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Some classical operators in a new vanishing generalized Orlicz-Morrey space

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Abstract. We study mapping properties of some classical operators of harmonic analysis-maximal, singular and fractional operators- in a new vanishing subspace of generalized Orlicz-Morrey spaces. We show that the vanishing property defining that subspace is preserved under the action of those operators.

Keywords. Orlicz-Morrey spaces, maximal functions, fractional operators, singular-type operators, vanishing properties

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1 Introduction

Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ play an important role in the study of local behaviour and regularity properties of solutions to PDE. It is well known that the Morrey spaces are non-separable if $\lambda > 0$. The lack of approximation tools for the entire Morrey space has motivated the introduction of appropriate subspaces like vanishing spaces. The definition of the vanishing Morrey spaces involves several vanishing conditions. Each condition generate a closed subspace of $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$. We use the notation of [1] and show these conditions as (V_0), (V_∞) and (V^*).

The space $V_0\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$, often called in the literature just by vanishing Morrey space, was already introduced in [3, 21, 22] motivated by regularity results of elliptic equations. The subspaces $V_\infty\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ and $V^{(*)}\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ were recently introduced in [1] to study the delicate problem in the approximation of Morrey functions by nice functions.

A natural step in the theory of functions spaces was to study Orlicz-Morrey spaces

$$\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$$

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where the "Morrey-type measuring" of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one.

We refer to [5,6,15,16] for the preservation of the vanishing property (V_0) of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ by some classical operators.

In this paper, we focus on the condition (V_∞). More precisely, the purpose of this paper is to introduce new vanishing Orlicz-Morrey space $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and to show that vanishing property (V_∞) is preserved under the action of maximal, singular and fractional operators.

We use the following notation: $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. The (Lebesgue) measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$ and χ_E denotes its characteristic function. We use C as a generic positive constant, i.e., a constant whose value may change with each appearance. The expression $A \lesssim B$ means that $A \le CB$ for some independent constant $C > 0$, and $A \approx B$ means $A \lesssim B \lesssim A$.

2 Preliminaries

We recall the definition of Young functions.

Definition 2.1 *A function $\Phi : [0,\infty] \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \Phi(\infty) = \infty$.*

A Young function Φ is said to satisfy the Δ_2-condition, denoted also by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \qquad r > 0$$

for some $C > 0$.

A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \qquad r \ge 0$$

for some $C > 1$.

Next we recall the generalized inverse of Young function Φ. For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = \infty).$$

Definition 2.2 (Orlicz Space) *For a Young function Φ, the Orlicz space $L^\Phi(\mathbb{R}^n)$ is defined by:*

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \, \textit{for some } k > 0 \right\}.$$

The space $L^\Phi_{\rm loc}(\mathbb{R}^n)$ is defined as the set of all measurable functions f such that $f\chi_B \in L^\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$. We refer to [2,19,20] for Orlicz spaces in some other settings.

$L^\Phi(\mathbb{R}^n)$ is a Banach space under the Luxemburg-Nakano norm

$$\|f\|_{L^\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\right\}.$$

For $\Omega \subset \mathbb{R}^n$, let

$$\|f\|_{L^\Phi(\Omega)} := \|f\chi_\Omega\|_{L^\Phi}.$$

A tacit understanding is that f is defined to be zero outside Ω.

In [4], the generalized Orlicz–Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz spaces and generalized Morrey spaces. The definition of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is as follows:

Definition 2.3 *Let φ be a positive measurable function on $(0,\infty)$ and Φ any Young function. The generalized Orlicz-Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is the space of functions $f \in L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{\Phi,\varphi}(f;x,r) < \infty,$$

where $\mathfrak{A}_{\Phi,\varphi}(f;x,r) := \frac{\Phi^{-1}(r^{-n})\|f\|_{L^{\Phi}(B(x,r))}}{\varphi(r)}$.

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost decreasing $\varphi : (0,\infty) \to (0,\infty)$ functions such that $t \in (0,\infty) \mapsto \frac{\varphi(t)}{\Phi^{-1}(t^{-n})}$ is almost increasing.

It will be assumed that the functions φ are of the class $\mathcal{G}_\Phi$ in the sequel. We refer to [9, Section 5] for more information about these spaces.

We consider the following subspace of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$:

Definition 2.4 *The vanishing generalized Orlicz-Morrey space $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ such that*

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}(f;x,r) = 0.$$

The vanishing subspace $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is nontrivial if $\varphi \in \mathcal{G}_\Phi$ satisfies the additional condition

$$\lim_{r\to\infty} \frac{\Phi^{-1}(r^{-n})}{\varphi(r)} = 0,$$

since then it contains bounded functions with compact support.

