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A new class of metrics and harmonicity on the cotangent bundle

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Abstract. In this paper, we study the harmonicity on cotangent bundle equipped with the new class of metrics [13]. We establish necessary and sufficient conditions under which a covector field is harmonic with respect to this metrics. Next we also construct some examples of harmonic covector fields.

Keywords. Horizontal lift, vertical lift, cotangent bundles, a new class of metrics ,harmonic maps.

Mathematics Subject Classification (2010): 53A45, 53C20, 58E20.

1 Introduction

In the field, one of the first works which deal with the cotangent bundles of a manifold as a Riemannian manifold is that of Patterson, E.M., Walker, A.G. [8], who constructed from an affine symmetric connection on a manifold a Riemannian metric on the cotangent bundle, which they call the Riemann extension of the connection. A generalization of this metric had been given by Sekizawa, M. [11] in his classification of natural transformations of affine connections on manifolds to metrics on their cotangent bundles, obtaining the class of natural Riemann extensions which is a 2-parameter family of metrics, and which had been intensively studied by many authors. On the other hand, inspired by the concept of g-natural metrics on tangent bundles of Riemannian manifolds, Ağca, F. considered another class of metrics on cotangent bundles of Riemannian manifolds, that he callad g-natural metrics [1]. Also, there are studies by other authors, Gezer, A., Altunbas, M.[2], Ocak, F., Kazimova, S. [6], Salimov, A.A., Ağca, F. [9], [10], Yano, K., Ishihara, S.[12], etc...

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X,Y) = \nabla^{\phi}_X d\phi(Y) - d\phi(\nabla_X Y). \tag{1.1}$$

Here ∇ is the Riemannian connection on M and ∇^{ϕ} is the pull-back connection on the pull-back bundle $\phi^{-1}TN$, and

$$\tau(\phi) = trace_g \nabla d\phi, \tag{1.2}$$

is the tension field of ϕ.

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The energy functional of ϕ is defined by

$$E(\phi) = \int_K e(\phi) dv_g, \quad (1.3)$$

such that K is any compact of M, where

$$e(\phi) = \frac{1}{2} trace_g h(d\phi, d\phi), \quad (1.4)$$

is the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional E. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d}{dt}\phi_t\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0} = -\int_K h(\tau(\phi), V) dv_g \quad (1.5)$$

Then ϕ is harmonic if and only if $\tau(\phi) = 0$.

One can refer to [3], [4], [5], [7] for background on harmonic maps.

The main idea in this note consists, in the study of harmonicity on cotangent bundle equipped with the new class of metrics [13]. We establish necessary and sufficient conditions under which a covector field is harmonic respect to this metrics (Theorem 4.2 and Theorem 4.3). We also construct some examples of harmonic covector fields and we give a formula for the construction of non trivial examples of covector fields (Theorem 4.4 and Corollary 4.2). After that we study the harmonicity of the map $\sigma : (M, g) \longrightarrow (T^*N, h^f)$ (Theorem 4.6 and Corollary 4.4).

2 Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{i=\overline{1,m}, \bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space $T^*_x M$, $x \in U$ with respect to the natural coframe dx^i. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s).

Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on T^*M, the vertical distribution $VT^*M = Ker(d\pi)$ and the horizontal distribution HT^*M that define a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \quad (2.1)$$

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be a local expressions in $U \subset M$ of a vector and covector (covector field) field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the horizontal and the vertical lifts of X and ω are defined, respectively by

$$X^H = X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial p_i}, \quad (2.2)$$

$$\omega^V = \omega_i \frac{\partial}{\partial p_i}, \quad (2.3)$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i}\}$, where Γ_{ij}^h are components of the Levi-Civita connection ∇ on M. (see [12] for more details).

From (2.1), (2.2) and (2.3) we have

$$d\pi(\omega^V) = 0\ ,\ \ d\pi(X^H) = X \circ \pi. \tag{2.4}$$

Lemma 2.1 *[12] Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of M satisfies the following*

1. $[\omega^V, \theta^V] = 0$,
2. $[X^H, \theta^V] = (\nabla_X \theta)^V$,
3. $[X^H, Y^H] = [X, Y]^H - (pR(X, Y)u)^V$,

for all vector fields $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$.

