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Notes on operators, integrability and the purity conditions of the Sasakian metric according to the almost paracomplex structure in $T(M^n)$

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Abstract. *There are different structures in tangent and cotangent bundle. One of them is the complete lifts of the $F(K,-(-)^{K+1})-$structure. Firstly, the $F(K,-(-)^{K+1})-$structure studyed in M^n to $T(M^n)$ by L. S. Das [8]. The purpose of this paper firstly is to obtain integrability conditions of the almost product (paracomplex) structure $\widetilde{\psi} = 2(-)^{K+1}\left(F^{K-1}\right)^C - I$ for the condition $\left(F^K\right)^C - (-)^{K+1} F^C = 0$. Later, we get the results of the Tachibana operators applied to vector fields according to the almost product structure $\widetilde{\psi}$ on tangent bundle $T(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the structure $\widetilde{\psi}$.*

Keywords. $F-$structure · Sasakian metric · Integrability · Tachibana Operators · Complete Lift · Tangent Bundle

Mathematics Subject Classification (2010): 53C25

1 Introduction

The idea of $F-$structure manifold on a differentiable manifold was initiated and developed by Yano [14], Ishiara and Yano [5], Goldberg [4] and among others. The horizontal and complete lifts from a differentiable manifold M^n of class C^∞ to its cotangent bundles have been studied by a lot of authors [1,3,10,12,16,17]. Yano and Ishihara [15] have studied lifts of an $F-$structure in the tangent and cotangent bundles. There are different structures in tangent and cotangent bundle. One of them is $F_a(K,1)-$structure. The horizontal and complete lift of $F_a(K,1)-$structure in the tangent bundle give by C. S. Prasad and P. K. S. Chauhan [11]. In addition, Upadhyay and Gupta have obtained some integrability conditions of $F(K,-(K-2))-$structure, satifying $F^K - F^{K-2} = 0$ [13]. In this paper, we investigate the complete lifts of $F(K,-(-)^{K+1})-$structure. Firstly, the $F(K,-(-)^{K+1})-$structure studyed in M^n to $T(M^n)$ by L. S. Das [8]. We calculate the Nijenhuis tensors of the almost product (paracomplex) structure $\widetilde{\psi} = 2(-)^{K+1}\left(F^{K-1}\right)^C - I$

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for the condition $\left(F^K\right)^C - (-)^{K+1} F^C = 0$. Later, we get the results of the Tachibana operators applied to vector fields according to the almost product structure $\widetilde{\psi}$ on tangent bundle $T(M^n)$. Finally, we inclued the purity conditions of Sasakian metric with respect to the structure $\widetilde{\psi}$.

Let F be a non zero tensor field of the type $(1,1)$ and of class C^∞ on M^n such that [7]

$$F^K - (-)^{K+1} F = 0 \;\; and \;\; F^\omega - (-)^{\omega+1} F \neq 0 \tag{1.1}$$

for $1 < \omega < K$, where K is a fixed positive integer greater than 2 The degree of the manifold being K, $(K \geq 3)$. Let us define operators on M^n by :

$$\widetilde{I} = (-)^{K+1} F^{K-1}, \widetilde{m} = I - (-)^{K+1} F^{K-1}, \tag{1.2}$$

where I denotes the identity operator on M^n. Thus from (1.1) and (1.2) the following result are obvious

$$\widetilde{1} + \widetilde{m} = I, \widetilde{I}^2 = \widetilde{1}, \widetilde{m}^2 = \widetilde{m}.$$

For F satisfying (1.1), there exists complementary disributions $\widetilde{L}$ and $\widetilde{M}$, corresponding to the projection operators $\widetilde{1}$ and $\widetilde{m}$, respectively.