Lastly, we define operators investigated in this paper.

Definition 2.5 *Let Φ any Young function. A sublinear operator T will be called Φ-admissible singular type operator, if:*

1) *T satisfies the size condition of the form*

$$\chi_{B(x,r)}(z)\left|T\Big(f\chi_{\mathbb{R}^n\setminus B(x,2r)}\Big)(z)\right| \le C\chi_{B(x,r)}(z)\int_{\mathbb{R}^n\setminus B(x,2r)} \frac{|f(y)|}{|y-z|^n}dy$$

for $x\in\mathbb{R}^n$ and $r>0$;

2) *T is bounded in $L^\Phi(\mathbb{R}^n)$.*

If $\Phi \in \nabla_2$ an example of Φ-admissible singular type operator is the Hardy-Littlewood maximal operator

$$Mf(x) := \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy.$$

If $\Phi \in \Delta_2 \cap \nabla_2$ the class above includes also Calderón-Zygmund operators S with "standard kernels" (cf. [12, p. 99]).

We refer to [18] for the boundedness of operators M and S in Orlicz spaces.

We also consider Riesz potential operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy$$

and the fractional maximal operator

$$M_\alpha f(x) = \sup_{r>0}\frac{1}{|B(x,r)|^{1-\alpha/n}}\int_{B(x,r)}|f(y)|dy,$$

where $0<\alpha<n$.

It is known that the fractional maximal operator M_α is dominated pointwise by the fractional integral operator I_α, that is,

$$M_\alpha f(x) \le CI_\alpha(|f|)(x), \quad x\in\mathbb{R}^n. \tag{2.1}$$

3 Auxiliary Estimates

The following Guliyev-type local estimates play an essential role in the proof of our results.

Lemma 3.1 *[17] Let Φ be a Young function. Then for the Φ-admissible singular type operator T the following inequality is valid*

$$\|Tf\|_{L^{\Phi}(B(x,r))} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_r^{\infty} \|f\|_{L^{\Phi}(B(x,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \tag{3.1}$$

for all $f \in L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ and any ball $B(x,r)$.

Lemma 3.2 *[4] Let $f \in L^{\Phi}_{\rm loc}(\mathbb{R}^n)$. Then for Young function $\Phi \in \nabla_2$*

$$\|Mf\|_{L^{\Phi}(B(x,r))} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \sup_{t>r} \Phi^{-1}(t^{-n}) \|f\|_{L^{\Phi}(B(x,t))} \tag{3.2}$$

with the implicit constant independent of $x \in \mathbb{R}^n$ and $r > 0$.

Lemma 3.3 *[10,13] Let $0 < \alpha < n$ and Φ, Ψ Young functions, $\Phi \in \nabla_2$. If Φ and Ψ satisfy the conditons*

$$r^{\alpha} \Phi^{-1}(r^{-n}) \lesssim \Psi^{-1}(r^{-n}) \tag{3.3}$$

and

$$\int_r^{\infty} \Phi^{-1}(t^{-n}) t^{\alpha} \frac{dt}{t} \lesssim \Psi^{-1}(r^{-n}), \tag{3.4}$$

then for all $f \in L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ and $B(x,r)$

$$\|I_{\alpha} f\|_{L^{\Psi}(B(x,r))} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \int_r^{\infty} \|f\|_{L^{\Phi}(B(x,t))} \Psi^{-1}(t^{-n}) \frac{dt}{t}. \tag{3.5}$$

Lemma 3.4 *[11,14] Let Φ, Ψ Young functions and $0 < \alpha < n$, $\Phi \in \nabla_2$. If Φ and Ψ satisfy the condition (3.3), then for all $f \in L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ and $B(x,r)$*

$$\|M_{\alpha} f\|_{L^{\Psi}(B(x,r))} \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>r} t^{\alpha} \Phi^{-1}(t^{-n}) \|f\|_{L^{\Phi}(B(x,t))}. \tag{3.6}$$

The following pointwise estimates also play an essential role in the proof of our results.

