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Anisotropic maximal operator with rough kernel and its commutators in generalized weighted anisotropic Morrey spaces

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Abstract. *Let $\Omega \in L_q(S^{n-1})$ be a homogeneous function of degree zero with $q > 1$. In this paper, we study the boundedness of the anisotropic maximal operator with rough kernels M^d_Ω and its commutators $[b, M^d_\Omega]$ on generalized weighted anisotropic Morrey spaces $M_{p,\varphi}(w)$. We find the sufficient conditions on the pair (φ_1, φ_2) with $q' \le p < \imath$, $p \ne 1$ and $w \in A_{p/q'}$ or $1 < p \le q$ and $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the operators M^d_Ω from one generalized weighted anisotropic Morrey space $M_{p,\varphi_1,d}(w)$ to another $M_{p,\varphi_2,d}(w)$ for $1 < p < \infty$. We find the sufficient conditions on the pair (φ_1, φ_2) with $b \in BMO(\mathbb{R}^n)$ and $q' \le p < \imath$, $p \ne 1$, $w \in A_{p/q'}$ or $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the operators $[b, M^d_\Omega]$ from $M_{p,\varphi_1,d}(w)$ to $M_{p,\varphi_2,d}(w)$ for $1 < p < \infty$. In all cases the conditions for the boundedness of the operators M^d_Ω, $[b, M^d_\Omega]$ are given in terms of supremal-type inequalities on (φ_1, φ_2) and w, which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ in r.*

Keywords. Anisotropic maximal operator; rough kernel; generalized weighted anisotropic Morrey spaces; commutator; A_p weights

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1 Introduction

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [9,10] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [11] states that the commutator operator $[b, K]f = K(bf) - b\,Kf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [13–15, 19, 28, 30]).

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The classical Morrey spaces were originally introduced by Morrey in [39] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [13, 14,16,19,23]. Guliyev, Mizuhara and Nakai [21,38,43] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [22,23,25,44]). Recently, Komori and Shirai [36] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [24] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$. In [24] Guliyev also studied the boundedness of the classical operators and its commutators in these spaces $M_{p,\varphi}(w)$, see also Guliyev et al. [3,15,17,26,29, 30,32–34].

Watson [45] and independently by Duoandikoetxea [18] established weighted L_p boundedness for the singular integral operators with rough kernels and their commutators.

Let $\mathbb{R}^n$ be the n-dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$, S^{n-1} denotes the unit sphere on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,r)$ denote the set $\mathbb{R}^n \backslash B(x,r)$. Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \big(t^{d_1}x_1, \dots, t^{d_n}x_n\big)$. By [6,12], the function $F(x,\rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points x, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([4,6,7,12]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1-x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n-x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let also $\Pi_d(x,r) = \{y \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i - y_i|^{1/d_i} < r\}$ denote the parallelopiped, ${}^{\complement}\mathcal{E}_d(x,r) = \mathbb{R}^n \setminus \mathcal{E}_d(x,r)$ be the complement of $\mathcal{E}_d(0,r)$. If $d = \mathbf{1} \equiv (1, \dots, 1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = B(x,r)$. Note that in the standard parabolic case $d = (1, \dots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $A_t = \operatorname{diag}\{t^{d_1}, \dots, t^{d_n}\}$. Suppose that Ω satisfies the following conditions.

(i) Ω is a A_t-homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(A_t x) \equiv \Omega\left(t^{d_1}x_1, \dots, t^{d_n}x_n\right) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The anisotropic maximal operator with rough kernel M_Ω^d is defined by

$$M_\Omega^d f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |\Omega(x-y)|\, |f(y)| dy.$$

The commutators generated by a suitable function b and the operator M_Ω^d is formally defined by

$$[b, M_\Omega^d] f = M_\Omega^d(bf) - b M_\Omega^d f.$$

It is obvious that when $\Omega \equiv 1$, M_Ω^d is the anisotropic maximal operator M^d. For $b \in L_1^{\rm loc}(\mathbb{R}^n)$ the commutator of the anisotropic maximal operator $M_{\Omega,b}^d$ is defined by

$$M_{\Omega,b}^d f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |\Omega(x-y)|\, |b(x)-b(y)|\, |f(y)| dy. \qquad (1.2)$$

Therefore, it will be an interesting thing to study the property of M_Ω. The main purpose of this paper is to show that anisotropic maximal operator with rough kernels M_Ω^d is bounded from one generalized weighted anisotropic Morrey space $M_{p,\varphi_1,d}(w)$ to another $M_{p,\varphi_2,d}(w)$, $1<p<\infty$. We find the sufficient conditions on the pair (φ_1,φ_2) with $b \in BMO(\mathbb{R}^n)$ and $q' \le p < 1$, $p \neq 1$, $w \in A_{p/q'}$ or $1<p\le q$, $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the commutator operators $[b, M_\Omega^d]$ from $M_{p,\varphi_1,d}(w)$ to $M_{p,\varphi_2,d}(w)$ for $1<p<\infty$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

Next we will give the weighted boundedness of anisotropic maximal operator M_Ω^d with rough kernel and its commutator. In their proof, the weighted boundedness of the anisotropic maximal operator M_Ω^d with rough kernel (for its definition, see (1.2)) is needed, while the latter itself is of great significance.