Theorem 1.1 *[9]If in M^n there is given a tensor field F $\left(F \neq 0, F^{K-1} \neq I\right)$ of type (1.1) and of class C^∞ such that $F^K - (-)^{K+1} F = 0$, then M^n admits an almost product structure $\Psi = 2(-)^{K+1} F^{K-1} - I$ where $\Psi = \widetilde{1} - \widetilde{m}$.*

Proof. If $F^{K-1} \neq I$, we have $\Psi = \widetilde{1} - \widetilde{m} = 2(-)^{K+1} F^{K-1} - I$

Also,

$$\begin{aligned}\Psi^2 &= 4(-)^{2K+2} F^{2K-2} + I - 4(-)^{K+1} F^{K-1} \\ &= 4F^K F^{K-2} + I - 4(-)^{K+1} F^{K-1} \\ &= 4F^{K-1} + I - 4F^{K-1} \;\; from \;\; (1.1) \\ &= I.\end{aligned}$$

Hence Ψ is an almost product structure.

1.1. Complete lift on $F(K, -(-)^{K+1})-$structure in tangent bundle

Definition 1.1 *Let F be a a non null tensor field of the type $(1,1)$ and of class C^∞ on M^n such that [7]*

$$F^K - (-)^{K+1} F = 0 \;\; and \;\; F^\omega - (-)^{\omega+1} F \neq 0 \tag{1.3}$$

for $1 < \omega < K$, where K is a fixed positive integer greater than 2. Such a structure on M^n has been called $F(K, -(-)^{K+1})-$ structure of rank "r", where the rank of F is constant on M^n and is equal to "r". In this case M^n is called an $F(K, -(-)^{K+1})$ manifold.

Let M be an $n-$dimensional differentiable manifold of class C^∞ and $T_p(M^n)$ the tangent space at a point p of M_n and $T(M^n) = p \in M^n \cup T_p(M^n)$ is the tangent bundle over the manifold M^n.

Let us denote by $T_s^r(M^n)$, the set of all tensor fields of class C^∞ and of type (r,s) in M^n and $T(M^n)$ be the tangent bundle over M^n. The complete lift F^C of an element of $T_1^1(M^n)$ with local components F_i^h has components of the form [17] $F^C : \begin{pmatrix} F_i^h & 0 \\ \delta_i^h & F_i^h \end{pmatrix}$.

Theorem 1.2 *For $F \in T_1^1(M^n)$, the complete lift F^C of F is an $F(K, -(-)^{K+1})-$ structure if it is for F also. Then F is of rank r if F^C is of rank $2r$ [7].*

Proof. Let $F \in T_1^1(M^n)$ satisfying (1.3). Then we have [17]

$$(FG)^C = F^C G^C. \tag{1.4}$$

Replacing G by F in (1.4) we obtain

$$(FF)^C = F^C G^C \tag{1.5}$$

or,

$$\left(F^2\right)^C = \left(F^C\right)^2. \tag{1.6}$$

Now putting $G = F^{K-1}$ in (1.4) since G is $(1,1)$ tensor field therefore F^{K-1} is also $(1,1)$ so we obtain $\left(FF^{K-1}\right)^C = F^C\left(F^{K-1}\right)^C$ which in view of (1.6) becomes

$$\left(F^K\right)^C = \left(F^C\right)^K. \tag{1.7}$$

Taking complete lift on both sides of equation (1.3) we get

$$\left(F^K\right)^C - \left((-)^{K+1} F\right)^C = 0 \tag{1.8}$$

which in consequence of equation (1.7) gives

$$\left(F^C\right)^K - (-)^{K+1} F^C = 0. \tag{1.9}$$

Thus equation (1.3) and (1.9) are equivalent.

2 Main Results

2.1. Integrability Conditions of Almost Product Structure on Tangent Bundle

Definition 2.1 *Let F be an almost product(paracomplex) structure on M_n, i.e., $F^2 = I$. We say that F is integrable if the Nijenhuis tensor N_F of F is identically equal to zero. The Nijenhuis tensor N_F is defined by*

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y] \tag{2.1}$$

for any $X, Y \in \Im_0^1(M_n)$ [2, 12].

In addition the structures are called as an almost complex structure for $F^2 = -I$ and dual structure for $F^2 = 0$. The condition of $N_F(X, Y) = N(X, Y) = 0$ is essential to integrability condition in these structures.