Let (M, g) be a Riemannian manifold, we define the map

$$\begin{aligned} \sharp : \Im_1^0(M) &\to \Im_0^1(M) \\ \omega &\mapsto \sharp\omega \end{aligned}$$

by for all $X \in \Im_0^1(M)$, $g(\sharp\omega, X) = \omega(X)$, the map $\sharp$ is $C^\infty(M)$-isomorphism.
Locally for all $\omega = \omega_i dx^i \in \Im_1^0(M)$, we have $\sharp\omega = g^{ij}\omega_i \frac{\partial}{\partial x^j}$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).
For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space T_x^*M by $g^{-1}(\omega, \theta) = g(\sharp\omega, \sharp\theta) = g^{ij}\omega_i\theta_j$.
If ∇ be the Levi-Civita connection of (M, g) we have

$$\nabla_X(\sharp\omega) = \sharp(\nabla_X \omega), \tag{2.5}$$

$$Xg^{-1}(\omega, \theta) = g^{-1}(\nabla_X \omega, \theta) + g^{-1}(\omega, \nabla_X \theta), \tag{2.6}$$

for all $X \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$.

From now on, we noted $\sharp\omega$ by $\tilde{\omega}$ for all $\omega \in \Im_1^0(M)$.

3 A new class of metrics on the cotangent bundle

Definition 3.1 *[13] Let (M, g) be a Riemannian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function on M. On the cotangent bundle T^*M, we define a new class of metrics noted g^f by*

$$g^f(X^H, Y^H) = g(X, Y)^V = g(X, Y) \circ \pi, \tag{3.1}$$

$$g^f(X^H, \theta^V) = 0, \tag{3.2}$$

$$g^f(\omega^V, \theta^V) = fg^{-1}(\omega, p)g^{-1}(\theta, p), \tag{3.3}$$

where $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$.

Theorem 3.1 *[13] Let (M,g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the new class of metrics. If ∇ (resp ∇^f) denote the Levi-Civita connection of (M,g) (resp (T^*M, g^f)), we have:*

$$(1)\ \nabla^f_{X^H} Y^H = (\nabla_X Y)^H,$$
$$(2)\ \nabla^f_{X^H} \theta^V = (\nabla_X \theta)^V + \frac{1}{2f} X(f)\theta^V,$$
$$(3)\ \nabla^f_{\omega^V} Y^H = \frac{1}{2f} Y(f)\omega^V,$$
$$(4)\ \nabla^f_{\omega^V} \theta^V = \frac{-1}{2} g^{-1}(\omega,p) g^{-1}(\theta,p)(grad\, f)^H + \frac{1}{r^2} g^{-1}(\omega,\theta)\mathcal{P}^V,$$

for all $X,Y \in \Im^1_0(M)$ and $\omega,\theta \in \Im^0_1(M)$, where $\mathcal{P}^V$ the canonical vertical vector field on T^*M and R denote the curvature tensor of (M,g).

4 A new class of metrics and Harmonicity.

41 Harmonicity of a covector field $\omega : (M,g) \longrightarrow (T^*M, g^f)$

Now we study the harmonicity of section $\omega : (M,g) \longrightarrow (T^*M, \tilde{g})$ i.e covector field ω on M, and we give the necessary and sufficient conditions under which a covector field is harmonic with respect to the new class of metrics g^f.

Lemma 4.1 *[13] Let (M,g) be a Riemannian manifold. If $\omega \in \Im^0_1(M)$ is a covector field (1-form) on M and $(x,p) \in T^*M$ such that $\omega_x = p$, then we have:*

$$d_x\omega(X_x) = X^H_{(x,p)} + (\nabla_X \omega)^V_{(x,p)}.$$

where $X \in \Im^1_0(M)$.

Proof. Let (U, x^i) be a local chart on M in $x \in M$ and $(\pi^{-1}(U), x^i, p_i)$ be the induced chart on T^*M, if $X_x = X^i(x)\frac{\partial}{\partial x^i}|_x$ and $\omega_x = \omega_i(x) dx^i|_x = p$, then

