Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **43** (1), 1-9 (2023).

Boundary value problem in an infinite strip for one characteristic equation degenerating into elliptic one

Mahir M. Sabzaliev, Arzu I. Mamedova

Received: 11.06.2022 / Revised: 23.12.2022 / Accepted: 01.01.2023

Abstract. *A complete asymptotic solution of the boundary value problem in an infinite strip is constructed for a one-characteristic third-order equation degenerating into an elliptic equation, and the remainder is estimated.*

Keywords. asymptotics · boundary layer type · function · remainder term

Mathematics Subject Classification (2010): 35B25, 35C20, 35C30

1 Introduction and problem statement

When studying some real phenomena with non-uniform transitions from one physical characteristics to other ones, we have to research singularly perturbed boundary value problems such problems have attracted attention of many prominent scientist as A.N. Tikhonov, L.S. Pontryagin, N.N.Bogolyubov, Yu. A. Mitropolsskii, V.Vazov, K. Friedrich, M.I. Vishik, L.A. Lusternik, O.A. Oleinik, E.F. Mishenko, N.Kh. Rozov, A.M.Il'in and others. But a great majority of the studied singularly perturbed partial differential equations were related to one of three classic types in bounded domains. Non –classic singularly perturbed differential equations have been little studied. The study of singularly perturbed boundary value problems for non-classic equations requires specific approaches from the author to their solution.

M.I. Vishik and L.A. Lusternik in [1] have introduced the so-called one-characteristic equations.

The equations of odd order $2k+1$ of the from

$$A_1(A_{2k}u) + B_{2k}u = f \tag{1.1}$$

were called by them one-characteristical equations if A_1 is a first order operator, A_{2k} is an elliptic operator of orders $2k$ while B_{2k} is any differential operator of order no more team

Mahir M. Sabzaliev,
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
Baku Business University, Baku, Azerbaijan
E-mail: sabzalievm@mail.ru

Arzu I. Mamedova
Azerbaijan State Economic University (UNEC), Baku, Azerbaijan
E-mail: arzu301@mail.ru

$2k$. Obviously, only the characteristics of the first order operator A_1 will be real characteristics of the equation (1.1) Mutual degenerations of one-characteristical and elliptic equations were studied in the same paper.

Asymptotics of the solutions of boundary value problems in an infinite strip were constructed in [2], [3] for third order one-characteristical equation degenerating into a hyperbolic equation. Complete asymptotics in a small parameter of the solution of boundary value problems in bounded and unbounded domains for a class of a singularly perturbed equations of odd order were constructed in the papers [4]-[6].

In the papers [7], [8] boundary value problems are studied for singularly perturbed one-characteristic equation degenerating into a parabolic and hyperbolic equation.

In the present paper, in the infinite strip $\Pi = \{(x,y)|0 \leq x \leq 1, -\infty < y < +\infty\}$ we consider the following boundary value problem for a third order one-characteristic equation degenerating,into an elliptic equation:

$$L_\varepsilon u \equiv \varepsilon \frac{\partial}{\partial x}(\Delta u) - \Delta u + au = f(x,y), \tag{1.2}$$

$$u|_{x=0} = 0, \quad u|_{x=1} = 0, \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0, \tag{1.3}$$

$$\lim_{|y|\to+\infty} u = 0, \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter, $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a Laplace operator, $a > 0$ is a constant, $f(x,y)$ the given function .

Our goal is to construct complete asymptotics in a small parameter of the solution of boundary value problem (1.2)-(1.4).

For that we carry out iterative processes.

2 Carrying out iterative processes

In the first iterative process, the approximate solution of the equation (1.2) is sought in the from

$$W = W_0 + \varepsilon W_1 + ... + \varepsilon^n W_n. \tag{2.1}$$

Having substituted the expression (2.1) for W in equation (1.2) and regrouping the terms with the same prowers with respect to ε, we obtain the following recurrently connected equations to determine the functions $W_i; i = 0, ..., n$:

$$-\Delta W_0 + aW_0 = f(x,y), \tag{2.2}$$

$$-\Delta W_k + aW_k = -\frac{\partial}{\partial x}(\Delta W_{k-1}), k = 1, 2, ..., n. \tag{2.3}$$

Obviously, it is impossible to use all three boundary conditions (1.3) for the equations (2.2), (2.3). For these equations we will use first two conditions from (1.3). Boundary conditions for equations (2.2), (2.3) at $x = 1$ will be written below.