Theorem 2.1 *[18] Suppose that Ω satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1<q\le\infty$. Then for every $q' \le p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$ or $1<p\le q$, $p\neq 1$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant C independent of f such that*

$$\|M_\Omega^d f\|_{L_{p,w}} \le C \|f\|_{L_{p,w}}.$$

Theorem 2.2 *[5] Suppose that Ω satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1<q\le\infty$. Let also $b \in BMO(\mathbb{R}^n)$. Then for every $q' \le p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$ or $1<p\le q$, $p\neq 1$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant C independent of f such that*

$$\|M_{\Omega,b}^d f\|_{L_{p,w}} \le C \|f\|_{L_{p,w}}.$$

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\Omega^d]$ and $M_{\Omega,b}^d$ are valid:

Let b be any non-negative locally integrable function. Then

$$|[b, M_\Omega^d] f(x)| \le M_{\Omega,b}^d f(x), \qquad x \in \mathbb{R}^n$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M_\Omega^d] f(x)| \le M_{\Omega,b}^d f(x) + 2b^-(x) M_\Omega^d f(x), \qquad x \in \mathbb{R}^n \qquad (2.1)$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ (see, for example, [1]).

In the sequel $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively. We also denote

$$\mathbb{A} = \{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow): \lim_{t\to 0^+} \varphi = 0\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|ug\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty),$$

The following theorem was proved in [8].

Theorem 2.3 *[8] Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,\cdot)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(\mathbb{R}_+)$ to $L_{\infty,v_2}(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \big(\|v_1\|^{-1}_{L_\infty(\cdot,1)}\big)\right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

3 Generalized weighted anisotropic Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [39] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [20, 37].

We recall that a weight function w is in the Muckenhoupt class A_p [40], $1 < p < \infty$, if

$$\begin{aligned}[w]_{A_p} :&= \sup_{\mathcal{E}} [w]_{A_p(\mathcal{E})} \\ &= \sup_{\mathcal{E}} \left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} w(x)dx\right)\left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} w(x)^{1-p'}dx\right)^{p-1} \end{aligned} \tag{3.1}$$

where the sup is taken with respect to all the anisotropic balls $\mathcal{E}$ and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls $\mathcal{E}$ using Hölder's inequality, we have that

$$[w]^{1/p}_{A_p(\mathcal{E})} = |\mathcal{E}|^{-1}\|w\|^{1/p}_{L_1(\mathcal{E})}\,\|w^{-1/p}\|_{L_{p'}(\mathcal{E})} \geq 1. \tag{3.2}$$

For $p = 1$, the class A_1 is defined by the condition $M^d w(x) \leq Cw(x)$ with $[w]_{A_1} = \sup\limits_{x\in\mathbb{R}^n} \frac{M^d w(x)}{w(x)}$, and for $p = \infty$ $A_\infty = \bigcup_{1\le p<\infty} A_p$ and $[w]_{A_1} = \inf\limits_{1\le p<1} [w]_{A_p}$.

Remark 3.1 It is known that

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]^{q'/p'}_{A_{p'/q'}(\mathcal{E})} = |\mathcal{E}|^{-1}\|w^{1-p'}\|^{q'/p'}_{L_1(\mathcal{E})}\,\|w^{q'/p}\|_{L_{(p'/q')'}(\mathcal{E})}.$$

Moreover, we can write $w^{1-p'} \in A_{p'/q'} \Rightarrow w^{1-p'} \in A_{p'}$ because of $w^{1-p'} \in A_{p'/q'} \subset A_{p'}$. Therefore, we get

$$\begin{aligned} w^{1-p'} \in A_{p'/q'} &\Rightarrow w^{1-p'} \in A_{p'} \\ &\Rightarrow [w^{1-p'}]^{1/p'}_{A_{p'}(\mathcal{E})} = |\mathcal{E}|^{-1}\|w^{1-p'}\|^{1/p'}_{L_1(\mathcal{E})}\,\|w^{1/p}\|_{L_p(\mathcal{E})}. \end{aligned} \tag{3.3}$$

But the opposite is not true.

Remark 3.2 Let's write $w^{1-p'} \in A_{p'/q'}$ and used the definitions A_p classes we get the following

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{\frac{q(p-1)}{p(q-1)}} = |\mathcal{E}|^{-1} \|w^{1-p'}\|_{L_1(\mathcal{E})}^{\frac{q(p-1)}{p(q-1)}} \|w^{q'/p}\|_{L_{(p'/q')'}(\mathcal{E})}$$
$$\Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'} = |\mathcal{E}|^{-\frac{q-1}{q}} \|w^{1-p'}\|_{L_1(\mathcal{E})}^{1/p'} \|w\|_{L_{\frac{q}{q-p}}(\mathcal{E})}^{1/p}, \tag{3.4}$$

where the following equalities are provided.

$$1-p' = -\frac{p'}{p}, \quad \frac{q'}{p} = \frac{q}{p(q-1)}, \quad \frac{q'}{p'} = \frac{q(p-1)}{p(q-1)}, \quad \left(\frac{q}{p}\right)' = \frac{q}{q-p}, \quad \left(\frac{p'}{q'}\right)' = \frac{p(q-1)}{q-p}.$$

Then from eq.(3.3) and eq.(3.4) we have

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'}$$
$$= |\mathcal{E}|^{\frac{1}{q}} [w^{1-p'}]_{A_{p'}(\mathcal{E})}^{1/p'} \|w^{1/p}\|_{L_p(\mathcal{E})}^{-1} \|w\|_{L_{\frac{q}{q-p}}(\mathcal{E})}^{1/p}. \tag{3.5}$$

Guliyev [24] introduced generalized weighted Morrey spaces $M^{p,\varphi}(w)$ as follows.

