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# On sharp constant in generalized Minkowski inequality on variable Lebesgue spaces

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**Abstract.** *In this paper we investigate a sharp constant in Minkowski inequality by the denseness of some subclass from variable Lebesgue spaces with mixed norm. In previous works, the constant in the generalized Minkowski inequality was greater than one. But in the presented note, it is proved that this constant is equal to one. And also, it is prove that the Lebesgue space with mixed norm can be introduced using the space of Bochner integrable functions.*

**Keywords.** Variable Lebesgue space with mixed norm, generalized Minkowski inequality, denseness of finite sums

**Mathematics Subject Classification (2010):** 26D15, 52E40

## 1 Introduction

It is well known that the variable Lebesgue space in the literature for the first time was studied by Orlicz [23] in 1931. In [23], Hölders inequality for variable discrete Lebesgue space was proved. Orlicz also considered the variable Lebesgue space on the real line, and proved the Hölder inequality in this setting. However, this paper is essentially the only contribution of Orlicz to the study of the variable Lebesgue spaces (see also [20]). The next step in the development of the variable Lebesgue spaces came two decades later in the work of Nakano [21, 22]. Somewhat later, a more explicit version of such spaces, namely modular function spaces, were investigated by Musielak and others Polish mathematicians (see [19]). In particular, the variable Lebesgue spaces were objects of interest during the last three decades (see [10, 11, 16, 17]). The modern period in the study of variable Lebesgue spaces begun with the foundational papers of Sharapudinov [27] from 1979 and Kovacik and Rakosnik [18] from 1991. Interest in the variable Lebesgue spaces has increased since the 1990's because of their use in a variety of applications. Foremost among these is the mathematical modeling of electrorheological fluids, namely, fluids whose viscosity changes when exposed to an electric field: Diening and Růžička [12-15], Acerbi and Mingione [1, 2]. The variable Lebesgue spaces have also been used to model the behavior of other physical

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problems. Some examples include quasi-Newtonian fluids (see Zhikov [29]), the thermistor problem (see [30]), fluid flow in porous media (see [3], [4]), magnetostatics (see [9]), and the study of image processing (see [8]).

In this paper we established a sharp constant in Minkowski inequality by the denseness of some subclass from variable Lebesgue spaces with mixed norm. In previous works, the constant in the generalized Minkowski inequality was greater than one (see [5, 6, 10, 25, 26]). But in the presented note, it is proved that this constant is equal to one. And also, it is shown that the Lebesgue space with mixed norm coincides with the space of Bochner integrable functions.

The paper is organized as follows. Section 2 contains problem statement and some preliminaries along with the standard ingredients used in the proofs. The main result is proved in Section 3.

## 2 Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$, $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^n$ and $x^{(i)} = \Big(0, \dots, 0, x_i, \dots, x_n\Big)$, $i = 1, \dots, n$. Suppose that $\mathbf{p}(x) = \left(p_1\left(x^{(1)}\right), p_2\left(x^{(2)}\right), \dots, p_n\left(x^{(n)}\right)\right)$ is a vector function defined on $\mathbb{R}^n$ with Lebesgue measurable components $p_i\left(x^{(i)}\right)$ such that $1 \le p_i\left(x^{(i)}\right) < \infty$. Further in this paper all sets and functions are supposed to be Lebesgue measurable and $x^{(1)} = x$, $x^{(2)} = (0, x_2, \dots, 0, x_n)$, ..., $x^{(n)} = (0, \dots, 0, x_n)$.

Throughout this paper $\overline{p}_i = \operatorname*{ess\,sup}_{x^{(i)} \in \mathbb{R}^{n-i+1}} p_i(x^{(i)})$, $\underline{p}_i = \operatorname*{ess\,inf}_{x^{(i)} \in \mathbb{R}^{n-i+1}} p_i(x^{(i)})$, and $p_n\left(x^{(n)}\right) = p_n\left(x_n\right)$. We denote by $\mathbf{p}^{'}(x) = \left(p_1^{'}(x), p_2^{'}\left(x^{(2)}\right), \dots, p_n^{'}\left(x^{(n)}\right)\right)$ the conjugate exponent vector-function defined by $\dfrac{1}{\mathbf{p}(x)} + \dfrac{1}{\mathbf{p}^{'}(x)} = \mathbf{1}$, $x \in \mathbb{R}^n$, i.e., $\dfrac{1}{p_i\left(x^{(i)}\right)} + \dfrac{1}{p_i^{'}\left(x^{(i)}\right)} = 1$, $i = 1, \dots, n$ and $\mathbf{1} = (1, \dots, 1)$.