With this choice of boundary conditions with respect to x for the equations (2.2), (2.3) on the boundary L_ε the third boundary condition from (1.3) will be lost. To compensate the lost boundary condition, we should construct a boundary layer type function near the boundary $x = 1$. This time, the first iterative process by means of which the functions $W_i; k = 0, 1, ..., n$ will be constructed and iterative process that serves to construct boundary layer functions near the boundary $x = 1$ will be embedded to each other. Therefore, for finding boundary conditions at $x = 1$ for the equations (2.2), (2.3), at first we must write equations whose solutions will be boundary layer functions near $x = 1$.

The first iterative process is carried out on the basis of splitting or (1.2) of the operator L_ε, that will be called the first splitting of the operator L_ε. For carrying out on-other iterative process by means of which a boundary layer function will be constructed near the boundary $x = 1$, in a new splitting of the operator L_ε should be written near this boundary. In order to write a new spliting near the boundary $x = 1$, we make substation of variables $1 - x = \varepsilon t, y = y$. The new splitting of the operator L_ε in new coordinates (t, y) is be of the from

$$L_{\varepsilon,1} \equiv \varepsilon^2 \left\{ -\left(\frac{\partial^3}{\partial t^3} + \frac{\partial^2}{\partial t^2} \right) + \varepsilon^2 \left(-\frac{\partial^3}{\partial t \partial y^2} - \frac{\partial^2}{\partial y^2} + a \right) \right\}.$$

We look for a boundary layer function V near the boundary $x = 1$ in the form

$$V = \varepsilon(V_0 + \varepsilon V_1 + ... + \varepsilon^n V_n), \tag{2.4}$$

as an approximate solution of the equation

$$L_{\varepsilon,1} V = 0. \tag{2.5}$$

Having substituted the expression for (2.4) V from to (2.5) and comparing the terms at the same prowers with respect to ε, we have

$$\frac{\partial^3 V_0}{\partial t^3} + \frac{\partial^2 V_0}{\partial t^2} = 0, \tag{2.6}$$

$$\frac{\partial^3 V_1}{\partial t^3} + \frac{\partial^2 V_1}{\partial t^2} = 0, \tag{2.7}$$

$$\frac{\partial^3 V_i}{\partial t^3} + \frac{\partial^2 V_i}{\partial t^2} = \frac{\partial^3 V_{i-2}}{\partial t \partial y^2} + \frac{\partial^2 V_{i-2}}{\partial y^2} - aV_{i-2}, \quad i = 2, 3, ..., n. \tag{2.8}$$

The iterative processes described above are inter connected with boundary conditions. To reveal this relations we require that the sum $W + V$ satisfy all boundary conditions (1.3). Considering that due to the smoothing functions, the boundary layer functions V_j will equal zero for $x = 0$, we obtain

$$W_0|_{x=0} = 0, \, W_0|_{x=1} = 0, \tag{2.9}$$

$$W_k|_{x=0} = 0, \, W_k|_{x=1} = - \left. V_{k-1}\right|_{t=0}, k = 1, 2, ..., n, \tag{2.10}$$

$$\left. \frac{\partial V_i}{\partial t} \right|_{t=0} = \left. \frac{\partial W_i}{\partial x} \right|_{x=1}, \quad i = 0, 1, ..., n. \tag{2.11}$$

We call the problem (2.2), (2.9) a degenerated problem corresponding to the problem (1.2)-(1.4). We have the following lemma.