$$\begin{aligned}
d_x\omega(X_x) &= X^i(x)\frac{\partial}{\partial x^i}|_{(x,p)} + X^i(x)\frac{\partial \omega_j}{\partial x^i}(x)\frac{\partial}{\partial p_j}|_{(x,p)}\\
&= X^i(x)\frac{\partial}{\partial x^i}|_{(x,p)} + \omega_k(x)\Gamma^k_{ji}(x)X^j(x)\frac{\partial}{\partial p_i}|_{(x,p)}\\
&\quad -\omega_k(x)\Gamma^k_{ji}(x)X^j(x)\frac{\partial}{\partial p_i}|_{(x,p)} + X^i(x)\frac{\partial \omega_j}{\partial x^i}(x)\frac{\partial}{\partial p_j}|_{(x,p)}\\
&= X^i(x)\frac{\partial}{\partial x^i}|_{(x,p)} + p_k\Gamma^k_{ji}(x)X^j(x)\frac{\partial}{\partial p_i}|_{(x,p)}\\
&\quad +X^i(x)\frac{\partial \omega_j}{\partial x^i}(x)\frac{\partial}{\partial p_j}|_{(x,p)} - \omega_k(x)\Gamma^k_{ij}(x)X^i(x)\frac{\partial}{\partial p_j}|_{(x,p)}\\
&= X^H_{(x,p)} + X^i(x)\big[\frac{\partial \omega_j}{\partial x^i}(x) - \omega_k(x)\Gamma^k_{ij}(x)X^i(x)\big](dx^i)^V_{(x,p)}\\
&= X^H_{(x,p)} + (\nabla_X \omega)^V_{(x,p)}.
\end{aligned}$$

Hence we have the following Lemma.

Lemma 4.2 *Let (M^m, g) be a Riemannian m-dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the new class of metrics. If $\omega \in \Im_1^0(M)$, then the energy density associated to ω is given by:*

$$e(\omega) = \frac{m}{2} + \frac{f}{2} trace_g g^{-1}(\nabla\omega, \omega)^2. \tag{4.1}$$

Proof. Let $(x,p) \in T^*M$, $\omega \in \Im_1^0(M)$, $\omega_x = p$ and $(E_1, \cdots, E_m)$ be a local orthonormal frame on M, then:

$$\begin{aligned} e(\omega)_x &= \frac{1}{2} trace_g g^f(d\omega, d\omega)_{(x,p)} \\ &= \frac{1}{2}\sum_{i=1}^{m} g^f(d\omega(E_i), d\omega(E_i))_{(x,p)}. \end{aligned}$$

Using Lemma 4.1, we obtain:

$$\begin{aligned} e(\omega) &= \frac{1}{2}\sum_{i=1}^{m} g^f(E_i^H + (\nabla_{E_i}\omega)^V, E_i^H + (\nabla_{E_i}\omega)^V) \\ &= \frac{1}{2}\sum_{i=1}^{m} \left[(g^f(E_i^H, E_i^H) + g^f((\nabla_{E_i}\omega)^V, (\nabla_{E_i}\omega)^V))\right] \\ &= \frac{1}{2}\sum_{i=1}^{m} \left[g(E_i, E_i) + f g^{-1}(\nabla_{E_i}\omega, \omega)^2\right] \\ &= \frac{m}{2} + \frac{f}{2} trace_g g^{-1}(\nabla\omega, \omega)^2. \end{aligned}$$

A direct consequence of usual calculations using the Lemma 4.2 gives the following result.

Theorem 4.1 *Let (M^m, g) be a Riemannian m-dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the new class of metrics. If $\omega \in \Im_1^0(M)$, then the tension field associated to ω is given by:*

$$\begin{aligned} \tau(\omega) &= \frac{-1}{2}\Big[trace_g\big[g^{-1}(\nabla\omega, \omega)^2 grad\, f\big]\Big]^H \\ &\quad + \Big[trace_g\big[\nabla^2\omega + \frac{1}{f} df(*)(\nabla\omega) + \frac{1}{\|\omega\|^2} g^{-1}(\nabla\omega, \nabla\omega)\omega\big]\Big]^V. \end{aligned} \tag{4.2}$$

where $r^2 = g^{-1}(\omega, \omega) = \|\omega\|^2$.

Proof. Let $(x,p) \in T^*M$, $\omega \in \Im^0_1(M)$, $\omega_x = p$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla^M_{E_i} E_i)_x = 0$, then

$$\begin{aligned}
\tau(\omega)_x &= trace_g(\nabla d\omega)_x \\
&= \sum_{i=1}^{m} \{\nabla^{\omega}_{E_i} d\omega(E_i) - d\omega(\nabla^M_{E_i} E_i)\}_x \\
&= \sum_{i=1}^{m} \{\nabla^f_{d\omega(E_i)} d\omega(E_i)\}_{(x,p)} \\
&= \sum_{i=1}^{m} \{\nabla^f_{(E_i^H + (\nabla_{E_i}\omega)^V)} (E_i^H + (\nabla_{E_i}\omega)^V)\}_{(x,p)} \\
&= \sum_{i=1}^{m} \{\nabla^f_{E_i^H} E_i^H + \nabla^f_{E_i^H} (\nabla_{E_i}\omega)^V + \nabla^f_{(\nabla_{E_i}\omega)^V} (E_i)^H \\
&\quad + \nabla^f_{(\nabla_{E_i}\omega)^V} (\nabla_{E_i}\omega)^V\}_{(x,p)}.
\end{aligned}$$