By $L_{(p_1(x), x_1)}\left(\mathbb{R}^n\right)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_1 > 0$

$$\left(I_{p_1(\cdot), x_1} f\right)(x_2, \dots, x_n) = \int_{\mathbb{R}} \left(\frac{|f(x)|}{\lambda_1}\right)^{p_1(x)} dx_1 < \infty.$$

The expression

$$\|f\|_{L_{(p_1(x), x_1)}(\mathbb{R}^n)} = \|f\|_{p_1(\cdot), x_1} = \left\{\lambda > 0 : \int_{\mathbb{R}} \left(\frac{|f(x)|}{\lambda}\right)^{p_1(x)} dx_1 \le 1\right\}$$

is a norm in $L_{(p_1(x), x_1)}\left(\mathbb{R}^n\right)$ with respect to the variable $x_1$. It is obvious that the result is a function of variables $x_2, \dots, x_n$. So, $\|f\|_{p_1(\cdot), x_1} = \|f\|_{p_1(\cdot), x_1}\left(x_2, \dots, x_n\right)$.

Further, by $L_{\left(p_1(x), p_2\left(x^{(2)}\right), x_1, x_2\right)}\left(\mathbb{R}^n\right)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_2 > 0$

$$\left(I_{p_2, x_2} f\right)(x_3, \dots, x_n) = \int_{R} \left(\frac{\|f\|_{p_1(\cdot), x_1}(x_2, \dots, x_n)}{\lambda_2}\right)^{p_2(x^{(2)})} dx_2 < \infty.$$

The expression

$$\|f\|_{L_{\left(p_1(x),\, p_2\left(x^{(2)}\right),\, x_1,\, x_2\right)}(\mathbb{R}^n)} = \left\| \|f\|_{p_1(\cdot),\, x_1} \right\|_{p_2(\cdot),\, x_2}$$

$$= \left\{ \mu > 0 : \int_{\mathbb{R}} \left( \frac{\|f\|_{p_1(\cdot),\, x_1}}{\mu} \right)^{p_2\left(x^{(2)}\right)} dx_2 \le 1 \right\}$$

is a norm in $L_{\left(p_1(x),\, p_2\left(x^{(2)}\right),\, x_1,\, x_2\right)}(\mathbb{R}^n)$ with respect to the variable $x_2$. It is obvious that the result is a function of variables $x_3, \ldots, x_n$. So, $\left\| \|f\|_{p_1(\cdot),\, x_1} \right\|_{p_2(\cdot),\, x_2} = \left\| \|f\|_{p_1(\cdot),\, x_1} \right\|_{p_2(\cdot),\, x_2} (x_3, \ldots, x_n)$.

Next, by $L_{\left(p_1(x),\, p_2\left(x^{(2)}\right),\ldots,p_n(x_n),\, x_1,\, x_2,\ldots,x_n\right)}(\mathbb{R}^n)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_{n-1}>0$

$$\left(I_{p_n,\, x_n} f\right)(x_n) = \int_{\mathbb{R}} \left( \frac{\|\ldots\| \, \|f\|_{p_1(\cdot),\, x_1} \|_{p_2(\cdot),\, x_2} \cdots \|_{p_{n-1}(\cdot),\, x_{n-1}}}{\lambda_{n-1}} \right)^{p_n(x_n)} dx_n < \infty.$$

The expression

$$\|f\|_{L_{\mathbf{p}(x)}(\mathbb{R}^n)} = \|\ldots\| \, \|f\|_{p_1(\cdot),\, x_1} \|_{p_2(\cdot),\, x_2} \cdots \|_{p_n(\cdot),\, x_n}$$

$$= \left\{ \nu > 0 : \int_{\mathbb{R}} \left( \frac{\|\ldots\| \, \|f\|_{p_1(\cdot),\, x_1} \|_{p_2(\cdot),\, x_2} \cdots \|_{p_{n-1}(\cdot),\, x_{n-1}}}{\nu} \right)^{p_n(x_n)} dx_n \le 1 \right\}$$

defines a norm in $L_{\mathbf{p}(x)}(\mathbb{R}^n)$.