The Nijenhuis tensor N_F is defined local coordinates by

$$N_{ij}^k \partial_k = (F_i^s \partial_s^k F_j^k - F_j^l \partial_l F_i^k - \partial_i F_j^l F_l^k + \partial_j F_i^s F_s^k)\partial_k,$$

where $X = \partial_i$, $Y = \partial_j$, $F \in \Im_1^1(M^n)$,i.e., $F^2 = I$.

Proposition 2.1 *Let F be a tensor field $F(F \neq 0, F^{K-1} \neq I)$ of type $(1,1)$ and of class C^{∞} such that $\left(F^{K}\right) - (-)^{K+1} F = 0$. $T\left(M^{n}\right)$ be its tangent bundle such that $\left(F^{K}\right)^{C} - (-)^{K+1} F^{C} = 0$ for $F^{C} \in \Im_{1}^{1} T(M^{n})$. Then, $T(M^{n})$ admits an almost product structure defined by*

$$\widetilde{\psi} = 2\,(-)^{K+1}\left(F^{K-1}\right)^{C} - I, \tag{2.2}$$

where $\tilde{l} = (-)^{K+1}\left(F^{K-1}\right)^{C}$, $\widetilde{m} = I - (-)^{K+1}\left(F^{K-1}\right)^{C}$ and $\widetilde{\psi} = \tilde{l} - \widetilde{m}$.

Proof. Let X be arbitrary vector field on M^{n}. For $X^{C} \in \Im_{0}^{1} T(M^{n})$, we get

$$\begin{aligned}
\widetilde{\psi}^{2} X^{C} &= \widetilde{\psi}\left(\widetilde{\psi}\left(X^{C}\right)\right) = \widetilde{\psi}\left(2\,(-)^{K+1}\left(F^{K-1}X\right)^{C} - X^{C}\right) \\
&= 2\,(-)^{K+1}\left(F^{K-1}\right)^{C} 2\,(-)^{K+1}\left(F^{K-1}X\right)^{C} \\
&\quad -2\,(-)^{K+1}\left(F^{K-1}\right)^{C} X^{C} - 2\,(-)^{K+1}\left(F^{K-1}X\right)^{C} + X^{C} \\
&= 4\left(F^{K-1}\right)^{C}\left(F^{K-1}\right)^{C} X^{C} - 4\,(-)^{K+1}\left(F^{K-1}\right)^{C} X^{C} + X^{C} \\
&= 4X^{C} - 4X^{C} + X^{C} \\
&= X^{C}.
\end{aligned}$$

Hence, $\widetilde{\psi}$ is an almost product structure on $T(M^{n})$.

Theorem 2.1 *Let $N_{\widetilde{\psi}}(X^{C}, Y^{C})$ be the Nijenhuis tensor of almost product (paracomplex) structure $\widetilde{\psi}$ of type $(1,1)$ defined by (2.2) on $T(M^{n})$. Then the almost paracomplex structure $\widetilde{\psi}$ on $T(M^{n})$ is integrable if and only if $N_{F^{K-1}}\left(X, Y\right) = 0$.*