Using Theorem 3.1, we obtain

$$\begin{aligned}
\tau(\omega) &= \sum_{i=1}^{m} \Big[(\nabla_{E_i} E_i)^H + (\nabla_{E_i}\nabla_{E_i}\omega)^V + \frac{1}{2f} E_i(f)(\nabla_{E_i}\omega)^V \\
&\quad + \frac{1}{2f} E_i(f)(\nabla_{E_i}\omega)^V - \frac{1}{2} g^{-1}(\nabla_{E_i}\omega,\omega)^2 (grad\, f)^H \\
&\quad + \frac{1}{\|\omega\|^2} g^{-1}(\nabla_{E_i}\omega, \nabla_{E_i}\omega)\omega^V\Big] \\
&= \sum_{i=1}^{m} \{-\frac{1}{2} g^{-1}(\nabla_{E_i}\omega,\omega)^2 (grad\, f)^H + (\nabla_{E_i}\nabla_{E_i}\omega)^V \\
&\quad + \frac{1}{f} E_i(f)(\nabla_{E_i}\omega)^V + \frac{1}{\|\omega\|^2} g^{-1}(\nabla_{E_i}\omega, \nabla_{E_i}\omega)\omega^V\} \\
&= \frac{-1}{2}\Big[trace_g\big[g^{-1}(\nabla\omega,\omega)^2 grad\, f\big]\Big]^H \\
&\quad + \Big[trace_g\big[\nabla^2\omega + \frac{1}{f} df(*)(\nabla\omega) + \frac{1}{\|\omega\|^2} g^{-1}(\nabla\omega,\nabla\omega)\omega\big]\Big]^V.
\end{aligned}$$

From that, we have the following result.

Theorem 4.2 *Let (M^m, g) be a Riemannian m-dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the new class of metrics. If $\omega \in \Im^0_1(M)$, then ω is harmonic covector field if and only if the following conditions are verified*

$$trace_g\big[g^{-1}(\nabla\omega,\omega)^2 grad\, f\big] = 0, \tag{4.3}$$

$$trace_g\big[\nabla^2\omega + \frac{1}{f} df(*)(\nabla\omega) + \frac{1}{\|\omega\|^2} g^{-1}(\nabla\omega,\nabla\omega)\omega\big] = 0. \tag{4.4}$$

where $g^{-1}(\omega,\omega) = \|\omega\|^2$.

Proof. The statement is a direct consequence of Theorem 4.1.

The direct consequence of Theorem 4.2 is the following Corollary.

Corollary 4.1 *Let (M^m, g) be a Riemannian m-dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the new class of metrics. If $\omega \in \Im^1_0(M)$, then ω is a parallel covector field $($i.e $\nabla\omega = 0)$ then ω is harmonic.*

The necessary and sufficient condition under which a covector field is harmonic with respect to the new class of metrics g^f is given in the following theorem.

Theorem 4.3 *Let (M^m, g) be a Riemannian compact m-dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the new class of metrics. If $\omega \in \Im^0_1(M)$, then ω is harmonic covector field if and only if ω is parallel.*

Proof. If ω is parallel from Corollary 4.1, we deduce that ω is harmonic covector field. Inversely, let φ_t be a compactly supported variation of ω defined by:

$$\begin{aligned} \mathbb{R} \times M &\longrightarrow T^*_x M \\ (t, x) &\longmapsto \varphi_t(x) = (1+t)\omega_x \end{aligned}$$

From lemma 4.2 we have:

$$e(\varphi_t) = \frac{m}{2} + \frac{(1+t)^4}{2} f\, trace_g g^{-1}(\nabla\omega, \omega)^2,$$

$$E(\varphi_t) = \frac{m}{2} Vol(M) + \frac{(1+t)^4}{2} \int_M f\, trace_g g^{-1}(\nabla\omega, \omega)^2 dv_g$$

ω is harmonic, then we have:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} E(\varphi_t)|_{t=0} \\ &= \frac{\partial}{\partial t}\Big[\frac{m}{2} Vol(M)\Big]_{t=0} + \frac{\partial}{\partial t}\Big[\frac{(1+t)^4}{2} \int_M f\, trace_g g^{-1}(\nabla\omega, \omega)^2 dv_g\Big]_{t=0} \\ &= \int_M 2f\, trace_g g^{-1}(\nabla\omega, \omega)^2 dv_g \end{aligned}$$

which gives

$$g^{-1}(\nabla\omega, \omega)^2 = 0,$$

hence $\nabla\omega = 0$.