Lemma 3.5 *[7] Let Φ be a Young function, $\varphi \in \mathcal{G}_{\Phi}$ and $\beta \in (0,1)$. If the conditions*

$$r^{\alpha} \lesssim \varphi(r)^{\beta - 1}, \tag{3.7}$$

and

$$\int_r^{\infty} t^{\alpha} \varphi(t) \frac{dt}{t} \lesssim \varphi(r)^{\beta} \tag{3.8}$$

hold, then there exist a positive constant C such that, for all $f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and for every $x \in \mathbb{R}^n$

$$I_{\alpha} f(x) \le C (Mf(x))^{\beta} \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}}. \tag{3.9}$$

Lemma 3.6 *[8] Let Φ be a Young function, $\varphi \in \mathcal{G}_{\Phi}$ and $\beta \in (0,1)$. If the condition (3.7) holds, then there exist a positive constant C such that, for all $f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and for every $x \in \mathbb{R}^n$*

$$M_{\alpha} f(x) \le C (Mf(x))^{\beta} \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}}. \tag{3.10}$$

4 Main Results

In this section, we show that the subspace $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is invariant with respect to sublinear Φ-admissible singular type operators. Moreover, we show that the vanishing property V_∞ is preserved under the action of fractional operators I_α and M_α.

Theorem 4.1 *Let Φ be a Young function and $\varphi \in \mathcal{G}_\Phi$ satisfy the condition*

$$\int_r^\infty \varphi(t)\frac{dt}{t} \leq C\varphi(r) \tag{4.1}$$

where C does not depend on r. Then Φ-admissible singular type operator T is bounded on $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

Proof. Since the generalized Orlicz-Morrey norm inequalities are already known [17, Theorem 18], it remains to show that $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is invariant with respect to T:

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi}(f;x,r)=0 \quad\Longrightarrow\quad \lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi}(Tf;x,r)=0.$$

If $f \in V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ then for any $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ such that

$$\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi}(f;x,t) < \epsilon \qquad \text{for every } t \geq R.$$

Using inequality (3.1) and the condition (4.1), we get

$$\mathfrak{A}_{\Phi,\varphi}(Tf;x,r) \lesssim \frac{1}{\varphi(r)}\int_r^\infty \Phi^{-1}(t^{-n})\|f\|_{L^\Phi(B(x,t))}\frac{dt}{t} \lesssim \epsilon$$

for any $x \in \mathbb{R}^n$ and every $r \geq R$ (with the implicit constants independent of x and r). Therefore

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi}(Tf;x,r)=0$$

and hence $Tf \in V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

Corollary 4.1 *Let Φ be a Young function and $\varphi \in \mathcal{G}_\Phi$ satisfy the condition (4.1). Then maximal operator M and singular operator S is bounded on $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ under the conditions $\Phi \in \nabla_2$ and $\Phi \in \Delta_2 \cap \nabla_2$, respectively.*

In view of (3.2) we can give a better result for maximal operator. More precisely, in the following result, we do not need the condition (4.1).

Theorem 4.2 *Let Φ be a Young function with $\Phi \in \nabla_2$ and $\varphi \in \mathcal{G}_\Phi$. Then the maximal operator M is bounded on $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.*

Proof. Since M is bounded in $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ (cf. [4, Theorem 4.6]) we only have to show that it preserves the vanishing property (V_∞). This can be done as in proof of Theorem 4.1, but now using the estimate (3.2).

Theorem 4.3 *(Spanne type result) Let Φ, Ψ be Young functions, $\Phi \in \nabla_2$, $\varphi_1 \in \mathcal{G}_\Phi$ and $\varphi_2 \in \mathcal{G}_\Psi$. Suppose that the conditions (3.3), (3.4) and*

$$\int_r^\infty \frac{\Psi^{-1}(t^{-n})}{\Phi^{-1}(t^{-n})}\varphi_1(t)\frac{dt}{t} \lesssim \varphi_2(r) \tag{4.2}$$

hold, then the operator I_α is bounded from $V_\infty\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V_\infty\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. Since I_α is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ into $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$[13, Theorem 14], we only need to check the action of I_α. Hence, it remains to show that

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi_1}(f;x,r)=0 \quad\Longrightarrow\quad \lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Psi,\varphi_2}(I_\alpha f;x,r)=0.$$

This can be done as in proof of Theorem 4.1, but now using the estimate (3.5).

Theorem 4.4 *(Spanne-type result) Let Φ,Ψ be Young functions, $\Phi\in\nabla_2$, $\varphi_1\in\mathcal{G}_\Phi$ and $\varphi_2\in\mathcal{G}_\Psi$. Suppose that the conditions (3.3) and*

$$\sup_{r<t<\infty}\frac{\Psi^{-1}(t^{-n})}{\Phi^{-1}(t^{-n})}\varphi_1(t)\lesssim\varphi_2(r) \tag{4.3}$$

hold, then the operator M_α is bounded from $V_\infty\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V_\infty\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. Since M_α is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ into $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ [14, Theorem 4.4], we only need to check the action of M_α. Hence, it remains to show that

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi_1}(f;x,r)=0 \quad\Longrightarrow\quad \lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Psi,\varphi_2}(M_\alpha f;x,r)=0.$$

This can be done as in proof of Theorem 4.1, but now using the estimate (3.6).