Definition 3.1 *[24] Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted anisotropic Morrey space, the space of all functions $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi,d}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}} \, \|f\|_{L_{p,w}(\mathcal{E}(x,r))},$$

where $L_{p,w}(\mathcal{E}(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(\mathcal{E}(x,r))} \equiv \|f\chi_{\mathcal{E}(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{\mathcal{E}(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi,d}(w)$ we denote the weak generalized weighted anisotropic Morrey space of all functions $f \in WL_{p,w}^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi,d}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}} \, \|f\|_{WL_{p,w}(\mathcal{E}(x,r))} < \infty,$$

where $WL_{p,w}(\mathcal{E}(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(\mathcal{E}(x,r))} \equiv \|f\chi_{\mathcal{E}(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in\mathcal{E}(x,r):\, |f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

Remark 3.3 (1) If $w \equiv 1$, then $M_{p,\varphi,d}(1) = M_{p,\varphi,d}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(\mathcal{E}(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi,d}(w) = L_{p,\kappa,d}(w)$ is the weighted anisotropic Morrey space.

(3) If $\varphi(x,r) \equiv v(\mathcal{E}(x,r))^{\frac{\kappa}{p}} w(\mathcal{E}(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi,d}(w) = L_{p,\kappa,d}(v,w)$ is the two weighted anisotropic Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi,d}(w) = L_{p,\lambda,d}(\mathbb{R}^n)$ is the classical anisotropic Morrey space and $WM_{p,\varphi,d}(w) = WL_{p,\lambda,d}(\mathbb{R}^n)$ is the weak anisotropic Morrey space.

(5) If $\varphi(x,r) \equiv w(\mathcal{E}(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi,d}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

The following statement, was proved in [35].

Theorem 3.1 *Let $1 \le p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{t>r} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(\mathcal{E}(x,\tau))^{\frac{1}{p}}}{w(\mathcal{E}(x,t))^{\frac{1}{p}}} \le C\, \varphi_2(x,r), \tag{3.6}$$

where C does not depend on x and r. Then the operator M is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.

The following statement, was proved in [35], see also [24].

Theorem 3.2 *Let $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{t>r} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(\mathcal{E}(x,\tau))^{\frac{1}{p}}}{w(\mathcal{E}(x,t))^{\frac{1}{p}}} \le C\, \varphi_2(x,r), \tag{3.7}$$

where C does not depend on x and r. Then the operator M_b^d is bounded from $M_{p,\varphi_1,d}(w)$ to $M_{p,\varphi_2,d}(w)$.

Note that, in the case $w = 1$ Theorem 3.1 was proved in [27,42], see also [2].

4 Anisotropic maximal operator with rough kernels M^d_Ω in the spaces $M_{p,\varphi,d}(w)$

In the following lemma we get Guliyev weighted local estimate (see, for example, [21, 23] in the case $w = 1$ and [24] in the case $w \in A_p$) for the operator T_Ω.

Lemma 4.1 *Suppose that Ω be satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$.*

If $q' \le p < \infty$, $p \ne 1$ and $w \in A_{p/q'}$, then the inequality

$$\|M^d_\Omega f\|_{L_{p,w}(\mathcal{E}(x,r))} \lesssim w(\mathcal{E}(x,r))^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, w(\mathcal{E}(x,t))^{-\frac{1}{p}}$$

holds for any anisotropic ball $\mathcal{E}(x,r)$, and for all $f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^n)$.

If $1 < p \le q$, $p \ne 1$ and $w^{1-p'} \in A_{p'/q'}$, then the inequality

$$\|M^d_\Omega f\|_{L_{p,w}(\mathcal{E}(x,r))} \lesssim \|w\|^{1/p}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,r))} \sup_{t>2r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, \|w\|^{-1/p}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))}$$

holds for any anisotropic ball $\mathcal{E}(x,r)$, and for all $f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^n)$.

Proof. Let Ω be satisfies the condition (1.1) and $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$.

Note that

$$\|\Omega(x - \cdot)\|_{L_q(\mathcal{E}(x,t))} \le c_0\, \|\Omega\|_{L_q(S^{n-1})}\, |\mathcal{E}(0, t + |x - x_0|)|^{\frac{1}{q}}, \tag{4.1}$$

where $c_0 = (nv_n)^{-1/q}$ and $v_n = |\mathcal{E}(0,1)|$ (see, [27]).