As an application to the above, we give the following examples.

Example 1 Let $\mathbb{R}^2$ endowed with the Riemannian metric in in polar coordinates defined by:

$$g_{\mathbb{R}^2} = dr^2 + r^2 d\theta.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},\ \Gamma^1_{22} = -r,$$

then we have,

$$\nabla_{\frac{\partial}{\partial r}} dr = 0,\ \nabla_{\frac{\partial}{\partial r}} d\theta = -\frac{1}{r} d\theta,\ \nabla_{\frac{\partial}{\partial \theta}} dr = r d\theta,\ \nabla_{\frac{\partial}{\partial \theta}} d\theta = -\frac{1}{r} dr.$$

The covector field $\omega = \cos\theta dr - r\sin\theta d\theta$ is harmnic because ω is parallel, indeed

$$\nabla_{\frac{\partial}{\partial r}} \omega = \cos\theta \nabla_{\frac{\partial}{\partial r}} dr - \sin\theta d\theta - r\sin\theta \nabla_{\frac{\partial}{\partial r}} d\theta = 0,$$

$$\nabla_{\frac{\partial}{\partial \theta}} \omega = -\sin\theta dr + \cos\theta \nabla_{\frac{\partial}{\partial \theta}} dr - r\cos\theta d\theta - r\sin\theta \nabla_{\frac{\partial}{\partial \theta}} d\theta = 0,$$

i.e $\nabla\omega = 0$, then ω is harmonic.

Example 2 Let $\mathbb{S}^2 \times \mathbb{R}$ endowed with the product of canonical metric

$$g = d\alpha^2 + sin^2(\alpha)d\beta^2 + dt^2,$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma^2_{12} = \Gamma^2_{21} = \cot\alpha, \; \Gamma^1_{22} = -\sin\alpha\cos\alpha.$$

Then, $\nabla_{\frac{\partial}{\partial\alpha}} dt = \nabla_{\frac{\partial}{\partial\psi}} dt = \nabla_{\frac{\partial}{\partial t}} dt = 0$, because $\Gamma^3_{ij} = 0$, for all $i, j = 1, 2, 3$.
The covector field $\omega = dt$ is harmnic because ω is parallel.

Example 3 Let $\mathbb{S}^1$ (Riemannian compact manifold) equipped with the metric:

$$g_{\mathbb{S}^1} = e^x dx^2.$$

The Christoffel symbols of the Levi-cita connection are given by:

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1}\right) = \frac{1}{2}.$$

The covector field $\omega = f(x)dx$, $f \in C^\infty(\mathbb{S}^1)$ is harmonic if and only if ω is parallel,

$$\begin{aligned}\nabla\omega = 0 &\Leftrightarrow f'(x) - \frac{1}{2}f(x) = 0\\ &\Leftrightarrow f(x) = k\exp(\frac{x}{2}) \; , \; k \in \mathbb{R}\\ &\Leftrightarrow \omega = k\exp(\frac{x}{2})dx \; , \; k \in \mathbb{R}.\end{aligned}$$

Remark 4.1 In general , using Corollary 4.1 and Theorem 4.3, we can construct many examples for harmonic covector fields.

Now we study a special case on the flat Riemannian manifold which is the real euclidean space $(\mathbb{R}^m, g_0)$.

Theorem 4.4 *Let* $(\mathbb{R}^m, g_0)$ *the real euclidean space and* $(T^*\mathbb{R}^m, g_0^f)$ *its cotangent bundle equipped with the new class of metrics. If* $\omega = (\omega_1, \cdots, \omega_m) \in \Im^0_1(\mathbb{R}^m)$*, then* ω *is harmonic covector field if and only if the following conditions are verified*

$$\omega = constant \;\; or \;\; f = constant, \tag{4.5}$$

$$\sum_{i=1}^m \Big(\frac{\partial^2\omega_k}{\partial(x^i)^2} + \frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial\omega_k}{\partial x^i} + \frac{1}{\|\omega\|^2}\sum_{j=1}^m (\frac{\partial\omega_j}{\partial x^i})^2\omega_k\Big) = 0. \tag{4.6}$$

for all $k = \overline{1,m}$*, where* $g^{-1}(\omega,\omega) = \|\omega\|^2$.

