frac{Ct^{\beta}}{\beta} \cdot \omega((r,x),t) \left\|f\right\|_{\mathcal{M}_{1,\omega,\alpha}},$$

and thus if $Ct^{-\frac{2\alpha+2+n}{q}} \omega((r,x),t) \left\|f\right\|_{\mathcal{M}_{1,\omega,\alpha}} = \beta$, then $|F_2(r,x)| \leq \beta$ and consequently, $|\left\{(s,y) \in B((0,0),t) \ : \ \mathcal{T}_{(r,x)} |F_2(s,y)| > \beta \right\}|_{\alpha} = 0$.

Finally

$$\left|\left\{(s,y) \in B((0,0),t) \ : \ \mathcal{T}_{(r,x)} |I^{\beta}_{\alpha} f(y)| > 2\beta \right\}\right|_{\alpha}$$
$$\leq \frac{C}{\beta} \omega((r,x),t) t^{\beta} \left\|f\right\|_{\mathcal{M}_{1,\omega,\alpha}}$$
$$= C\omega^q((r,x),t) \left(\frac{\|f\|_{\mathcal{M}_{1,\omega,\alpha}}}{\beta} \right)^q.$$

The Theorem 3.2 is proved.

6 The Littlewood-Paley g-function

In this section we define and study the g-littlewood function associated with Laplace differential operator inthe generalized Morrey space $\mathcal{M}_{p,w,\alpha}(\mathbb{R}^{n+1})$.

In the following we recall some facts and definitions. The Poisson integrals $u_t(f)$, $t>0$ and $f\in \mathrm{S}_*(\mathbb{R}^{n+1})$ is defined by

$$u_t(f)(r,x)=p_t*f(r,x),\quad (r,x)\in\mathbb{R}^{n+1},$$

where p_t is the Poisson kernel given by

$$p_t(r,x)=\mathfrak{F}^{-1}(e^{-t|(\mu,\lambda)|})(r,x)=\frac{2^{\alpha+1}\Gamma(\alpha+\frac{n+3}{2}}{\pi^{\frac{n+1}{2}}\Gamma(\alpha+1)(t^2+r^2+\|x\|^2)},$$

see [4], Proposition 3.1.

Proposition 6.1 *Let $f\in \mathrm{S}_*(\mathbb{R}^{n+1})$ be a positive function and $p>1$, then we have*

i) For $|(r,x)|=\sqrt{r^2+x^2}$ large we have

$$u_t(r,x)\le C(t^2+r^2+|x|^2)^{(-\alpha+\frac{n+2}{2})},$$

ii)

$$\frac{\partial u}{\partial t}(r,x)\le Ct^{-2\alpha+n+3},$$

iii)

$$\frac{\partial u}{\partial r}(r,x)\le C(t^2+r^2+|x|^2)^{(-\alpha+\frac{n+3}{2})},$$

iv)

$$\frac{\partial u}{\partial x_i}(r,x)\le C(t^2+r^2+|x|^2)^{(-\alpha+\frac{n+3}{2})},\ 1\le i\le n.$$

Proposition 6.2 *Let Φ a positive, non increasing in $L^1(d\nu_\alpha)$ locally integrable function on $\mathbb{R}^{n+1}$, we have*

$$\sup_{t>0}\Phi_t*_\alpha f(r,x)\le\|\Phi_t\|_{1,\alpha}M_\alpha(f)(r,x),$$

where Φ_t is the dilation of Φ given by

$$\Phi_t(r,x)=t^{-(2\alpha+n+2}\Phi(\frac{r}{t},\frac{x}{t}).$$

Definition 6.1 *We define the g-functions associated with Laplace-Bessel differential operators for $f\in \mathrm{S}_*(\mathbb{R}^{n+1})$ by the following*

$$\forall\ (r,x)\in\mathbb{R}^{n+1},\ g(f)(r,x)=\Big(\int_0^\infty|\nabla u_t(r,x)|^2tdt\Big)^{1/2},$$

where u_t is the Poisson integral and

$$\big|\nabla u_t(r,x)\big|^2=\big|\frac{\partial u}{\partial t}(r,x)\big|^2+\big|\frac{\partial u}{\partial r}(r,x)\big|^2+\sum_{i=1}^n\big|\frac{\partial u}{\partial x_i}(r,x)\big|^2.$$