Proof. For $X, Y \in \Im_{0}^{1}(M^{n})$, $F \in \Im_{1}^{1}(M^{n})$, we get

$$\begin{aligned}
N_{\widetilde{\psi}}\left(X^{C}, Y^{C}\right) &= \left[\widetilde{\psi}X^{C}, \widetilde{\psi}Y^{C}\right] - \widetilde{\psi}\left[\widetilde{\psi}X^{C}, Y^{C}\right] - \widetilde{\psi}\left[X^{C}, \widetilde{\psi}Y^{C}\right] + \widetilde{\psi}^{2}\left[X^{C}, Y^{C}\right] \\
&= [(2\,(-)^{K+1}\left(F^{K-1}\right)^{C} - I)X^{C}, (2\,(-)^{K+1}\left(F^{K-1}\right)^{C} - I)Y^{C}] \\
&\quad -(2\,(-)^{K+1}\left(F^{K-1}\right)^{C} - I)[(2\,(-)^{K+1}\left(F^{K-1}\right)^{C} - I)X^{C}, Y^{C}] \\
&\quad -(2\,(-)^{K+1}\left(F^{K-1}\right)^{C} - I)[X^{C}, (2\,(-)^{K+1}\left(F^{K-1}\right)^{C} - I)Y^{C}] + [X, Y]^{C} \\
&= 4\left[F^{K-1}X, F^{K-1}Y\right]^{C} - 2\,(-)^{K+1}\left[F^{K-1}X, Y\right]^{C} - 2\,(-)^{K+1}\left[X, F^{K-1}Y\right]^{C} \\
&\quad + [X, Y]^{C} - (2\,(-)^{K+1}\left(F^{K-1}\right)^{C} - I)(2\,(-)^{K+1}\left[\left(F^{K-1}X\right), Y\right]^{C} - [X, Y]^{C}) \\
&\quad -(2\,(-)^{K+1}\left(F^{K-1}\right)^{C} - I)(2\,(-)^{K+1}\left[X, \left(F^{K-1}Y\right)\right]^{C} - [X, Y]^{C}) + [X, Y]^{C} \\
&= 4(\left[F^{K-1}X, F^{K-1}Y\right]^{C} - F^{K-1}\left[F^{K-1}X, Y\right] - F^{K-1}\left[X, F^{K-1}Y\right] \\
&\quad + (-)^{K+1}\left(F^{K-1}\left[X, Y\right]\right))^{C}.
\end{aligned}$$

From the condition of $\left(F^{K}\right)^{C} - (-)^{K+1} F^{C} = 0$, we have

$$\begin{aligned}
&= 4(\left[F^{K-1}X, F^{K-1}Y\right] - F^{K-1}\left[F^{K-1}X, Y\right] - F^{K-1}\left[X, F^{K-1}Y\right] \\
&\quad + \left(F^{K-1}\right)^{2}\left[X, Y\right])^{C} \\
&= 4\left(N_{F^{K-1}}\left(X, Y\right)\right)^{C}.
\end{aligned}$$

If $N_{F^{K-1}}\left(X, Y\right) = 0$, then $N_{\widetilde{\psi}} = 0$. The theorem is proved.

Theorem 2.2 *Let $N_{\widetilde{\psi}}(X^C, Y^V)$ be the Nijenhuis tensor of almost product (paracomplex) structure $\widetilde{\psi}$ of type $(1,1)$ defined by (2.2) on $T(M^n)$. Then the almost paracomplex structure $\widetilde{\psi}$ on $T(M^n)$ is integrable if and only if $N_{F^{K-1}}(X,Y) = 0$.*

Proof.

$$\begin{aligned}
N_{\widetilde{\psi}}\left(X^C, Y^V\right) &= \left[\widetilde{\psi}X^C, \widetilde{\psi}Y^V\right] - \widetilde{\psi}\left[\widetilde{\psi}X^C, Y^V\right] - \widetilde{\psi}\left[X^C, \widetilde{\psi}Y^V\right] + \widetilde{\psi}^2\left[X^C, Y^V\right] \\
&= [(2(-)^{K+1}\left(F^{K-1}\right)^C - I)X^C, (2(-)^{K+1}\left(F^{K-1}\right)^C - I)Y^V] \\
&\quad -(2(-)^{K+1}\left(F^{K-1}\right)^C - I)[(2(-)^{K+1}\left(F^{K-1}\right)^C - I)X^C, Y^V] \\
&\quad -(2(-)^{K+1}\left(F^{K-1}\right)^C - I)[X^C, (2(-)^{K+1}\left(F^{K-1}\right)^C - I)Y^V] \\
&\quad + \left[X^C, Y^V\right] \\
&= 4\left[F^{K-1}X, F^{K-1}Y\right]^V - 2(-)^{K+1}\left[X, F^{K-1}Y\right]^V \\
&\quad -2(-)^{K+1}\left[F^{K-1}X, Y\right]^V + [X,Y]^V - 4\left(F^{K-1}\left[F^{K-1}X, Y\right]\right)^V \\
&\quad +2(-)^{K+1}\left(F^{K-1}[X,Y]\right)^V + 2(-)^{K+1}\left[F^{K-1}X, Y\right]^V - [X,Y]^V \\
&\quad -4\left(F^{K-1}\left[X, F^{K-1}Y\right]\right)^V + 2(-)^{K+1}\left(F^{K-1}[X,Y]\right)^V \\
&\quad +2(-)^{K+1}\left[X, F^{K-1}Y\right]^V - [X,Y]^V + [X,Y]^V \\
&= 4(\left[F^{K-1}X, F^{K-1}Y\right]^V - F^{K-1}\left[F^{K-1}X, Y\right] - F^{K-1}\left[X, F^{K-1}Y\right] \\
&\quad + (-)^{K+1}\left(F^{K-1}[X,Y]\right))^V.
\end{aligned}$$