Proof. Let $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1,m}}$ be a canonical frame on $\mathbb{R}^m$. Using Theorem 4.2, we have:
$\tau(\omega) = 0$ equivalent the following conditions (4.3) and (4.4) are verified

$$\begin{aligned}(4.3) &\Leftrightarrow trace_g\big[g^{-1}(\nabla\omega,\omega)^2 grad\, f\big] = 0\\ &\Leftrightarrow \sum_{i=1}^m g^{-1}(\nabla_{\frac{\partial}{\partial x^i}}\omega,\omega)^2 = 0 \;\; or \;\; grad\, f = 0\\ &\Leftrightarrow \sum_{i,j=1}^m (\frac{\partial\omega_j}{\partial x^i}\omega_j)^2 = 0 \;\; or \;\; f = constant\\ &\Leftrightarrow \frac{\partial\omega_j}{\partial x^i} = 0 \;, \; for\ all\ i,j = \overline{1,m} \;\; or \;\; f = constant\\ &\Leftrightarrow \omega = constant \;\;\; or \;\; f = constant.\end{aligned}$$

$$
\begin{aligned}
(4.4) \Leftrightarrow & \; trace_g\big[\nabla^2\omega + \frac{1}{f}df(*)(\nabla\omega) + \frac{1}{r^2}g^{-1}(\nabla\omega,\nabla\omega)\omega\big] = 0 \\
\Leftrightarrow & \sum_{i=1}^{m}\Big[\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^i}}\omega + \frac{1}{f}df(\frac{\partial}{\partial x^i})\nabla_{\frac{\partial}{\partial x^i}}\omega + \frac{1}{\|\omega\|^2}g^{-1}(\nabla_{\frac{\partial}{\partial x^i}}\omega,\nabla_{\frac{\partial}{\partial x^i}}\omega)\omega\Big] = 0 \\
\Leftrightarrow & \sum_{i=1}^{m}\Big\{\sum_{k=1}^{m}\Big(\frac{\partial^2\omega_k}{\partial(x^i)^2}dx^k + \frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial\omega_k}{\partial x^i}dx^k + \frac{1}{\|\omega\|^2}(\frac{\partial\omega^k}{\partial x^i})^2\sum_{j=1}^{m}\omega_j dx^j\Big)\Big\} = 0 \\
\Leftrightarrow & \sum_{i=1}^{m}\Big\{\sum_{k=1}^{m}\Big(\frac{\partial^2\omega_k}{\partial(x^i)^2} + \frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial\omega_k}{\partial x^i} + \frac{1}{\|\omega\|^2}\sum_{j=1}^{m}(\frac{\partial\omega^j}{\partial x^i})^2\omega_k\Big)dx^k\Big\} = 0 \\
\Leftrightarrow & \sum_{i=1}^{m}\Big(\frac{\partial^2\omega_k}{\partial(x^i)^2} + \frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial\omega_k}{\partial x^i} + \frac{1}{\|\omega\|^2}\sum_{j=1}^{m}(\frac{\partial\omega^j}{\partial x^i})^2\omega_k\Big) = 0
\end{aligned}
$$

for all $k = \overline{1,m}$.

From that, we have

Example 4 If $\mathbb{R}^m$ is endowed with the canonical metric, then any constant covector field ω on $\mathbb{R}^m$ is harmonic.

Corollary 4.2
Let $(\mathbb{R}^m, g_0)$ the real euclidean space, $(T^*\mathbb{R}^m, g_0^f)$ its cotangent bundle equipped with the new class of metrics and $\omega = (\omega_1, \cdots, \omega_m) \in \Im_1^0(\mathbb{R}^m)$. If f is a constant function, then ω is a harmonic covector field if and only if for all $k = \overline{1,m}$:

$$
\sum_{i=1}^{m}\Big(\frac{\partial^2\omega_k}{\partial(x^i)^2} + \frac{1}{\|\omega\|^2}\sum_{j=1}^{m}(\frac{\partial\omega_j}{\partial x^i})^2\omega_k\Big) = 0, \tag{4.7}
$$

for all $k = \overline{1,m}$, where $g^{-1}(\omega,\omega) = \|\omega\|^2$.

Corollary 4.3
Let $(\mathbb{R}^m, g_0)$ the real euclidean space, $(T^*\mathbb{R}^m, g_0^f)$ its cotangent bundle equipped with the new class of metrics and $\omega = (\omega_1, \cdots, \omega_m) \in \Im_1^0(\mathbb{R}^m)$. If $f \neq constant$, then ω is a harmonic covector field if and only if ω is constant.

Remark 4.2
Using Corollary 4.2, we can construct many examples of non trivial harmonic vector fields.