**Remark 2.1** Let $\mathbf{p}(x) = (p_1, \ldots, p_n) = \mathbf{p} \ge 1$, i.e. $1 \le p_i \le \infty$, $i = 1, \ldots, n$. It is well known that Lebesgue spaces with mixed norm were introduced and studied in [7]. The variable Lebesgue spaces with mixed norm were introduced and studied in [5] and [6].

Suppose that $\Omega \subset \mathbb{R}^n$ is a measurable set and $f : \Omega \mapsto \mathbb{R}$. We define the norm in the space $L_{\mathbf{p}(x)}(\Omega)$ by

$$\|f\|_{L_{\mathbf{p}(\cdot)}(\Omega)} = \|f \, \chi_\Omega\|_{L_{\mathbf{p}(\cdot)}(\mathbb{R}^n)},$$

where $\chi_\Omega(x)$ is the characteristic function of $\Omega$.

**Remark 2.2** Let $\mathbf{p}(x) = (p_1, \ldots, p_n) = \mathbf{p} \ge 1$. Then $L_{\mathbf{p}(x)}(\mathbb{R}^n)$ coincides with the usual mixed norm Lebesgue spaces.

**Remark 2.3** Let $p_1(x_1, \ldots, x_n) = p_2\left(x^{(2)}\right) = \cdots = p_n\left(x^{(n)}\right) = p(x_n)$, i.e. $\mathbf{p}(x) = (p(x_n), \ldots, p(x_n))$. Then $L_{\mathbf{p}(x)}(\mathbb{R}^n) = L_{p(x_n)}(\mathbb{R}^n)$.

**Theorem 2.1** *Let* $\boldsymbol{p}(x) = (p_1(x), \ldots, p_n(x_n))$, $\boldsymbol{q}(x) = (q_1(x), \ldots, q_n(x_n))$ *and* $\boldsymbol{r}(x) = (r_1(x), \ldots, r_n(x_n))$. *If* $1 \le \underline{p}_i \le p_i\left(x^{(i)}\right) \le q_i\left(x^{(i)}\right) \le \overline{q}_i < \infty$ *and* $\frac{1}{r_i\left(x^{(i)}\right)} = \frac{1}{p_i\left(x^{(i)}\right)} - \frac{1}{q_i\left(x^{(i)}\right)}$, $i = 1, \ldots, n$, *then the inequality*

$$\|fg\|_{L_{\boldsymbol{p}(\cdot)}(\Omega)} \le \prod_{i=1}^{n} \left( A_i + B_i + \left\|\chi_{\Omega_{2,i}}\right\|_{L_\infty(\Omega)} \right)^{1/\underline{p}_i} \|f\|_{L_{\boldsymbol{q}(\cdot)}(\Omega)} \|g\|_{L_{\boldsymbol{r}(\cdot)}(\Omega)}$$

*holds for every* $f \in L_{\boldsymbol{q}(x)}(\Omega)$, $g \in L_{\boldsymbol{r}(x)}(\Omega)$, *where*

$$A_i = \operatorname*{ess\,sup}_{x\in\Omega_{1,i}} \frac{p_i\left(x^{(i)}\right)}{q_i\left(x^{(i)}\right)} \text{ and } B_i = \operatorname*{ess\,sup}_{x\in\Omega_{1,i}} \frac{q_i\left(x^{(i)}\right) - p_i\left(x^{(i)}\right)}{q_i\left(x^{(i)}\right)},$$

$$\Omega_{1,i} = \left\{x \in \Omega:\ p_i\left(x^{(i)}\right) < q_i\left(x^{(i)}\right)\right\}, \text{ and } \Omega_{2,i} = \left\{x \in \Omega:\ p_i\left(x^{(i)}\right) = q_i\left(x^{(i)}\right)\right\}.$$

**Remark 2.4** Note that in the case $p_1(x) = p_2\left(x^{(2)}\right) = \ldots = p_n(x_n) = 1$, Theorem 2.1 was proved in [5] (see, also [6]). The proof of Theorem 2.1 is similar with same modifications.