Theorem 6.1 *Let $p \in]1,2]$ and $w(r,x)$ positive measurable weight function on $\mathbb{R}^{n+1} \times [0,\infty[$ satsfying*

$$\int_{\varepsilon}^{\infty} w(r,x,\tau)\frac{d\tau}{\tau} \leq Cw(r,x,\varepsilon),$$

where C does not depend on (r,x) and ε then there exists a positive constant $C_{p,\alpha}$ such that for all $f \in \mathcal{M}_{p,w,\alpha}(\mathbb{R}^{n+1})$ we have

$$\|g(f)\|_{\mathcal{M}_{p,w,\alpha}} \leq C_{p,\alpha}\|f\|_{\mathcal{M}_{p,w,\alpha}}.$$

Proof. Since the operator g is semi-linear operator then it suffices to proof theorem for all positive function.

case $1 < p < 2$.

By the same way as [4], we obtain

$$|g(f)(r,x)|^p \leq \Big(\frac{1}{p(p-1)}\Big)^{\frac{p}{2}}\Big(\widetilde{\mathcal{M}}_\alpha(f)(r,x)\Big)^{\frac{p(2-p)}{2}}\Big(I_\alpha(f)(r,x)\Big)^{\frac{p}{2}}, \qquad (6.1)$$

where

$$\widetilde{\mathcal{M}}_\alpha(f)(r,x) = \sup_{t>0}|u_t(r,x)|$$

and

$$I_\alpha(f)(r,x) = \int_0^{\infty} |\Delta_\alpha u_t^p(r,x)|tdt,$$

with

$$\Delta_\alpha = \frac{\partial^2 u_t}{\partial t^2} + \Delta_{n,\alpha}.$$

Lemma 3 allows us to get

$$\int_{B(0,0),\varepsilon} |\mathcal{T}_{(r,x)}g(f)(s,y)|^p d\nu_\alpha(s,y)$$

$$= \int_{E((r,x),0,\varepsilon)} |\mathcal{T}_{(r,x)}g(f)(\sqrt{\tau^2+\overline{\tau}^2},z)|^p\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz.$$

Thus, using relation (6.1) we obtain

$$\int_{B(0,0),\varepsilon} |\mathcal{T}_{(r,x)}g(f)(s,y)|^p d\nu_\alpha(s,y)$$

$$\leq \Big(\frac{1}{p(p-1)}\Big)^{\frac{p}{2}}\int_{E((r,x),0,\varepsilon)}\Big(\widetilde{\mathcal{M}}_\alpha(f)(\sqrt{\tau^2+\overline{\tau}^2},z)\Big)^{\frac{p(2-p)}{2}}$$

$$\times\Big(I_\alpha(f)(\sqrt{\tau^2+\overline{\tau}^2},z)\Big)^{\frac{p}{2}}\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz.$$

Applying Hölder inequality we get

$$\int_{B(0,0),\varepsilon} |\mathcal{T}_{(r,x)}g(f)(s,y)|^p d\nu_\alpha(s,y)$$

$$\leq \Big(\frac{1}{p(p-1)}\Big)^{\frac{p}{2}}\Big(\int_{E((r,x),0,\varepsilon)}\Big(\widetilde{\mathcal{M}}_\alpha(f)(\sqrt{\tau^2+\overline{\tau}^2},z)\Big)^p\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz\Big)^{\frac{2-p}{2}}$$
$$\times\Big(\int_{E((r,x),0,\varepsilon)}|I_\alpha(f)(\sqrt{\tau^2+\overline{\tau}^2},z)|\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz\Big)^{\frac{p}{2}}. \tag{6.2}$$

In the first hand using the same method as [4] we have

$$\int_{E((r,x),0,\varepsilon)}|I_\alpha(f)(\sqrt{\tau^2+\overline{\tau}^2},z)|\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz$$
$$=\int_{E((r,x),0,\varepsilon)}\Big(\int_0^\infty \Delta_\alpha u_t^p((\sqrt{\tau^2+\overline{\tau}^2},z)tdt\Big)\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz$$
$$\leq\int_{E((r,x),0,\varepsilon)}|f(\sqrt{\tau^2+\overline{\tau}^2},z))|^p\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz$$
$$=\int_{B((0,0),\varepsilon)}|\mathcal{T}_{(r,x)}f(s,y)|^p d\nu_\alpha(s,y)\leq\varepsilon^{2\alpha+n+2}(w((r,x),\varepsilon))^p\|f\|^p_{\mathcal{M}_{p,w,\alpha}}.$$