For arbitrary $x_0 \in \mathbb{R}^n$, set $\mathcal{E} = \mathcal{E}(x,r)$ for the ball centered at x_0 and of radius r, $2\mathcal{E} = \mathcal{E}(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2\mathcal{E}}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2\mathcal{E})}}(y), \quad r > 0 \tag{4.2}$$

and have

$$\|M^d_\Omega f\|_{L_{p,w}(\mathcal{E})} \le \|M^d_\Omega f_1\|_{L_{p,w}(\mathcal{E})} + \|M^d_\Omega f_2\|_{L_{p,w}(\mathcal{E})}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $M^d_\Omega f_1 \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of M^d_Ω in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q' \le p < \infty$, $p \ne 1$ (see Theorem 2.2) it follows that

$$\begin{aligned}\|M^d_\Omega f_1\|_{L_{p,w}(\mathcal{E})} &\le \|M^d_\Omega f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,\|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\approx \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,\|f\|_{L_{p,w}(2\mathcal{E})} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,w(\mathcal{E})^{\frac{1}{p}}\sup_{t>2r}\|f\|_{L_{p,w}(\mathcal{E}(x,t))}w(\mathcal{E}(x,t))^{-\frac{1}{p}}.\end{aligned}$$

Let z be an arbitrary point in $\mathcal{E} \equiv \mathcal{E}(x,r)$. If $\mathcal{E}(z,t) \cap {}^{\complement}\mathcal{E}(x,2r) \ne \varnothing$, then $t > r$. Indeed, if $y \in \mathcal{E}(z,t) \cap {}^{\complement}\mathcal{E}(x,2r)$, then we get $t > \rho(y-z) \ge \rho(x-y) - \rho(x-z) > 2r - r = r$.

On the other hand, $\mathcal{E}(z,t) \cap {}^{\complement}\mathcal{E}(x,2r) \subset \mathcal{E}(x,2t)$. Indeed, if $y \in \mathcal{E}(z,t) \cap {}^{\complement}\mathcal{E}(x,2r)$, then we get $\rho(x-y) \le \rho(y-z) + \rho(x-z) < t + r < 2t$. Hence, for all $z \in \mathcal{E}$

$$\begin{aligned}M^d_\Omega f_2(z) &= \sup_{t>0}|\mathcal{E}(z,t)|^{-1}\int_{\mathcal{E}(z,t)}|\Omega(z-y)|\,|f_2(y)|dy \\ &\le \sup_{t>r}|\mathcal{E}(x,2t)|^{-1}\int_{\mathcal{E}(z,t)\cap{}^{\complement}\mathcal{E}(x,2r)}|\Omega(z-y)|\,|f(y)|dy \\ &\le \sup_{t>r}|\mathcal{E}(x,2t)|^{-1}\int_{\mathcal{E}(x,2t)}|\Omega(z-y)|\,|f(y)|dy \\ &= \sup_{t>2r}|\mathcal{E}(x,t)|^{-1}\int_{\mathcal{E}(x,t)}|\Omega(z-y)|\,|f(y)|dy.\end{aligned}$$

By applying Hölder's inequality for $q' \le p < \infty$, $p \ne 1$ and $w \in A_{p/q'}$, we get

$$\begin{aligned}&M^d_\Omega f_2(z) \le \sup_{t>2r}|\mathcal{E}(x,t)|^{-1}\int_{\mathcal{E}(x,t)}|\Omega(z-y)|\,|f(y)|dy \\ &\lesssim \sup_{t>2r}|\mathcal{E}(x,t)|^{-1}\|\Omega(z-\cdot)\|_{L_q(\mathcal{E}(x,t))}\,\|f\|_{L_{q'}(\mathcal{E}(x,t))} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\sup_{t>2r}|\mathcal{E}(x,t)|^{-1}\|f\|_{L_{p,w}(\mathcal{E}(x,t))}\,\|w^{-q'/p}\|^{\frac{1}{q'}}_{L_{(p/q')'}(\mathcal{E}(x,t))}\,|\mathcal{E}(0,t+|x-z|)|^{\frac{1}{q}} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\sup_{t>2r}|\mathcal{E}(x,t)|^{-1}\|f\|_{L_{p,w}(\mathcal{E}(x,t))}\,w(\mathcal{E}(x,t))^{-\frac{1}{p}}\,|\mathcal{E}(x,t)|^{\frac{1}{q'}}\,|\mathcal{E}(0,t+r)|^{\frac{1}{q}} \\ &\approx \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\sup_{t>2r}\|f\|_{L_{p,w}(\mathcal{E}(x,t))}w(\mathcal{E}(x,t))^{-\frac{1}{p}}.\end{aligned} \tag{4.3}$$

Moreover, for all $q' \le p < \infty$, $p \ne 1$ the inequality

$$\|M^d_\Omega f_2\|_{L_{p,w}(\mathcal{E})} \lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,w(\mathcal{E})^{\frac{1}{p}}\sup_{t>2r}\|f\|_{L_{p,w}(\mathcal{E}(x,t))}w(\mathcal{E}(x,t))^{-\frac{1}{p}}$$

is valid. Thus

$$\|M_\Omega^d f\|_{L_{p,w}(\mathcal{E})} \lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}}\, w(\mathcal{E})^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))} w(\mathcal{E}(x,t))^{-\frac{1}{p}}.$$

Let also $1 < p \le q$, $p \ne 1$ and $w^{1-p'} \in A_{p'/q'}$. Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $M_\Omega^d f_1 \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of M_Ω^d in $L_{p,w}(\mathbb{R}^n)$ for $w^{1-p'} \in A_{p'/q'}$ and $1 < p < q$ (see Theorem 2.2) it follows that

$$\|M_\Omega^d f_1\|_{L_{p,w}(\mathcal{E})} \le \|M_\Omega^d f_1\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}}\, \|f_1\|_{L_{p,w}(\mathbb{R}^n)}$$

$$\approx \|\Omega\|_{L_q(S^{n-1})}\, [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}}\, \|f\|_{L_{p,w}(2\mathcal{E})}.$$