**Lemma 2.1** *Let* $\boldsymbol{p}(x) = (p_1(x), \ldots, p_n(x_n))$, $\boldsymbol{q}(x) = (q_1(x), \ldots, q_n(x_n))$ *and* $\boldsymbol{r}(x) = (r_1(x), \ldots\ r_n(x_n))$. *If* $1 \le \underline{p}_i \le p_i\left(x^{(i)}\right) \le q_i\left(x^{(i)}\right) \le \overline{q}_i < \infty$ *and the condition* $\frac{1}{r_i\left(x^{(i)}\right)} = \frac{1}{p_i\left(x^{(i)}\right)} - \frac{1}{q_i\left(x^{(i)}\right)}$, $i = 1, \ldots, n$ *is satisfied, then the following statements are equivalent:*

*a)* $L_{\boldsymbol{q}(x)}(\Omega) \hookrightarrow L_{\boldsymbol{p}(x)}(\Omega)$;

*b)* $\|1\|_{L_{r(\cdot)}(\Omega)} = \left\|\ldots\|\,\|\chi_\Omega\|\,\|_{r_1(\cdot),x_1\ldots}\right\|_{r_n(\cdot),x_n} < \infty$.

**Remark 2.5** Note that particular case of Lemma 2.1 was proved in [6]. The sufficiency of condition b) of Lemma 2.1 follows immediately from the Theorem 2.1. The necessity of condition b) is proved similar to the case of usual variable Lebesgue space.

We need the following Theorem.

**Theorem 2.2** *[5, 6, 10, 25] Let* $1 \le \overline{p} \le q(y) \le \overline{q} < \infty$ *for all* $x \in \Omega_1 \subset \mathbb{R}^n$ *and* $y \in \Omega_2 \subset \mathbb{R}^m$. *Then the inequality*

$$\left\|\|f\|_{L_{p(\cdot)}(\Omega_1)}\right\|_{L_{q(\cdot)}(\Omega_2)} \le C_{p,q} \left\|\|f\|_{L_{q(\cdot)}(\Omega_2)}\right\|_{L_{p(\cdot)}(\Omega_1)}$$

*is valid, where* $C_{p,q} = \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + \frac{\overline{p}}{\underline{q}} - \frac{\underline{p}}{\overline{q}}\right)\left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta 2}\|_\infty\right)$, $\underline{q} = \operatorname*{ess\,inf}_{y\in\Omega_2} q(y)$, $\overline{q} = \operatorname*{ess\,sup}_{y\in\Omega_2} q(y)$, $\Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2:\ p(x) = q(y)\}$, $\Delta_2 = (\Omega_1 \times \Omega_2)\setminus\Delta_1$ *and* $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *is a Lebesgue measurable function.*

## 3 Main results.

Let $I_m \subset \mathbb{R}^n$ be a countably family of sets of finite Lebesgue measure of the form $I_m = A_{1m} \times \ldots \times A_{nm}$, $|A_{im}| < \infty$, $i = 1, \ldots, n$, and such that $\bigcup_{m=1}^{\infty} I_m = \mathbb{R}^n$. We shall denote by $H(\mathbb{R}^n)$ the class of simple functions of the form $\sum_{i=1}^{n} c_i\, \chi_{A_i}$, where the sets $A_i = (a_{1i},\ b_{1i}) \times \ldots \times (a_{ni},\ b_{ni})$ are parallelepipeds and there exists an $I_m$ such that for every $i = 1, \ldots, n$, $A_i \subset I_m$. It is obvious that the set $H(\mathbb{R}^n)$ is dense in $L_{\mathbf{p}(x)}(\mathbb{R}^n)$ for $1 \le \mathbf{p}(x) \le \overline{\mathbf{p}} < \infty$.