Lemma 1. *Let the function $f(x, y)$ in Π have continuous derivatives with respect to x the $(n+2)$-th order, inclusively, and with respect to the variable y be infinitely differentiable, and for any pair of non-negative numbers l, k satisfy the inequality of the form*

$$\sup_y (1 + |y|^l) \left| \frac{\partial^k f(x, y)}{\partial x^{k_1} \partial y^{k_2}} \right| = C^{(1)}_{l_{k_1,k_2}} < +\infty, \tag{2.12}$$

where $C^{(1)}_{lk_1,k_2} > 0$ is a constant, $k = k_1 + k_2$ moreover $k_1 \leq n + 2$, k_2 is arbitrary . Then there exists a unique solution of the problem (2.2), (2.9) and the function $W(x, y)$ satisfies the condition

$$\sup_y (1 + |y|^l) \left| \frac{\partial^k W_0(x, y)}{\partial x^{k_1} \partial y^{k_2}} \right| = C^{(2)}_{l_{k_1,k_2}} < +\infty, \tag{2.13}$$

where $C^{(2)}_{l_{k_1,k_2}} > 0$ is a constant, $k = k_1 + k_2$ moreover $k_1 \leq n+3, k_2$ is arbitrary.

Proof. Using the Fourier transformation with respect to the variable y, we reduce the problem (2.2),(2.9) to the problem

$$\frac{d^2\widetilde{W}_0}{dx^2} - (a+\lambda^2)\widetilde{W}_0 = \widetilde{f}, \tag{2.14}$$

$$\widetilde{W}_0\Big|_{x=0} = 0, \quad \widetilde{W}_0\Big|_{x=1} = 0, \tag{2.15}$$

where

$$\widetilde{W}_0(x,\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-i\lambda y} W_0(x,y)dy,$$

$$\widetilde{f}(x,\lambda) = -\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-i\lambda y} f(x,y)dy.$$

The solution of problem (2.14),(2.15) is written in the form

$$\widetilde{W}_0(x,\lambda) = \int\limits_0^1 \widetilde{f}(t,\lambda)G(x,t,\lambda)dt, \tag{2.16}$$

where $G(x,t,\lambda)$ is the Green function of this problem and has the following form

$$G(x,t,\lambda) = \begin{cases} \frac{1}{2k(e^{-2k}-1)} \left[e^{-k(2-x+t)} - e^{-k(2-x-t)} - e^{-k(x+t)} + e^{-k(x-t)}\right] & for\ t \leq x, \\ \frac{1}{2k(e^{-2k}-1)} \left[e^{-k(t-x)} - e^{-k(x+t)} - e^{-k(2-x-t)} + e^{-k(2+x-t)}\right] & for\ t \geq x, \end{cases}$$

where $k(\lambda) = \sqrt{a+\lambda^2}$.

Applying the inverse Fourier transformation to $\widetilde{W}_0(x,\lambda)$, we obtain the solution of the problem (2.2), (2.9) in the form

$$W_0(x,\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{i\lambda y} \widetilde{W}_0(x,\lambda)d\lambda.$$

Obviously, to prove the Lemma is suffices to show that the function $\widetilde{W}_0(x,\lambda)$ and all its derivatives with respect to x to the $(n+3)-$ th order, inclusively belong to the Schwarts space of repialy decreasing functions as $|\lambda| \to +\infty$. Further, we will denote this space by S_λ. Thus we have to prove the validity of the following inequality

$$\sup_y (1+|\lambda|^l) \left| \frac{\partial^k \widetilde{W}_0(x,\lambda)}{\partial x^{k_1} \partial \lambda^{k_2}} \right| = C^{(3)}_{l_{k_1,k_2}} < +\infty, \tag{2.17}$$

where $C^{(3)}_{l_{k_1,k_2}} > 0$ is a constant, $k = k_1 + k_2$ where $k_1 \leq n+3, k_2$ is arbitrary.