Therefore

$$\Big(\int_{E((r,x),0,\varepsilon)}|I_\alpha(f)(\sqrt{\tau^2+\overline{\tau}^2},z)|\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz\Big)^{\frac{p}{2}}$$
$$\leq\varepsilon^{(\alpha+n+2)p/2}(w((r,x),\varepsilon))^{p^2/2}\|f\|^{p^2/2}_{\mathcal{M}_{p,w,\alpha}}. \tag{6.3}$$

On the other hand using Proposition 2.6 and the fact that $\|p_t\|_{1,\alpha}=1$ (see [4]) we deduce

$$\int_{E((r,x),0,\varepsilon)}\Big(\widetilde{\mathcal{M}}_\alpha(f)(\sqrt{\tau^2+\overline{\tau}^2},z)\Big)^p\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz$$
$$\leq\|p_t\|^p_{1,\alpha}\Big(\int_{E((r,x),0,\varepsilon)}\mathcal{M}_\alpha(f)(\sqrt{\tau^2+\overline{\tau}^2},z)\Big)^p\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz$$
$$\leq\int_{B((0,0),\varepsilon)}|\mathcal{T}_{(r,x)}\mathcal{M}_\alpha(f)(s,y)|^p d\nu_\alpha(s,y)$$
$$\leq\varepsilon^{2\alpha+n+2}(w((r,x),\varepsilon))^p\|\mathcal{M}_\alpha(f)\|^p_{\mathcal{M}_{p,w,\alpha}}.$$

From Theorem 4.1.we deduce

$$\Big(\int_{E((r,x),0,\varepsilon)}\Big(\widetilde{\mathcal{M}}_\alpha(f)(\sqrt{\tau^2+\overline{\tau}^2},z)\Big)^p\overline{\tau}^{2\alpha+1}d\overline{\tau}d\tau dz\Big)^{\frac{2-p}{2}}$$
$$\leq\varepsilon^{(2\alpha+n+2)p(2-p)/2}(w((r,x),\varepsilon))^{p(2-p)/2}\|f\|^{p(2-p)/2}_{\mathcal{M}_{p,w,\alpha}}. \tag{6.4}$$

Relations (6.2), (6.3) and (6.4) involve that

$$\int_{B(0,0),\varepsilon}|\mathcal{T}_{(r,x)}g(f)(s,y)|^p d\nu_\alpha(s,y)\leq\varepsilon^{2\alpha+n+2}(w(r,x),\varepsilon)^p\|f\|^p_{\mathcal{M}_{p,w,\alpha}}$$

which gives that

$$\|g(f)\|_{\mathcal{M}_{p,w,\alpha}}\leq C\|f\|_{\mathcal{M}_{p,w,\alpha}}.$$

Note that, in the case $p = 2$ we have

$$\int_{B(0,0),\varepsilon} |\mathcal{T}_{(r,x)} g(f)(s,y)|^2 d\nu_\alpha(s,y)$$
$$= \int_{E((r,x),0,\varepsilon)} |\mathcal{T}_{(r,x)} g(f)(\sqrt{\tau^2+\overline{\tau}^2}, z)|^2 \overline{\tau}^{2\alpha+1} d\overline{\tau} d\tau dz$$
$$= \frac{1}{2} \int_{E((r,x),0,\varepsilon)} \Big(\int_0^\infty \Delta_\alpha u_t^2(\sqrt{\tau^2+\overline{\tau}^2}, z) t dt \Big) \overline{\tau}^{2\alpha+1} d\overline{\tau} d\tau dz,$$

see [4], relation (5.1).

Using also ([4], Proposition 3.7) we obtain

$$\int_{B(0,0),\varepsilon} |\mathcal{T}_{(r,x)} g(f)(s,y)|^2 d\nu_\alpha(s,y)$$
$$\leq \frac{1}{2} \int_{E((r,x),0,\varepsilon)} |f(\sqrt{\tau^2+\overline{\tau}^2}, z)|^2 \overline{\tau}^{2\alpha+1} d\overline{\tau} d\tau dz$$
$$\leq \frac{1}{2} \int_{B(0,0),\varepsilon} |\mathcal{T}_{(r,x)} f(s,y)|^2 d\nu_\alpha(s,y).$$

This implies that

$$\|g(f)\|_{\mathcal{M}_{p,w,\alpha}} \leq \frac{1}{\sqrt{2}} \|f\|_{\mathcal{M}_{p,w,\alpha}}.$$

The theorem is proved.

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