Remark 4.1 Although fractional maximal function is pointwise dominated by the Riesz potential (cf. (2.1)), and consequently, the results for the former could be derived from the results for the latter, we consider them separately, because we are able to study the fractional maximal operator under weaker assumptions than it derived from the results for the potential operator. More precisely, condition (4.3) is weaker than (4.2). Indeed, (4.2) implies (4.3):

We first note that $\Psi^{-1}(\tau)/\tau$ is decreasing, since $\Psi^{-1}(0)=0$ and Ψ^{-1} concave. By this fact, we have

$$\Psi^{-1}(s^{-n})\approx\Psi^{-1}(s^{-n})s^n\int_s^\infty\frac{dt}{t^{n+1}}\lesssim\int_s^\infty\Psi^{-1}(t^{-n})\frac{dt}{t}.$$

It follows from this inequality and $\varphi_1\in\mathcal{G}_\Phi$

$$\begin{aligned}\varphi_2(r)&\gtrsim\int_r^\infty\frac{\varphi_1(t)}{\Phi^{-1}(t^{-n})}\Psi^{-1}(t^{-n})\frac{dt}{t}\\&\gtrsim\int_s^\infty\frac{\varphi_1(t)}{\Phi^{-1}(t^{-n})}\Psi^{-1}(t^{-n})\frac{dt}{t}\\&\gtrsim\frac{\varphi_1(s)}{\Phi^{-1}(s^{-n})}\int_s^\infty\Psi^{-1}(t^{-n})\frac{dt}{t}\\&\approx\frac{\varphi_1(s)}{\Phi^{-1}(s^{-n})}\Psi^{-1}(s^{-n}),\end{aligned}$$

where we took $s\in(r,\infty)$, so that

$$\sup_{r<s<\infty}\frac{\Psi^{-1}(s^{-n})}{\Phi^{-1}(s^{-n})}\varphi_1(s)\lesssim\varphi_2(r).$$

Moreover, we do not need the condition (3.4) for the boundedness of fractional maximal operator.

The theorems below provide Adams-type results for the action of the operators I_α and M_α on vanishing subspace.

Theorem 4.5 *(Adams type result) Let Φ be a Young function with $\Phi \in \nabla_2$ and let $\varphi \in \mathcal{G}_\Phi$. Let $\beta \in (0,1)$ and define $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$. If conditions (3.7) and (3.8) hold, then I_α is bounded from $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $V_\infty\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof. The Adams-type boundedness of the operator I_α in generalized Orlicz-Morrey spaces follows from [7, Theorem 4.2]. To show the preservation of vanishing property, we make use of the pointwise estimate (3.9). From that estimate and scaling property $\|(Mf)^\beta\|_{L^\Psi(B(x,r))} = \|(Mf)\|^\beta_{L^\Phi(B(x,r))}$, we get

$$\mathfrak{A}_{\Psi,\eta}(I_\alpha f; x, r) \lesssim \big(\mathfrak{A}_{\Phi,\varphi}(Mf; x, r)\big)^\beta \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}} \tag{4.4}$$

for all $r > 0$ and $x \in \mathbb{R}^n$. If $f \in V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, then $Mf \in V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ by Theorem 4.2. Consequently, we have $I_\alpha f \in V_\infty\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$ taking into account estimate (4.4).

Theorem 4.6 *(Adams type result) Let Φ be a Young function with $\Phi \in \nabla_2$ and let $\varphi \in \mathcal{G}_\Phi$. Let $\beta \in (0,1)$ and define $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$. If the condition (3.7) holds, then M_α is bounded from $V_\infty\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $V_\infty\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof. The Adams-type boundedness of the operator M_α in generalized Orlicz-Morrey spaces follows from [8, Theorem 3.2]. So, we only need to check the preservation of vanishing property. This can be done as in proof of Theorem 4.5, but now using the estimate (3.10).

Remark 4.2 We find it important to underline once again the results for the fractional maximal operator are obtained under weaker assumptions than derived from Thereom 4.5. More precisely, we do not need the condition (3.8).