The following lemma hold.

**Lemma 3.1** *Let* $1 \le \mathbf{p}(x) \le \overline{\mathbf{p}} < \infty$. *Then*

$$L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\ldots,\, p_n(x_n))}(\mathbb{R}^n) = L_{p_n(x_n)}\left(L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\, \ldots,\, p_{n-1}(x_{n-1},\, x_n))}\right).$$

*Proof.* Let $f \in L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\ldots,\, p_n(x_n))}(\mathbb{R}^n)$. There there exists $g(x)$ such that $g = f$ a.e. and $\left\| \ldots \|g\|_{p_1,x_1} \ldots \right\|_{p_{n-1},\, x_{n-1}} < \infty$ for every $x_n \in \mathbb{R}$. Writing $y = (x_1, \ldots, x_{n-1})$, $g\,(y, x_n) = g_{x_n}(y)$ may be interpreted as a function of $x_n$, whose values are functions from $L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\, \ldots,\, p_{n-1}(x_{n-1},\, x_n))}$. Next, there exists a sequence $\left\{g^{(m)}\right\}$ of functions from $H(\mathbb{R}^n)$ with the following two properties:

$$\left\| g^{(m)} - g \right\|_{L_{\mathbf{p}(x)}(\mathbb{R}^n)} \to 0$$

and

$$\left\| g^{(m)}(\cdot, x_n) - g\,(\cdot, x_n) \right\|_{L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\, \ldots,\, p_{n-1}(x_{n-1},\, x_n))}} \to 0 \quad \text{for a.e. } \ x_n \in \mathbb{R}.$$

Since $g^{(m)}\,(y, x_n) = g^{(m)}_{x_n}(y)$ is a simple function with values in $L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\, \ldots,\, p_{n-1}(x_{n-1},\, x_n))}$, it follows from (1) that $g_{x_n}(y)$ is a strongly measurable function. On the other hand, we have

$$\|f\|_{L_{\mathbf{p}(x)}(\mathbb{R}^n)} = \|g\|_{L_{\mathbf{p}(x)}(\mathbb{R}^n)} = \left\| \|f\,(\cdot, x_n)\|_{L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\, \ldots,\, p_{n-1}(x_{n-1},\, x_n))}} \right\|_{p_n,\, x_n}.$$

So, $\|f\|_{L_{\mathbf{p}(x)}(\mathbb{R}^n)}$ is the $p_n$-Bochner norm of the function $g_{x_n}(y)$.

Now, let $g_{x_n}(y) \in L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\, \ldots,\, p_{n-1}(x_{n-1},\, x_n))}$ for every $x_n \in \mathbb{R}$. Suppose that $g$ has a finite $p_n$-Bochner norm. Since it is strongly measurable, there exists a sequence of countably-valued functions $\left\{ g^{(m)}_{x_n}(y) \right\}$, uniformly convergent (a.e.) to $g_{x_n}(y)$ (see, [28]). Then, for every set $E$ of finite measure in $\mathbb{R}$, we have

$$\left\| \left\| \chi_E\,(x_n) \left( g^{(i)}_{x_n}(\cdot) - g^{(j)}_{x_n}(\cdot) \right) \right\|_{L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\, \ldots,\, p_{n-1}(x_{n-1},\, x_n))}} \right\|_{p_n,\, x_n} \to 0, \ \text{ for } i, j \to \infty.$$