At first we consider the case $k_1 = 0$. From the explicit expression of the Green function $G(x,t,\lambda)$ it follows that this function has any order bounded derivatives with respect to λ i.e.

$$\left| \frac{\partial^k G(x,t,\lambda)}{\partial \lambda^k} \right| \leq M_k, k = 0,1,\dots . \tag{2.18}$$

From (2.16),(2.12) and (2.18) we obtain

$$\left|\frac{\partial^k \widetilde{W}_0}{\partial \lambda^{k_2}}\right| = \left|\int_0^1 \sum_{i=0}^{k} C_k^i \frac{\partial^i \widetilde{f}}{\partial \lambda^i} \frac{\partial^{k-i} G}{\partial \lambda^{k-i}} dt\right|$$

$$\leq \int_0^1 \sum_{i=0}^{k} C_k^i \frac{C_{l0i}^{(4)}}{\left(1+|\lambda|^l\right)} M_{k-i} dt = \frac{C_{l0i}^{(3)}}{\left(1+|\lambda|^l\right)},$$

where $C_{l0i}^{(3)} = \sum_{i=0}^{k} C_{l0i} M_{k-i}$, i.e. the function $\widetilde{W}_0(x,\lambda)$ belongs to the space S_λ.

We prove the validity of (2.17) for $k_1 = 1$. To this end ,at first we note that from the explicit expression of $G(x,t,\lambda)$ it follows that

$$\left|\frac{\partial^{k+1} G}{\partial x \partial \lambda^k}\right| \leq N_k; k = 0, 1, \ldots .$$

Using the last relation similar to wow it has been done for $\widetilde{W}_0(x,\lambda)$, we obtain $\frac{\partial^k \widetilde{W}_0(x,\lambda)}{\partial x} \in$ $\in S_\lambda$.

To show the validity of the relation (2.19) for $2 \leq k \leq n+3$ we differentiate the both hand sides of (2.14) $k_1 - 2$ times with respect to x

$$\frac{\partial^{k_1} \widetilde{W}_0}{\partial x^{k_1}} = (a+\lambda^2) \frac{\partial^{k_1-2} \widetilde{W}_0}{\partial x^{k_1-2}} + \frac{\partial^{k_1-2} \widetilde{f}}{\partial x^{k_1-2}},\ 2 \leq k_1 \leq n+3. \tag{2.19}$$

Since $\widetilde{W}_0 \in S_\lambda$,and the function $a + \lambda^2$ has a polynomial growth with respect to λ then from (2.19) for $k_1 = 2$ it follows that $\frac{\partial^2 \widetilde{W}_0}{\partial x^2} \in S_\lambda$, i.e. the relation (2.17) is valid for $k_1 = 2$. Continuing the reasoning from (2.19), finally we obtain that $\frac{\partial^{n+3} \widetilde{W}_0}{\partial x^{n+3}} \in S_\lambda$, i.e. the relation (2.17) is valid for $k_1 = n+3$.

Lemma 1 is proved.

It follows from (2.13) that the function $W_0(x,y)$ satisfies the condition $\lim\limits_{|y| \to +\infty} W_0(x,y) = 0$ as well.

Knowing the function W_0 from (2.6) and from (2.11) for $i = 0$ we can determine the function V_0. The function V_0 will be a boundary layer type solution of the equation (2.6) satisfying the condition

$$\left.\frac{\partial V_0}{\partial t}\right|_{t=0} = \left.\frac{\partial W_0}{\partial x}\right|_{x=1}. \tag{2.20}$$

The characteristic equation corresponding to the ordinary differential equation (2.6), besides zero roots has one negative root: $k = -1$.

This fact provides regularity of degeneration of problem (1.2)-(1.4) on the boundary $x = 1$.

The boundary layer type solution of the problem (2.20), (2.21)is of the form :

$$V_0 = -\frac{\partial W_0(1,y)}{\partial x} e^{-t}. \tag{2.21}$$

From (2.3) and from (2.10) for $k = 1$ we obtain that the function $W_1(x,y)$ is determined from the following boundary value problem :

$$-\varDelta W_1 + aW_1 = f_1(x,y), \tag{2.22}$$

$$W_1|_{x=0} = 0\ ,\ W_1|_{x=1} = -\ V_0|_{t=0}\ , \tag{2.23}$$

where $f_1(x,y) = -\frac{\partial}{\partial x}(\Delta W_0)$. Using the Fourier transformation with respect to the variable y we reduce the problem (2.22), (2.23) to the problem

$$\frac{d^2\widetilde{W}_1}{dx^2} - (a+\lambda^2)\widetilde{W}_1 = \widetilde{f}_1(x,\lambda), \tag{2.24}$$

$$\widetilde{W}_1\Big|_{x=0} = 0\ ,\ \widetilde{W}_1\Big|_{x=1} = \varphi_1(\lambda), \tag{2.25}$$

where $\widetilde{W}_1(x,\lambda), \widetilde{f}_1(x,\lambda)$ is Fourier transformation of the function $W_1(x,y)$ and $-f_1(x,y)$ respectively, $\varphi_1(\lambda)$ is determined by the following formula

$$\varphi_1(\lambda) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{-i\lambda y}\frac{\partial W_0(1,y)}{\partial x}dy. \tag{2.26}$$