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References

1. Almeida, A., Samko, S.: *Approximation in Morrey spaces*, J. Funct. Anal., 272, 2392-2411 (2017).
2. Armutcu, H., Omarova, M.: *Some characterizations of BMO spaces via maximal commutators in Orlicz spaces over Carleson curves*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., 42 (1), Mathematics, 47-60 (2022).
3. Chiarenza, F., Franciosi, M.: *A generalization of a theorem by C. Miranda*, Ann. Mat. Pura Appl. IV, 161, 285-297 (1992).
4. Deringoz, F., Guliyev, V.S., Samko, S.: *Boundedness of maximal and singular operators on generalized Orlicz-Morrey spaces*, Operator Theory, Operator Algebras and Applications, Oper. Theory Adv. Appl., 242, 139-158, (2014).
5. Deringoz, F., Guliyev, V.S., Samko, S.: *Boundedness of the maximal operator and its commutators on vanishing generalized Orlicz-Morrey spaces*, Ann. Acad. Sci. Fenn. Math., 40, 535-549 (2015).
6. Deringoz, F., Guliyev, V.S., Samko, S.: *Vanishing generalized Orlicz-Morrey spaces and fractional maximal operator*, Publ. Math. Debrecen, 90 (1-2), 125-147 (2017).

7. Deringoz, F., Guliyev, V.S., Hasanov, S.G.: *Characterizations for the Riesz potential and its commutators on generalized Orlicz-Morrey spaces*, J. Inequal. Appl., Paper No. 248, 22 pp. (2016).
8. Deringoz, F., Guliyev, V.S., Hasanov, S.G.: *A characterization for Adams-type boundedness of the fractional maximal operator on generalized Orlicz-Morrey spaces*, Integral Transforms Spec. Funct., 28 (4), 284-299 (2017).
9. Deringoz, F., Guliyev, V.S., Nakai, E., Sawano, Y., Shi, M.: *Generalized fractional maximal and integral operators on Orlicz and generalized Orlicz-Morrey spaces of the third kind*, Positivity, 727-757 (2019).
10. Deringoz, F.: *Spanne-Guliyev type characterization for fractional integral operator and its commutators in generalized Orlicz-Morrey spaces on spaces of homogeneous type*, Operator theory and harmonic analysisOTHA 2020. Part I. New general trends and advances of the theory, Springer Proc. Math. Stat., 357, 143-159, Springer, Cham, (2021).
11. Deringoz, F., Dorak, K., Guliyev, V.S.: *Characterization of the boundedness of fractional maximal operator and its commutators in Orlicz and generalized Orlicz-Morrey spaces on spaces of homogeneous type*, Anal. Math. Phys., 11, no. 2, Paper No. 63, 30 pp. (2021).
12. Duoandikoetxea, J.: Fourier analysis, *Amer. Math. Soc. Graduate Studies*, 29, (2001).
13. Guliyev, V.S., Deringoz, F.: *On the Riesz potential and its commutators on generalized Orlicz-Morrey spaces*, J. Funct. Spaces, 2014, 1-11 (2014).
14. Guliyev, V.S., Deringoz, F.: *Boundedness of fractional maximal operator and its commutators on generalized Orlicz-Morrey spaces*, Complex Anal. Oper. Theory, 9, 1249-1267 (2015).
15. Guliyev, V.S., Deringoz, F., Hasanov, J.: *Φ-admissible singular operators and their commutators on vanishing generalized Orlicz-Morrey spaces*, J. Inequal. Appl., 2014:143, 1-18 (2014).
16. Guliyev, V.S., Deringoz, F., Hasanov, J.: *(Φ,Ψ)-admissible potential operators and their commutators on vanishing Orlicz-Morrey spaces*, Collect. Math., 67, 133-153 (2016).
17. Hasanov, J.J.: *Φ-admissible sublinear singular operators and generalized Orlicz-Morrey spaces*, J. Funct. Spaces Appl., 2014, 1-7 (2014).
18. Kokilashvili, V., Krbec, M. M.: Weighted Inequalities in Lorentz and Orlicz Spaces, *World Scientific, Singapore* (1991).
19. Muslumova, F., *Maximal commutators in Orlicz spaces for the Dunkl operator on the real line*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., 42 (1), Mathematics, 151-161 (2022).
20. Omarova, M.: *Nonsingular integral on weighted Orlicz spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., 41 (1), Mathematics, 138-145 (2021).
21. Vitanza, C.: Functions with vanishing Morrey norm and elliptic partial differential equations, *in: Proceedings of Methods of Real Analysis and Partial Differential Equations, 147-150, Springer, Capri* (1990).
22. Vitanza, C.: *Regularity results for a class of elliptic equations with coefficients in Morrey spaces*, Ricerche Mat., 42 (2), 265-281 (1993).