Because of the measurability of $g^{(m)}\,(y, x_n)$ in $\mathbb{R}^n$, the sequence $\left\{ g^{(m)}\,\chi_E \right\}$ converges in $L_{\mathbf{p}(x)}(\mathbb{R}^n)$, and there exists subsequence (that we denote again $\left\{g^{(m)}\right\}$) converging a.e. $x \in \mathbb{R}^n$. This process provides us with measurable function $f(y,x) = f_{x_n}(y)$ such that for almost every $x_n \in \mathbb{R}$, $f(y,x) = g_{x_n}(y)$ a.e. $y \in \mathbb{R}^{n-1}$. Next, $g$ coincides with $f$ as an element of $L_{p_n(x_n)}\left( L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\, \ldots,\, p_{n-1}(x_{n-1},\, x_n))} \right)$ and $f \in L_{\mathbf{p}(x)}(\mathbb{R}^n)$. Thus, the space $L_{p_n(x_n)}\left( L_{(p_1(x),\, p_2(x_2,\ldots,x_n),\, \ldots,\, p_{n-1}(x_{n-1},\, x_n))} \right)$ is contained in $L_{\mathbf{p}(x)}(\mathbb{R}^n)$.

This completes the proof.

The sum of the form $\sum\limits_{i=1}^{N} \chi_{E_i}(x) g_i\,(y)$ is dense in variable Lebesgue spaces with mixed norm (see, [26]). Here $\chi_{E_i}$ is the characteristic function of pairwise disjoint sets $E_i$, $i = 1, \ldots, N$. The following theorem holds.

**Theorem 3.1** *Let $1 \le \underline{p} \le p(x) \le \overline{p} < \infty$ and $x \in \mathbb{R}^m$. Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to R$ and Lebesgue measurable function. Then the following inequality holds*

$$\left\| \int_{\mathbb{R}^m} f(x, \cdot) dx \right\|_{L_{p(\cdot)}(\mathbb{R}^n)} \le \int_{\mathbb{R}^m} \|f(x, \cdot)\|_{L_{p(\cdot)}(\mathbb{R}^n)} dx. \tag{3.1}$$

**Proof.** Since it is from, it suffices to prove the inequality for function in the collection form $\sum\limits_{i=1}^{N}\chi_{E_i}(x)g_i\,(y)$. We have $L_{p(\cdot)}\,(L_1(\mathbb{R}^m))$, it suffices to prove the inequality (2.1) for function in the collection $B$.

Let $= \left\{f : f\,(x, y) = \sum\limits_{i=1}^{N}\chi_{E_i}\,(\mathrm{x})\;g_i(y)\right\}$. Here $E_i \subset \mathbb{R}^n, |E_i| < \infty$ and $E_i \cap E_j = \emptyset, i \neq j, (i, j = \overline{1, N})$ and $g_i \in L_{p(\cdot)}(\mathbb{R}^n)$.

We have

$$\left\|\int_{R^m}\left(\sum_{i=1}^{N}\chi_{E_i}\,(x)\;g_i(\cdot)\right)dx\right\|_{L_{p(\cdot)}(\mathbb{R}^n)} = \left\|\sum_{j=1}^{N}\int_{E_j}\sum_{i=1}^{N}\chi_{E_i}\,(x)\;g_i(\cdot)dx\right\|_{L_{p(\cdot)}(\mathbb{R}^n)}$$

$$= \left\|\sum_{i=1}^{N}\int_{E_i}g_i(\cdot)dx\right\|_{L_{p(\cdot)}(\mathbb{R}^n)} = \left\|\sum_{i=1}^{N}g_i(\cdot)\,|E_i|\right\|_{L_{p(\cdot)}(\mathbb{R}^n)} \leq \sum_{i=1}^{N}|E_i|\|g_i\|_{L_{p(\cdot)}(\mathbb{R}^n)}.$$

Further, one has

$$\int_{\mathbb{R}^m}\left\|\sum_{i=1}^{N}\chi_{E_i}(x)\;g_i(\cdot)\right\|_{L_{p(\cdot)}(\mathbb{R}^n)}dx = \sum_{j=1}^{N}\int_{E_j}\left\|\sum_{i=1}^{N}\chi_{E_i}(x)\;g_i(\cdot)\right\|_{L_{p(\cdot)}(\mathbb{R}^n)}dx$$

$$= \sum_{i=1}^{N}\int_{E_j}\|g_i\|_{L_{p(\cdot)}(\mathbb{R}^n)}dx = \sum_{i=1}^{N}|E_i|\;\|g_i\|_{L_{p(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof.

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