The solution of the problem (2.24),(2.25) is written in the form

$$\widetilde{W}_1(x,\lambda) = \overline{W}_1(x,\lambda) + \int\limits_0^1 \widetilde{f}_1(t,\lambda)G(x,t,\lambda)dt, \tag{2.27}$$

where

$$\overline{W}_1(x,\lambda) = \frac{\varphi_1(\lambda)}{1-e^{-2k(\lambda)}}\left[e^{-k(\lambda)(1-x)} - e^{-k(\lambda)(1+x)}\right].$$

It is seen from (2.26) that $\overline{W}_1(x,\lambda)$ and all its derivatives with respect to x belong to the space S_λ. Therefore, since $\frac{\partial^k \widetilde{f}_1(x,\lambda)}{\partial x^k} \in S_\lambda, k = 0,1,...,n,$ then by the Lemma 1, from the expression (2.27) for $\widetilde{W}_1(x,\lambda)$ it follows that the function $\widetilde{W}_1(x,\lambda)$ and all its derivatives with respect to x to the $(n+2)-$ th order inclusively, belong to the space S_λ. Consequently, the function $W_1(x,\lambda)$ being the inverse Fourier transformation with respect to the variable λ for $\widetilde{W}_1(x,\lambda)$, itself with all its derivatives with respect to x belongs to the space S_y. Therefore, the function $W_1(x,y)$ satisfies the condition $\lim\limits_{|y|\to+\infty} W_1(x,y) = 0$ as well.

Further we determine the function V_1 from (2.7) and (2.11) for $i = 1$.Obviously, V_1 is determined by the following formula:

$$V_1 = -\frac{\partial W_1(1,y)}{\partial x}e^{-t}.$$

Then, from (2.3) and (2.10) for $k = 2$ we determine the function $W_2(x,y)$. For $W_2(x,\lambda)$ the following condition is fulfilled:$\frac{\partial^k W_2}{\partial x^k} \in S_y, k = 0,1,...,n+1$. After determining W_2 from (2.8) and from (2.11) for $i = 2$ we determine the function V_2 as a boundary layer type solution of the following problem :

$$\frac{\partial^3 V_2}{\partial t^3} + \frac{\partial^3 V_2}{\partial t} = \frac{\partial^3 V_0}{\partial t\partial y^2} + \frac{\partial^2 V_0}{\partial y^2} - aV_0, \tag{2.28}$$

$$\frac{\partial V_2}{\partial t}\Big|_{t=0} = \frac{\partial W_2}{\partial x}\Big|_{x=1}. \tag{2.29}$$

It can be easily shown that the boundary layer type solution of the problem (2.28),(2.29) is of the form:

$$V_2 = \left[a(t+1)\frac{\partial W_0(1,y)}{\partial x} - \frac{\partial W_2(1,y)}{\partial x} \right] e^{-t}.$$

Assume that we have already constructed the functions W_j, V_{j-1} for $0 \le j \le i-1$ and for them the following induction hypotheses fulfilled:

1) The functions W_j satisfy the condition (2.13) for $k_1 \le n+3-j$;

2) The function V_{j-1} is of boundary layer character in the vicinity $x=1$, more exactly, is of the form:

$$V_{j-1} = e^{-t} \sum_{s=0}^{l} c_{j-1,s}(y) t^s,$$

where $c_{j-1,s}(y)$ are expressed by $\frac{\partial W_r(1,y)}{\partial x}, r \le j-1$ and their derivatives with respect to y.