If $1 < p \le q, p \ne 1$ and $w^{1-p'} \in A_{p'/q'}$, then Minkowski theorem and Hölder inequality,

$$\|M_\Omega^d f_2\|_{L_{p,w}(\mathcal{E})} \le \left(\int_{\mathcal{E}} \left(\sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |\Omega(x-y)|\, |f(y)| dy \right)^p w(x) dx \right)^{\frac{1}{p}}$$

$$\le \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} \|\Omega(\cdot - y)\|_{L_{p,w}(\mathcal{E})} |f(y)|\, dy$$

$$\lesssim \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} \|\Omega(\cdot - y)\|_{L_q(\mathcal{E})}\, \|w\|_{L_{(q/p)'}(\mathcal{E})}^{\frac{1}{p}}\, |f(y)|\, dy$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|_{L_{(q/p)'}(\mathcal{E})}^{\frac{1}{p}} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |\mathcal{E}(0, r + \rho(x-y))|^{\frac{1}{q}}\, |f(y)|\, dy$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|_{L_{(q/p)'}(\mathcal{E})}^{\frac{1}{p}} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \|f\|_{L_1(\mathcal{E}(x,t))}\, |\mathcal{E}(0, r+t)|^{\frac{1}{q}}$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|_{L_{\frac{q}{q-p}}(\mathcal{E})}^{\frac{1}{p}} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, \|w^{-p'/p}\|_{L_1(\mathcal{E}(x,t))}^{\frac{1}{p'}}\, |\mathcal{E}(x,t)|^{\frac{1}{q}}$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|_{L_{\frac{q}{q-p}}(\mathcal{E})}^{\frac{1}{p}} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, \|w^{1-p'}\|_{L_1(\mathcal{E}(x,t))}^{\frac{1}{p'}}\, |\mathcal{E}(x,t)|^{\frac{1}{q}}$$

is obtained. By applying (3.3) for $\|w^{1-p'}\|_{L_1(\mathcal{E}(x,t))}^{\frac{1}{p'}}$ and (3.5) for $\|w\|_{L_{\frac{q}{q-p}}(\mathcal{E})}^{\frac{1}{p}}$ we have the following inequality

$$\|M_\Omega^d f_2\|_{L_{p,w}(\mathcal{E})}$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}}\, \|w\|_{L_{\frac{q}{q-p}}(\mathcal{E})}^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, \|w\|_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))}^{-\frac{1}{p}}$$

is valid. Thus

$$\|M_\Omega^d f\|_{L_{p,w}(\mathcal{E})} \lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}}\, \|w\|_{L_{\frac{q}{q-p}}(\mathcal{E})}^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, \|w\|_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))}^{-\frac{1}{p}}.$$

Thus we complete the proof of Lemma 4.1.

Theorem 4.1 *Suppose that Ω be satisfies the condition* (1.1) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. *Let also, for* $q' \le p < \infty$, $w \in A_{p/q'}$ *the pair* (φ_1, φ_2) *satisfies the condition* (3.6) *and for* $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$ *the pair* (φ_1, φ_2) *satisfies the condition*

$$\sup_{t>r} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) \|w\|^{1/p}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))}} \le C\, \varphi_2(x,r)\, \frac{w(\mathcal{E}(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,r))}}, \tag{4.4}$$

where C does not depend on x and r.

Then the operator M^d_Ω is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover

$$\|M^d_\Omega f\|_{M_{p,\varphi_2,d}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. When $q' \le p < \infty$, $w \in A_{p/q'}$, by Lemma 4.1 and Theorem 2.3 with $\nu_2(r) = \varphi_2(x,r)^{-1}, \nu_1(r) = \varphi_1(x,r)^{-1} w(\mathcal{E}(x,r))^{-\frac{1}{p}}, g(r) = \|f\|_{L_{p,w}(\mathcal{E}(x,r))}$ and $w(r) = w(\mathcal{E}(x,r))^{-\frac{1}{p}}$ we have

$$\begin{aligned}
\|M^d_\Omega f\|_{M_{p,\varphi_2,d}(w)} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|M^d_\Omega f\|_{L_{p,w}(\mathcal{E}(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, w(\mathcal{E}(x,t))^{-\frac{1}{p}} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(\mathcal{E}(x,r))} \\
&= \|f\|_{M_{p,\varphi_1,d}(w)}.
\end{aligned}$$

For the case of $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 4.1 and Theorem 2.3 with $\nu_2(r) = \varphi_2(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,r))}, \nu_1(r) = \varphi_1(x,r)^{-1} w(\mathcal{E}(x,r))^{-\frac{1}{p}}, g(r) = \|f\|_{L_{p,w}(\mathcal{E}(x,r))}$ and $w(r) = \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,r))}$ we have

$$\begin{aligned}
&\|M^d_\Omega f\|_{M_{p,\varphi_2,d}(w)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|M^d_\Omega f\|_{L_{p,w}(\mathcal{E}(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E})} \sup_{t>r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(\mathcal{E}(x,r))} \\
&= \|f\|_{M_{p,\varphi_1,d}(w)}.
\end{aligned}$$

Remark 4.1 Note that, if $\Omega \equiv 1$, Theorem 4.1 were proved in [33].