Example 5
If $\mathbb{R}^n$ is endowed with the canonical metric and $T^*\mathbb{R}^m$ its cotangent bundle equipped with the new class of metrics such as $f =$ constant. From corollary 4.2, we deduce that.
If $\omega = (h(x_1), 0, \cdots, 0) \in \Im_1^0(\mathbb{R}^m)$ is a harmonic covector field if and only if the function h is solution of differential equation

$$
h'' + \frac{(h')^2}{h} = 0, \tag{4.8}
$$

i.e $h(x_1) = \pm\sqrt{ax_1 + b}$, where $a, b \in \mathbb{R}$.

42 Harmonicity of the map $\sigma : (M, g) \longrightarrow (T^*N, h^f)$

Now we study the harmonicity of the map $\sigma : (M, g) \longrightarrow (T^*N, h^f)$ and we give the necessary and sufficient conditions under which this map is harmonic with respect to the new class of metrics h^f.

Lemma 4.3 *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds and σ be a map that covers φ, ($\varphi = \pi_N \circ \sigma$) defined by*

$$\begin{aligned} \sigma : M &\longrightarrow T^*N \\ x &\longmapsto (\varphi(x), q) \end{aligned}$$

where $q \in T^*_{\varphi(x)}N$ and $\pi_N : T^*N \to N$ is the canonical projection, then

$$d\sigma(X) = (d\varphi(X))^H + (\nabla^{\varphi}_X \sigma)^V, \tag{4.9}$$

for all $X \in \Im^1_0(M)$.

Proof. Let $x \in M$, $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(N)$ such that $\omega_{\varphi(x)} = q \in T^*_{\varphi(x)}N$. Using Lemma 4.1, we obtain:

$$\begin{aligned} d_x\sigma(X_x) &= d_x(\omega \circ \varphi)(X_x) \\ &= d_{\varphi(x)}\omega(d_x\varphi(X_x)) \\ &= (d\varphi(X))^H_{(\varphi(x),q)} + (\nabla_{d\varphi(X)}\omega)^V_{(\varphi(x),q)} \\ &= (d\varphi(X))^H_{(\varphi(x),q)} + (\nabla^{\varphi}_X \sigma)^V_{(\varphi(x),q)}. \end{aligned}$$

Theorem 4.5 *Let (M^m, g), (N^n, h) be two Riemannian manifolds, $f : N \to]0, +\infty[$ be a strictly positive smooth function on N, (T^*N, h^f) the cotangent bundle of N equipped with the new class of metrics and $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map. The tension field of the map*

$$\begin{aligned} \sigma : (M, g) &\longrightarrow (T^*N, h^f) \\ x &\longmapsto (\varphi(x), q) \end{aligned}$$

such that $q \in T^*_{\varphi(x)}N$ is given by

$$\begin{aligned} \tau(\sigma) = &\Big[\tau(\varphi) - \frac{1}{2} trace_g\big[h^{-1}(\nabla^{\varphi}\sigma, \sigma)^2 grad\, f\big]\Big]^H \\ &+ \Big[trace_g\big[(\nabla^{\varphi})^2\sigma + \frac{1}{f} df(d\varphi(*))(\nabla^{\varphi}\sigma) + \frac{1}{\|\sigma\|^2} h^{-1}(\nabla^{\varphi}\sigma, \nabla^{\varphi}\sigma)\sigma\big]\Big]^V \end{aligned} \tag{4.10}$$

where $h^{-1}(\sigma, \sigma) = \|\sigma\|^2$.

Proof. Let $x \in M$ and $(E_1, \cdots, E_m)$ be a local orthonormal frame on M such that $(\nabla^M_{E_i} E_i)_x = 0$ and $\sigma(x) = (\varphi(x), q), q \in T^*_{\varphi(x)}N$, we have

$$\begin{aligned}
\tau(\sigma)_x &= trace_g(\nabla d\sigma)_x \\
&= \sum_{i=1}^m \{\nabla^\sigma_{E_i} d\sigma(E_i)\}_{(\varphi(x),q)} \\
&= \sum_{i=1}^m \{\nabla^{T^*N}_{d\sigma(E_i)} d\sigma(E_i)\}_{(\varphi(x),q)} \\
&= \sum_{i=1}^m \{\nabla^{T^*N}_{(d\varphi(E_i))^H} (d\varphi(E_i))^H + \nabla^{T^*N}_{(d\varphi(E_i))^H} (\nabla^\varphi_{E_i}\sigma)^V \\
&\quad + \nabla^{T^*N}_{(\nabla^\varphi_{E_i}\sigma)^V} (d\varphi(E_i))^H + \nabla^{T^*N}_{(\nabla^\varphi_{E_i}\sigma)^V} (\nabla^\varphi_{E_i}\sigma)^V\}_{(\varphi(x),q)}.
\end{aligned}$$