In the same way that we determined W_0, W_1, we determine the function W_i as the solution of the problem

$$-\Delta W_i + a W_i = -\frac{\partial}{\partial x}\left(\Delta W_{i-1}\right),$$

$$W_i|_{x=0} = 0, \quad W_i|_{x=1} = -\left. V_{i-1}\right|_{t=0}.$$

For the arguments carried out in the construction of W_1 and from Lemma 1 it follows that the function W_i will satisfy the condition (2.13) for $k_1 \le n+3-i$. Hence, in particular it follows that the functions W_i satisfy the boundary condition

$$\lim_{|y|\to+\infty} W_i = 0, \ i = 0,1,...,n \tag{2.30}$$

as well.

The function V_i is determined as a boundary layer type solution of problem (2.7),(2.11).

Recall that by virtue of the second part of the hypotheses, the right hand side of the equation (2.7) is of the form $e^{-t}\sum\limits_{s=0}^{l} P_s(y)t^s$, where $P_s(y)$ is expressed by $c_{j-2,s}(y)$ and $c''_{j-2,s}(y)$. Hence it follows that one can also find the solution of the problem (2.7),(2.11) in the form $e^{-t}\sum\limits_{s=0}^{l} Q_s(y)t^s$, where $Q_s(y)$ is expressed by the function $\frac{\partial W_r(1,y)}{\partial x}, r \le i$.

Note that from obvious expressions of the functions $V_0, V_1, ..., V_n$ it follows that these functions satisfy the following conditions as well:

$$\lim_{|y|\to+\infty} V_i = 0, \ i = 0,1,...,n. \tag{2.31}$$

Having multiplied all the functions V_j by the smoothing functions, for the obtained new functions we keep the previous notation.

Thus, we constructed the sum $\widetilde{u} = W + V$ that approximately satisfies the equation (1.2) in the sense

$$L_\varepsilon \widetilde{u} = O(\varepsilon^{n+1}). \tag{2.32}$$

It follows from (2.9)-(2.11),(2.30),(2.31) that the function $\widetilde{u}$ satisfies the following boundary conditions:

$$\widetilde{u}|_{x=0} = 0, \quad \widetilde{u}|_{x=1} = \varepsilon\varphi(y), \quad \left.\frac{\partial \widetilde{u}}{\partial x}\right|_{x=1} = 0, \tag{2.33}$$

where $\varphi(y) = v_n|_{t=0}$. Obviously,

$$\lim_{|y|\to+\infty} \varphi(y) = 0. \tag{2.34}$$

Having denoted $u - \widetilde{u} = \varepsilon^n z$ we obtained the following asymptotic represention of the solution of the problem (1.2)-(1.4):

$$u = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{i=0}^{n} \varepsilon^{1+i} V_i + \varepsilon^n z, \tag{2.35}$$

where $\varepsilon^n z$ is a remainder.

3 Estimating of the remainder term and the main result

Acting on both sides of (2.35) by the appropriate splitting, of the operator L_ε and taking into account equations (1.2),(2.32) it is easy to see that z satisfies the equation

$$L_\varepsilon z = F, \tag{3.1}$$

where $F(\varepsilon, x, y) = h_1(\varepsilon, x, y) + h_2(\varepsilon, x, y)$ is a function uniformly bounded in Π with respect ε. to There

$$h_1(\varepsilon, x, y) = -\varepsilon \frac{\partial}{\partial x}(\Delta W_n),$$

while $h_2(\varepsilon, x, y)$ near the boundary $x = 1$ is of the form

$$h_2(\varepsilon, x, y) = \frac{\partial^3 V_{n-1}}{\partial t \partial y^2} + \frac{\partial^2 V_{n-1}}{\partial y^2} - aV_{n-1} + \varepsilon \left(\frac{\partial^3 V_n}{\partial t \partial y^2} + \frac{\partial^2 V_n}{\partial y^2} - aV_n \right).$$