5 Commutator of anisotropic maximal operator with rough kernels $[b, M^d_\Omega]$ in the spaces $M_{p,\varphi,d}(w)$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 5.1 *Suppose that $b \in L_1^{\rm loc}(\mathbb{R}^n)$, and let*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |\mathcal{E}(y) - b_{\mathcal{E}(x,r)}| dy < \infty,$$

where

$$b_{\mathcal{E}(x,r)} = \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} b(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{\rm loc}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Lemma 5.1 *[41] Let $w \in A_\infty$. Then the norm $\|\cdot\|_*$ is equivalent to the norm*

$$\|b\|_{*,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{w(\mathcal{E}(x,r))} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r),w}|\, w(y)\, dy,$$

where

$$b_{\mathcal{E}(x,r),w} = \frac{1}{w(\mathcal{E}(x,r))} \int_{\mathcal{E}(x,r)} b(y)\, w(y)\, dy.$$

The following lemma is proved in [24].

Lemma 5.2 *1 Let $w \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$ and $r_1, r_2 > 0$. Then,*

$$\Big(\frac{1}{w(\mathcal{E}(x,r_1))} \int_{\mathcal{E}(x,r_1)} |b(y) - b_{B(x,r_2),w}|^p w(y) dy\Big)^{\frac{1}{p}} \le C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big) \|b\|_*,$$

where $C > 0$ is independent of f, w, x, r_1 and r_2.

2 Let $w \in A_p$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$ and $r_1, r_2 > 0$. Then,

$$\Big(\frac{1}{w^{1-p'}(\mathcal{E}(x,r_1))} \int_{\mathcal{E}(x,r_1)} |b(y) - b_{\mathcal{E}(x,r_2),w}|^{p'}\, w(y)^{1-p'} dy\Big)^{\frac{1}{p'}} \le C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big) \|b\|_*,$$

where $C > 0$ is independent of b, w, x, r_1 and r_2.

Remark 5.1 ([31])

(1) The John-Nirenberg inequality : There are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in \mathcal{E} \ : \ |b(x) - b_{\mathcal{E}}| > \beta\}| \le C_1 |\mathcal{E}| e^{-C_2\beta/\|b\|_*}, \quad \forall \mathcal{E} \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}|^p dy\right)^{\frac{1}{p}} \tag{5.1}$$

for $1 < p < 1$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|b_{\mathcal{E}(x,r)} - b_{\mathcal{E}(x,t)}\right| \le C\|b\|_* \ln \frac{t}{r} \ \text{ for } \ 0 < 2r < t, \tag{5.2}$$

where C is independent of b, x, r and t.

In the following lemma we get Guliyev weighted local estimate (see, for example, [24]) for the maximal commutator operator $M_{\Omega,b}$.

Lemma 5.3 *Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Suppose that Ω be satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$.*

If $q' \le p < \infty$ and $w \in A_{p/q'}$, then the inequality

$$\|M^d_{\Omega,b} f\|_{L_{p,w}(\mathcal{E}(x,r))} \lesssim \|b\|_* \, w(\mathcal{E}(x,r))^{\frac{1}{p}} \sup_{t>2r} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(\mathcal{E}(x,t))} w(\mathcal{E}(x,t))^{-\frac{1}{p}}$$

holds for any ball $\mathcal{E}(x,r)$, and for all $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

If $1 < p < q$ and $w^{1-p'} \in A_{p'/q'}$, then the inequality

$$\|M^d_{\Omega,b} f\|_{L_{p,w}(\mathcal{E}(x,r))} \lesssim \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,r))} \sup_{t>2r} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(\mathcal{E}(x,t))} \, \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))}$$

holds for any ball $\mathcal{E}(x,r)$, and for all $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

Proof. Let $p \in (1,\imath)$ and $b \in BMO(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $\mathcal{E} = \mathcal{E}(x,r)$ for the ball centered at x and of radius r, $2\mathcal{E} = \mathcal{E}(x,2r)$. We represent f as (4.2) and have

$$\|M^d_{\Omega,b} f\|_{L_{p,w}(\mathcal{E})} \le \|M^d_{\Omega,b} f_1\|_{L_{p,w}(\mathcal{E})} + \|M^d_{\Omega,b} f_2\|_{L_{p,w}(\mathcal{E})}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $M^d_{\Omega,b} f_1 \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of $M_{\Omega,b}$ in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q' \le p < \infty$ (see Theorem 2.2) it follows that

$$\begin{aligned}\|M^d_{\Omega,b} f_1\|_{L_{p,w}(\mathcal{E})} &\le \|M^d_{\Omega,b} f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})} \, [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \, \|b\|_* \, \|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\approx \|\Omega\|_{L_q(S^{n-1})} \, [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \, \|b\|_* \, \|f\|_{L_{p,w}(2B)}.\end{aligned}$$

Let z be an arbitrary point in $\mathcal{E} \equiv \mathcal{E}(x,r)$. If $\mathcal{E}(z,t) \cap {}^{\complement}\mathcal{E}(x,2r) \ne \varnothing$, then $t > r$. Indeed, if $y \in \mathcal{E}(z,t) \cap {}^{\complement}\mathcal{E}(x,2r)$, then we get $t > \rho(y-z) \ge \rho(x-y) - \rho(x-z) > 2r - r = r$.