From the condition of $\left(F^K\right)^C - (-)^{K+1}F^C = 0$, we get

$$\begin{aligned}
&= 4(\left[F^{K-1}X, F^{K-1}Y\right]^V - F^{K-1}\left[F^{K-1}X, Y\right] - F^{K-1}\left[X, F^{K-1}Y\right] \\
&\quad + \left(F^{K-1}\right)^2[X,Y])^V \\
&= 4\left(N_{F^{K-1}}(X,Y)\right)^V .
\end{aligned}$$

If $N_{F^{K-1}}(X,Y) = 0$, then $N_\psi = 0$.

Theorem 2.3 *Let $N_{\widetilde{\psi}}(X^V, Y^V)$ be the Nijenhuis tensor of almost product (paracomplex) structure $\widetilde{\psi}$ of type $(1,1)$ defined by (2.2) on $T(M^n)$. Then the almost paracomplex structure $\widetilde{\psi}$ on $T(M^n)$ is integrable.*

Proof. If we put $\widetilde{\psi} = 2(-)^{K+1}\left(F^{K-1}\right)^C - I$ and taking account of lifting formulations, we obtain

$$\begin{aligned}
N_{\widetilde{\psi}}\left(X^V, Y^V\right) &= \left[\widetilde{\psi}X^V, \widetilde{\psi}Y^V\right] - \widetilde{\psi}\left[X^V, Y^V\right] - \widetilde{\psi}\left[X^V, \widetilde{\psi}Y^V\right] + \widetilde{\psi}^2\left[X^V, Y^V\right] . \\
&= 0.
\end{aligned}$$

Thus, $\widetilde{\psi}$ is integrable.

2.2. Tachibana operators applied to vector fields according to an almost paracomplex structure $\widetilde{\psi}$ on $T(M^n)$

Definition 2.2 *Let $\varphi \in \Im_1^1(M^n)$, and $\Im(M) = \sum_{r,s=0}^{\infty} \Im_s^r(M^n)$ be a tensor alebra over R. A map $\phi_\varphi \mid_{r+s\rangle 0}: *\Im(M^n) \to \Im(M^n)$ is called as Tachibana operatör or $\phi_\varphi$operatör on M^n if*

$a)$ ϕ_φ is linear with respect to constant coefficient,
$b)$ $\phi_\varphi : *\Im(M^n) \to \Im_{s+1}^r(M^n)$ for all r and s,
$c)$ $\phi_\varphi(KC\otimes L) = (\phi_\varphi K) \otimes L + K \otimes \phi_\varphi L$ for all $K, L \in *\Im(M^n)$,
$d)$ $\phi_{\varphi X} Y = -(L_Y \varphi)X$ for all $X, Y \in \Im_0^1(M^n)$, where L_Y is the Lie derivation with respect to Y (see [6]),
$e)$

$$(\phi_{\varphi X}\eta)Y = (d(\imath_Y \eta))(\varphi X) - (d(\imath_Y(\eta o \varphi)))X + \eta((L_Y \varphi)X) \tag{2.3}$$

$$= \phi X(\imath_Y \eta) - X(\imath_{\varphi Y}\eta) + \eta((L_Y\varphi)X)$$

for all $\eta \in \Im_1^0(M^n)$ and $X, Y \in \Im_0^1(M^n)$, where $\imath_Y \eta = \eta(Y) = \eta C\otimes Y, *\Im_s^r(M^n)$ the module of all pure tensor fields of type (r, s) on M^n with respect to the affinor field, $C\otimes$ is a tensor product with a contraction C [12].