It follows from (1.3), (1.4), (2.33), (2.34), (2.35) that z satisfies the following boundary conditions :

$$z|_{x=0} = 0,\ z|_{x=1} = -\varepsilon\varphi(y), \quad \left.\frac{\partial z}{\partial x}\right|_{x=1} = 0, \tag{3.2}$$

$$\lim_{|y|\to+\infty} z = 0, \tag{3.3}$$

We introduce a new unixliarly function by the formula

$$z_1 = z + \varepsilon x e^{1-x} \varphi(y). \tag{3.4}$$

Then the function z_1 will be the solution of the problem

$$L_\varepsilon z_1 = F_1, \tag{3.5}$$

$$z_1|_{x=0} = 0,\ z_1|_{x=1} = 0, \quad \left.\frac{\partial z_1}{\partial x}\right|_{x=1} = 0, \tag{3.6}$$

$$\lim_{|y|\to+\infty} z_1 = 0, \tag{3.7}$$

where $F_1 = F - \varepsilon L_\varepsilon \left[x e^{1-x} \varphi(y) \right]$.

We have the following lemma

Lemma 2. *For the solution of problem (3.5)-(3.7) the estimation*

$$\|z_1\|^2_{W_2^1(\Pi)} \le C_1 \varepsilon, \tag{3.8}$$

is valid, where the constant $C_1 > 0$ *is independent of* ε.

To prove Lemma 2 we have to multiply the both hand sides of (3.5) by z_1 and integrate by parts allowing for boundary condition (3.6)-(3.7). After some transformations we obtain the estimate (3.8).

Knowing the estimation for z_1, from the equality (3.4) we easily obtain the same estimation for z :

$$\|z\|^2_{W_2^1(\Gamma)} \leq C\varepsilon, \tag{3.9}$$

where the constant $C > 0$ is independent of ε.

Combining the results obtained above, we arrive at the following statement.

Theorem. *Let* $f(x, y)$ *be a given function in* Π, *with continuous derivatives with respect to* x *to the* $(n+1)-th$ *the order, inclusively, while with respect to the variable* y *is infinitely differentiable and satisfies the condition (2.12). Then for the solution of the boundary value problem (1.2)-(1.4) we have the asymptotic representation (2.35), where the functions* W_i *were determined by the first iterative process,* V_j *are boundary layer type functions near the boundary* $x = 1$ *defined by the second iterative process,* $\varepsilon^n z$ *is a remainder and the estimation (3.9) is valid for* z.

References

1. Vishik, M.I., Lusternik, L.A.: *Regular degeneration and boundary layer for linear differential equations with small paramete*, (Russian) Uspehi Mat. Nauk (N.S.) **12** 5 (77), 3-122 (1957).
2. Dzavadov M.G., Sabzaliev M.M.: *A boundary value problem for a monocharacteristic equation degenerating into a monocharacteristic one*, (Russian) Dokl. Akad. Nauk SSSR **247** (5), 1041-1046 (1979).
3. Sabzaliev M.M.: *Asymptotic behavior of the solution of a boundary value problem for a one-characteristic third-order differential equation with a small parameter*, (Russian) Akad. Nauk Azerbadzhan. SSR Dokl. **35** (7), 3-7 (1979).
4. Salimov Ya. Sh. Sabzalieva I.M.: *On boundary value problems for a class of singularly perturbed differential equations of nonclassical type*, (Russian) Dokl. Akad. Nauk **399** (4), 450-453 (2004).
5. Salimov Ya.Sh., Sabzalieva I.M.: *On boundary value problems for a class of singularly perturbed equations of arbitrary odd order*, (Russian) Differ. Uravn. **42** (5), 53-659 (2006).
6. Salimov Ya.Sh., Sabzaliyeva I.M. *On asymptotics of solution of boundary value problem for one class of equations of arbitrary odd order*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **20**, 125-132 (2004).
7. Sabzaliev M.M., Kerimova M.E.: *Asymptotics of the solution in a rectangle of a boundary value problem for one-characteristic differential equation degenerating into a parabolic equation*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **34** (4) Mathematics and Mechanics, 97-106 (2014).
8. Sabzaliev M.M., Kerimova M.E.: *On asymptotics of the solution of a boundary value problem in a semi-infinite strip for a quasilinear one-characteristic differential equation*, Proceedings of the 7-th International conference on control and optimization with industrial application, vol 2, 347-349 (2020).