On the other hand, $\mathcal{E}(z,t) \cap {}^{\complement}\mathcal{E}(x,2r) \subset \mathcal{E}(x,2t)$. Indeed, if $y \in \mathcal{E}(z,t) \cap {}^{\complement}\mathcal{E}(x,2r)$, then we get $\rho(x-y) \le \rho(y-z) + \rho(x-z) < t + r < 2t$. Hence, for all $z \in \mathcal{E}$

$$\begin{aligned} M^d_{\Omega,b} f_2(z) &= \sup_{t>0} |\mathcal{E}(z,t)|^{-1} \int_{\mathcal{E}(z,t)} |b(y)-b(z)| \, |\Omega(y-z)| \, |f_2(y)| \, dy \\ &= \sup_{t>0} |\mathcal{E}(z,t)|^{-1} \int_{\mathcal{E}(z,t) \cap {}^{\complement}\mathcal{E}(x,2r)} |b(y)-b(z)| \, |\Omega(y-z)| \, |f(y)| \, dy \\ &\lesssim \sup_{t>r} |\mathcal{E}(x,2t)|^{-1} \int_{\mathcal{E}(x,2t)} |b(y)-b(z)| \, |\Omega(y-z)| \, |f(y)| \, dy \\ &= \sup_{t>2r} |\mathcal{E}(x,2t)|^{-1} \int_{\mathcal{E}(x,t)} |b(y)-b(z)| \, |\Omega(y-z)| \, |f(y)| \, dy. \end{aligned}$$

Therefore, for all $z \in \mathcal{E}$ we have

$$M^d_{\Omega,b} f_2(z) \lesssim \sup_{t>2r} |\mathcal{E}(x,2t)|^{-1} \int_{\mathcal{E}(x,t)} |b(y)-b(z)|\,|\Omega(y-z)|\,|f(y)|\,dy.$$

By applying Hölder's inequality for $q' \le p < \infty$, $p \ne 1$ and $w \in A_{p/q'}$, we get

$$\begin{aligned}
&M^d_{\Omega,b} f_2(z) \le \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(y)-b(z)|\,|\Omega(z-y)|\,|f(y)|dy \\
&\lesssim \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \|\Omega(z-\cdot)\|_{L_q(\mathcal{E}(x,t))} \,\|(b(y)-b(z))\,f\|_{L_{q'}(\mathcal{E}(x,t))} \\
&\lesssim \|\Omega\|_{L_q(S^{n-1})} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \|f\|_{L_{p,w}(\mathcal{E}(x,t))} \,\|(b(y)-b(z))\,w^{-q'/p}\|^{\frac{1}{q'}}_{L_{(p/q')'}(\mathcal{E}(x,t))} |\mathcal{E}(0,t+\rho(x-z))|^{\frac{1}{q}} \\
&\lesssim \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, w(\mathcal{E}(x,t))^{-\frac{1}{p}}\, |\mathcal{E}(x,t)|^{\frac{1}{q'}}\, |\mathcal{E}(0,t+r)|^{\frac{1}{q}} \\
&\approx \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \sup_{t>2r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))} w(\mathcal{E}(x,t))^{-\frac{1}{p}}.
\end{aligned} \tag{5.3}$$

Moreover, for all $q' \le p < \infty$, $p \ne 1$ the inequality

$$\|M^d_{\Omega} f_2\|_{L_{p,w}(\mathcal{E})} \lesssim \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} w(\mathcal{E})^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))} w(\mathcal{E}(x,t))^{-\frac{1}{p}}$$

is valid. Thus

$$\|M_{\Omega} f\|_{L_{p,w}(\mathcal{E})} \lesssim \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} w(\mathcal{E})^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(\mathcal{E}(x,t))} w(\mathcal{E}(x,t))^{-\frac{1}{p}}.$$

If $1 < p \le q$, $p \ne 1$ and $w^{1-p'} \in A_{p'/q'}$, then Minkowski theorem and Hölder inequality,

$$\begin{aligned}
&\|M_{\Omega,b} f_2\|_{L_{p,w}(\mathcal{E})} \\
&\lesssim \Big(\int_B \Big(\sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(y)-b(z)|\,|\Omega(y-z)|\,|f(y)|dy \Big)^p w(z)\,dz \Big)^{\frac{1}{p}} \\
&\lesssim \Big(\int_B \Big(\sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(y)-b_{\mathcal{E},w}|\,|\Omega(y-z)|\,|f(y)|dy \Big)^p w(z)\,dz \Big)^{\frac{1}{p}} \\
&+ \Big(\int_B \Big(\sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(z)-b_{\mathcal{E},w}|\,|\Omega(y-z)|\,|f(y)|dy \Big)^p w(z)\,dz \Big)^{\frac{1}{p}} \\
&= J_1 + J_2.
\end{aligned}$$

Let us estimate J_1. Applying Hölder's inequality and by Lemma 5.2 we get

$$J_1 = \Big(\int_{\mathcal{E}} \Big(\sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(y) - b_{\mathcal{E},w}| \, |\Omega(y-z)| \, |f(y)| dy \Big)^p w(z)\, dz \Big)^{\frac{1}{p}}$$

$$\le \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} \|\Omega(y-\cdot)\|_{L_{p,w}(\mathcal{E})} \, |b(y) - b_{\mathcal{E},w}| \, |f(y)| \, dy$$