Remark 2.1 If $r = s = 0$, then from $c), d)$ and $e)$ of Definition 2.2 we have $\phi_{\varphi X}(\imath_Y \eta) = \phi X(\imath_Y \eta) - X(\imath_{\varphi Y}\eta)$ for $\imath_Y \eta \in \Im_0^0(M_n)$, which is not well-defined $\phi_\varphi-$operator. Different choices of Y and η leading to same function $f = \imath_Y \eta$ do get the same values. Consider $M = R^2$ with standard coordinates x, y. Let $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the function $f = 1$. This may be written in many different ways as $\imath_Y \eta$. Indeed taking $\eta = dx$, we may choose $Y = \frac{\partial}{\partial_x}$ or $Y = \frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$. Nov the right-hand side of $\phi_{\varphi X}(\imath_Y \eta) = \phi X(\imath_Y \eta) - X(\imath_{\varphi Y}\eta)$ is $(\phi X)1 - 0 = 0$ in the first case, and $(\phi X)1 - Xx = -Xx$ in the second case. For $X = \frac{\partial}{\partial_x}$, the latter expression is $-1 \neq 0$. Therefore, we put $r + s > 0$ [12].

Remark 2.2 From $d)$ of Definition 2.2 we have

$$\phi_{\varphi X} Y = [\varphi X, Y] - \varphi[X, Y].$$

By virtue of

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

for any $f, g \in \Im_0^0(M_n)$, we see that $\phi_{\varphi X} Y$ is linear in X, but not Y [12].

Theorem 2.4 *Let $\widetilde{\psi}$ be an almost paracomplex structure on $T(M^n)$, i.e., $\widetilde{\psi}^2 = I$ and $\phi_{\widetilde{\psi}}$ be the Tachibana operator, defined by Definition 2.2. If the Lie derivative $L_Y F^{K-1} = 0$, then all results with respect to the almost paracomplex structure $\widetilde{\psi}$ is zero.*

$$i)\ \phi_{\widetilde{\psi} X^C} Y^C = -2\,(-)^{K+1} \left(\left(L_Y F^{K-1}\right) X\right)^C,$$

$$ii)\ \phi_{\widetilde{\psi} X^C} Y^V = -2\,(-)^{K+1} \left(\left(L_Y F^{K-1}\right) X\right)^V,$$

$$iii)\ \phi_{\widetilde{\psi} X^V} Y^C = -2\,(-)^{K+1} \left(\left(L_Y F^{K-1}\right) X\right)^V,$$

$$iv)\ \phi_{\widetilde{\psi} X^V} Y^V = 0,$$

where $X, Y \in \Im_0^1(M^n)$, the complete lifts $X^C Y^C \in \Im_0^1(T(M^n))$ and the vertical lift $X^V, Y^V \in \Im_0^1(T(M^n))$.

Proof. $i)$

$$\begin{aligned}\phi_{\widetilde{\psi}X^C}Y^C &= -\left(L_{Y^C}\widetilde{\psi}\right)X^C = -L_{Y^C}\widetilde{\psi}X^C + \widetilde{\psi}L_{Y^C}X^C \\ &= -L_{Y^C}(2\,(-)^{K+1}\left(F^{K-1}X\right)^C - X^C) + \widetilde{\psi}\,[Y,X]^C \\ &= -2\,(-)^{K+1}\left(\left(L_Y F^{K-1}\right)X\right)^C - 2\,(-)^{K+1}\left(F^{K-1}\left(L_Y X\right)\right)^C \\ &\quad + \left(L_Y X\right)^C + 2\,(-)^{K+1}\left(F^{K-1}\left(L_Y X\right)\right)^C - \left(L_Y X\right)^C \\ &= -2\,(-)^{K+1}\left(\left(L_Y F^{K-1}\right)X\right)^C.\end{aligned}$$