From the theorem 3.1, we obtain:

$$\begin{aligned}
\tau(\sigma) &= \sum_{i=1}^m \Big[(\nabla^N_{d\varphi(E_i)} d\varphi(E_i))^H + (\nabla^N_{d\varphi(E_i)} \nabla^\varphi_{E_i}\sigma)^V \\
&\quad + \frac{1}{2f} d\varphi(E_i)(f)(\nabla^\varphi_{E_i}\sigma)^V + \frac{1}{2f} d\varphi(E_i)(f)(\nabla^\varphi_{E_i}\sigma)^V \\
&\quad - \frac{1}{2} h^{-1}(\nabla^\varphi_{E_i}\sigma, \sigma)^2 (grad\, f)^H + \frac{1}{\|\sigma\|^2} h^{-1}(\nabla^\varphi_{E_i}\sigma, \nabla^\varphi_{E_i}\sigma)\sigma^V\Big] \\
&= \sum_{i=1}^m \Big[(\nabla^\varphi_{E_i} d\varphi(E_i))^H + (\nabla^\varphi_{E_i}\nabla^\varphi_{E_i}\sigma)^V + \frac{1}{f} df(d\varphi(E_i))(\nabla^\varphi_{E_i}\sigma)^V \\
&\quad - \frac{1}{2} h^{-1}(\nabla^\varphi_{E_i}\sigma, \sigma)^2 (grad\, f)^H + \frac{1}{\|\sigma\|^2} h^{-1}(\nabla^\varphi_{E_i}\sigma, \nabla^\varphi_{E_i}\sigma)\sigma^V\Big] \\
&= \Big[\tau(\varphi) - \frac{1}{2} trace_g\big[h^{-1}(\nabla^\varphi\sigma, \sigma)^2 grad\, f\big]\Big]^H \\
&\quad + \Big[trace_g\big[(\nabla^\varphi)^2\sigma + \frac{1}{f} df(d\varphi(*))(\nabla^\varphi\sigma) + \frac{1}{\|\sigma\|^2} h^{-1}(\nabla^\varphi\sigma, \nabla^\varphi\sigma)\sigma\big]\Big]^V.
\end{aligned}$$

From Theorem 4.5 we obtain.

Theorem 4.6 *Let (M^m, g), (N^n, h) be two Riemannian manifolds, $f : N \to]0, +\infty[$ be a strictly positive smooth function on N, (T^*N, h^f) the cotangent bundle of N equipped with the new class of metrics and $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map. The map*

$$\begin{aligned}
\sigma : (M, g) &\longrightarrow (T^*N, h^f) \\
x &\longmapsto (\varphi(x), q)
\end{aligned}$$

such that $q \in T^*_{\varphi(x)}N$ is a harmonic if and only if the following conditions are verified

$$\tau(\varphi) = \frac{1}{2} trace_g\big[h^{-1}(\nabla^\varphi\sigma, \sigma)^2 grad\, f\big], \tag{4.11}$$

$$trace_g\big[(\nabla^\varphi)^2\sigma + \frac{1}{f} df(d\varphi(*))(\nabla^\varphi\sigma) + \frac{1}{\|\sigma\|^2} h^{-1}(\nabla^\varphi\sigma, \nabla^\varphi\sigma)\sigma\big] = 0. \tag{4.12}$$

where $h^{-1}(\sigma, \sigma) = \|\sigma\|^2$.

Corollary 4.4 *Let (M^m, g), (N^n, h) be two Riemannian manifolds, $f : N \to]0, +\infty[$ be a strictly positive constant on N, (T^*N, h^f) the cotangent bundle of N equipped with the new class of metrics and $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map. The map*

$$\begin{aligned} \sigma : (M, g) &\longrightarrow (T^*N, h^f) \\ x &\longmapsto (\varphi(x), q) \end{aligned}$$

is a harmonic if and only if φ is harmonic and

$$trace_g\big[(\nabla^{\varphi})^2\sigma + \frac{1}{\|\sigma\|^2} h^{-1}(\nabla^{\varphi}\sigma, \nabla^{\varphi}\sigma)\sigma\big] = 0.$$

where $h^{-1}(\sigma, \sigma) = \|\sigma\|^2$.

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