$$\lesssim \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} \|\Omega(y-\cdot)\|_{L_q(\mathcal{E})} \, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(\mathcal{E})} \, |b(y) - b_{\mathcal{E},w}| \, |f(y)| \, dy$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(\mathcal{E})} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |\mathcal{E}(0, r + \rho(x-y))|^{\frac{1}{q}} \, |b(y) - b_{\mathcal{E},w}| \, |f(y)| \, dy$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(\mathcal{E})} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1+\frac{1}{q}} \int_{\mathcal{E}(x,t)} |b(y) - b_{\mathcal{E},w}| \, |f(y)| \, dy$$

$$\le \|\Omega\|_{L_q(S^{n-1})} \, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(\mathcal{E})} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1+\frac{1}{q}} \Big(\int_{\mathcal{E}(x,t)} |b(y) - b_{\mathcal{E},w}|^{p'} w(y)^{1-p'} dy \Big)^{\frac{1}{p'}} \, \|f\|_{L_{p,w}(\mathcal{E}(x,t))}$$

$$\lesssim \|b\|_* \, \|\Omega\|_{L_q(S^{n-1})} \, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(\mathcal{E})} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1+\frac{1}{q}} \Big(1 + \ln \frac{t}{r} \Big) \|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(\mathcal{E}(x,t))} \, \|f\|_{L_{p,w}(\mathcal{E}(x,t))}$$

$$\lesssim \|b\|_* \, \|\Omega\|_{L_q(S^{n-1})} \, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E})} \sup_{t>2r} |\mathcal{E}(x,t)|^{-1+\frac{1}{q}} \Big(1 + \ln \frac{t}{r} \Big) \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))} \, |\mathcal{E}(x,t)|^{\frac{1}{q'}} \, \|f\|_{L_{p,w}(\mathcal{E}(x,t))}$$

$$= \|b\|_* \, \|\Omega\|_{L_q(S^{n-1})} \, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E})} \sup_{t>2r} |\mathcal{E}(x,t)|^{\frac{1}{q}} \, \ln \Big(e + \frac{t}{r} \Big) \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))} \, \|f\|_{L_{p,w}(\mathcal{E}(x,t))}.$$

In order to estimate J_2 note that

$$J_2 = \Big(\int_{\mathcal{E}} \Big(\sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(z) - b_{\mathcal{E},w}| \, |\Omega(y-z)| \, |f(y)| dy \Big)^p w(z)\, dz \Big)^{\frac{1}{p}}$$

$$\le \sup_{t>2r} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} \Big(\int_B \big| \big(b(z) - b_{\mathcal{E},w} \big) \, \Omega(y-z) \big|^p w(z) dz \Big)^{\frac{1}{p}} \, |f(y)| dy.$$

With similar techniques for $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$ can be achieved and the proof is finished.

Theorem 5.1 *Suppose that Ω be satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. Let $b \in BMO(\mathbb{R}^n)$. Let also, for $q' \le p < \infty$, $w \in A_{p/q'}$ the pair (φ_1, φ_2) satisfies the condition* (3.7) *and for $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$ the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} \Big(1 + \ln \frac{t}{r} \Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) \|w\|^{1/p}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))}} \frac{dt}{t} \le C \, \varphi_2(x,r) \, \frac{w(\mathcal{E}(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,r))}}, \quad (5.4)$$

where C does not depend on x and r.

Then the operator $M^d_{\Omega,b}$ is bounded from $M_{p,\varphi_1,d}(w)$ to $M_{p,\varphi_2,d}(w)$.

$$\|M^d_{\Omega,b} f\|_{M_{p,\varphi_2,d}(w)} \lesssim \|f\|_{M_{p,\varphi_1,d}(w)}.$$

Proof. When $q' \leq p < \infty$, $w \in A_{p/q'}$, by Lemma 5.3 and Theorem 2.3 with $\nu_2(r) = \varphi_2(x,r)^{-1}, \nu_1(r) = \varphi_1(x,r)^{-1} w(\mathcal{E}(x,r))^{-\frac{1}{p}}, g(r) = \|f\|_{L_{p,w}(\mathcal{E}(x,r))}$ and $w(r) = w(\mathcal{E}(x,r))^{-\frac{1}{p}} r^{-1}$ we have

$$\begin{aligned}
&\|M^d_{\Omega,b} f\|_{M^d_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|M^d_{\Omega,b} f\|_{L_{p,w}(\mathcal{E}(x,r))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^1 \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, w(\mathcal{E}(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(\mathcal{E}(x,r))} \\
&= \|b\|_* \|f\|_{M_{p,\varphi_1,d}(w)}.
\end{aligned}$$

For the case of $1 < p \leq q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 4.1 and Theorem 2.3 with $\nu_2(r) = \varphi_2(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,r))}, \nu_1(r) = \varphi_1(x,r)^{-1} w(\mathcal{E}(x,r))^{-\frac{1}{p}}, g(r) = \|f\|_{L_{p,w}(\mathcal{E}(x,r))}$ and $w(r) = \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,r))} r^{-1}$ we have

$$\begin{aligned}
&\|M^d_{\Omega,b} f\|_{M_{p,\varphi_2,d}(w)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|M^d_{\Omega} f\|_{L_{p,w}(\mathcal{E}(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E})} \\
&\times \int_r^1 \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(\mathcal{E}(x,t))}\, \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(\mathcal{E}(x,t))} \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(\mathcal{E}(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(\mathcal{E}(x,r))} = \|f\|_{M_{p,\varphi_1,d}(w)}.
\end{aligned}$$

Remark 5.2 Note that, if $\Omega \equiv 1$, Theorem 5.1 were proved in [33].

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