$ii)$

$$\begin{aligned}\phi_{\widetilde{\psi}X^C}Y^V &= -(L_{Y^V}\widetilde{\psi})X^C = -L_{Y^V}\widetilde{\psi}X^C + \widetilde{\psi}L_{Y^V}X^C \\ &= -L_{Y^V}(2\,(-)^{K+1}\left(F^{K-1}X\right)^C - X^C) + \widetilde{\psi}\,[Y,X]^V \\ &= -2\,(-)^{K+1}\left(\left(L_Y F^{K-1}\right)X\right)^V - 2\,(-)^{K+1}\left(F^{K-1}\left(L_Y X\right)\right)^V \\ &\quad +2\,(-)^{K+1}\left(F^{K-1}\left(L_Y X\right)\right)^V \\ &= -2\,(-)^{K+1}\left(\left(L_Y F^{K-1}\right)X\right)^V.\end{aligned}$$

$iii)$

$$\begin{aligned}\phi_{\widetilde{\psi}X^V}Y^C &= -(L_{Y^C}\widetilde{\psi})X^V = -L_{Y^C}\circ\Psi X^V + \widetilde{\psi}L_{Y^C}X^V \\ &= -L_{Y^C}(2\,(-)^{K+1}\left(F^{K-1}X\right)^V - X^V) + \widetilde{\psi}\left(L_Y X\right)^V \\ &= -2\,(-)^{K+1}\left(\left(L_Y F^{K-1}\right)X\right)^V - 2\,(-)^{K+1}\left(F^{K-1}\left(L_Y X\right)\right)^V \\ &\quad +2\,(-)^{K+1}\left(F^{K-1}\left(L_Y X\right)\right)^V \\ &= -2\,(-)^{K+1}\left(\left(L_Y F^{K-1}\right)X\right)^V.\end{aligned}$$

$iv)$

$$\begin{aligned}\phi_{\widetilde{\psi}X^V}Y^V &= -(L_{Y^V}\widetilde{\psi})X^V = -L_{Y^V}\widetilde{\psi}X^V + \widetilde{\psi}L_{Y^V}X^V \\ &= -L_{Y^V}(2\,(-)^{K+1}\left(F^{K-1}X\right)^V - X^V) \\ &= -L_{Y^V}2\,(-)^{K+1}\left(F^{K-1}X\right)^V + L_{Y^V}X^V \\ &= 0.\end{aligned}$$

2.3. The purity conditions of the Sasakian metrics with respect to the almost para-complex structure $\widetilde{\psi}$ on $T(M^n)$

There are well known classical examples of metrics on the tangent bundle $T(M^n)$ which can be constructed from a Riemannian metric g, namely the Sasakian metrics ${}^S g$ on tangent bundle $T(M^n)$, which is completely determined by its action the horizontal and vertical lifts of vector fields.

Definition 2.3 *The Sasakian metrics ${}^S g$ on the tangent bundle $T\left(M^n\right)$ over a Riemannian manifold $\left(M^n, g\right)$ is defined by three equations*

$$ {}^S g\left(X^V, Y^V\right) = \left(g\left(X, Y\right)\right)^V, \tag{2.4}$$

$$^Sg\left(X^V,Y^H\right)={}^Sg\left(X^H,Y^V\right)=0,$$
$$^Sg\left(X^H,Y^H\right)=\left(g\left(X,Y\right)\right)^V,$$

for all vector fields $X,Y\in\Im_0^1(M^n)$. It is obvius that the Sasakian metrics ^Sg is contained in the class of natural metrics [12, 15].

Theorem 2.5 *The Sasakian metrics ^Sg is pure with respect to $\widetilde{\psi}=2\left(-\right)^{K+1}\left(F^{K-1}\right)^C-I$ if $F=I$ and $\nabla F^{K-1}=0$, where I =ıdentity tensor field of type $(1,1)$.*

Proof. $S\left(\widetilde{X},\widetilde{Y}\right)={}^S g\left(\widetilde{\psi}\widetilde{X},\widetilde{Y}\right)-{}^S g\left(\widetilde{X},\widetilde{\psi}\widetilde{Y}\right)$ if $S\left(\widetilde{X},\widetilde{Y}\right)=0$, for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^V,Y^V or X^H,Y^H then $S=0$.

$i)$

$$\begin{aligned}S\left(X^V,Y^V\right)&={}^Sg\left(\widetilde{\psi}X^V,Y^V\right)-{}^S g\left(X^V,\widetilde{\psi}Y^V\right)\\&={}^Sg(2\left(-\right)^{K+1}\left(F^{K-1}X\right)^V-X^V,Y^V)\\&\quad-{}^Sg(X^V,2\left(-\right)^{K+1}\left(F^{K-1}Y\right)^V-Y^V)\\&=2-{}^{K+1S}g(\left(F^{K-1}X\right)^V,Y^V)-{}^S g\left(X^V,Y^V\right)\\&\quad+{}^Sg\left(X^V,Y^V\right)-2\left(-\right)^{K+1S}g(X^V,\left(F^{K-1}Y\right)^V)\\&=2\left(-\right)^{K+1}({}^Sg(\left(F^{K-1}X\right)^V,Y^V)-{}^S g(X^V,\left(F^{K-1}Y\right)^V))\\&=2\left(-\right)^{K+1}\left(g\left(\left(F^{K-1}X\right),Y\right)-g\left(X,\left(F^{K-1}Y\right)\right)\right)^V.\end{aligned}$$

$ii)$

$$\begin{aligned}S\left(X^V,Y^H\right)&={}^Sg\left(\widetilde{\psi}X^V,Y^H\right)-{}^S g\left(X^V,\widetilde{\psi}Y^H\right)\\&={}^Sg(2\left(-\right)^{K+1}\left(F^{K-1}X\right)^V-X^V,Y^V)\\&\quad-{}^Sg(X^V,2\left(-\right)^{K+1}\left(F^{K-1}\right)^CY^H-Y^H)\\&=-{}^Sg(X^V,2\left(-\right)^{K+1}\left(\gamma\left(\nabla_\gamma F^{K-1}\right)Y\right))\\&=-2\left(-\right)^{K+1}\,{}^Sg\left(X^V,\gamma\left(\left(\nabla F^{K-1}\right)Y\right)\right)\\&=-2\left(-\right)^{K+1}\,{}^Sg(X^V,\left(\left(\left(\nabla F^{K-1}\right)Y\right)U\right)^V)\\&=-2\left(-\right)^{K+1}\left(g\left(X,\left(\left(\nabla F^{K-1}\right)Y\right)U\right)\right)^V.\end{aligned}$$

$iii)$

$$\begin{aligned}S\left(X^H,Y^H\right)&={}^Sg\left(\widetilde{\psi}X^H,Y^H\right)-{}^S g\left(X^H,\widetilde{\psi}Y^H\right)\\&={}^Sg(2\left(-\right)^{K+1}\left(F^{K-1}\right)^CX^H-X^H,Y^H)\\&\quad-{}^Sg(X^H,2\left(-\right)^{K+1}\left(F^{K-1}\right)^CY^H-Y^H)\\&=2\left(-\right)^{K+1}\left(g\left(F^{K-1}X,Y\right)-g\left(X,F^{K-1}Y\right)\right)^V\\&\quad+2\left(-\right)^{K+1}\,{}^Sg(\left(\left(\left(\nabla F^{K-1}\right)X\right)U\right)^V,Y^H)\\&\quad-2\left(-\right)^{K+1}\,{}^Sg(X^H,\left(\left(\left(\nabla F^{K-1}\right)Y\right)U\right)^V)\\&=2\left(-\right)^{K+1}\left(g\left(F^{K-1}X,Y\right)-g\left(X,F^{K-1}Y\right)\right)^V.\end{aligned}$$

If $F = I$ and $\nabla F^{K-1} = 0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^V, Y^V or X^H, Y^H then $S = 0$. The theorem is proved.

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