Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **43** (1), 1-16 (2023).

# On uncountable $b$-frames in non-separable Hilbert spaces

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Received: 21.04.2022 / Revised: 13.12.2022 / Accepted: 01.01.2023

**Abstract.** *This work is dedicated to uncountable frames in non-separable Hilbert spaces associated with bilinear mappings. Bounded bilinear mapping is considered, and using this mapping, the concepts of uncountable b-Besselian system, b-frame and b-frame operator are introduced. Criteria for uncountable b-Besselness and b-frameness of system are proved, and some properties of uncountable b-frame operator are established. Stability and perturbation of uncountable b-frames in non-separable Hilbert spaces are studied. From the obtained results, in particular, corresponding results for tensors are derived.*

**Keywords.** non-separable Hilbert space · uncountable frame · frame operator · Noetherian perturbation

**Mathematics Subject Classification (2010):**

## 1 Introduction

In the context of applications to some problems in different branches of natural science, there has been an upsurge of interest lately in frames. They can be used in signalling processes, in data compression and data processing, in medicine, in physics, etc. The concept of frame in a Hilbert space has been introduced by R.J.Duffin and A.C.Schaeffer [1] in 1952. They studied the frame properties of the perturbed exponential system, as well as the frames in the abstract separable Hilbert spaces. Frames are playing an important role in the theory of wavelets and Gabor transforms. This field of frame theory has received rapid development after the fundamental work by I.Daubechies, A.Grossman, Y.Meyer [2]. Many authors have treated frames since then, such as N.M.Astafyeva [3], I.Daubechies [4], I.M.Dremin, O.V.Ivanov, V.A.Nechitailo [5], C.Chui [6], R.Coifman [7], etc. One of the most important fields of frame theory is the study of their generalizations in different structures. In [8], W.Sun introduced the concepts of $g$-frame and $g$-Riesz basis for separable Hilbert spaces. Many properties of ordinary frames have been extended to this case. $g$-frames have been studied also in [9-11]. Frames in tensor products of Hilbert spaces have been considered in [12]. Note that every element of a separable Hilbert space has, probably not unique, expansion with respect to the elements of the frame of the space. Such expansions in function

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spaces have been studied by Coifman R.R. and G.Weis in [13] by introducing the concept of atomic decomposition. Later, atomic decompositions in Banach spaces have been treated by Feichtinger H.G. and K.Grochenig in [14]. The concept of Banach frame in Banach spaces has been first introduced by K.Grochenig in [15]. Banach frames and atomic decompositions in Banach spaces have also been studied in [16-19]. $p$-frames in invariant subspaces of $L_p$ have been considered in [20]. For general spaces of sequences of scalars, these matters have been treated in [21]. The case of Banach spaces of sequences of frame vectors associated with bilinear mappings has been considered in [22-26].

One of the methods to obtain a frame is a method of perturbation and stability of frames. In this field, there are some results in the context of Paley-Wiener theorem on Riesz basicity of perturbed exponential system. For these and other results concerning frame theory we refer the readers to the monographs [27-30]. Along with discrete frames, the continuous and uncountable frames also play an important role in non-separable Hilbert spaces. Continuous frames have been introduced by S.T.Ali, J.P.Antoine and Gazeau J.P. in [31]. The works [32, 33] also can be attributed to this field. Uncountable frames and Riesz bases in non-separable Hilbert spaces have been considered in [34], where the concepts of uncountable Besselian system, uncountable frame, uncountable frame operator, uncountable Riesz basis in non-separable Hilbert spaces have been introduced and the known properties of ordinary frames and Riesz bases have been studied for them.

This work is dedicated to the study of uncountable Besselian system and uncountable frame in non-separable Hilbert spaces for bilinear mappings. In Section 2, the concept of uncountable b-Besselian system is introduced and a criterion for it is proved. In Section 3, the concepts of uncountable b-frame and uncountable b-frame operator in non-separable Hilbert space are introduced. Criterion for uncountable b-frameness and some properties of b-frame operator are proved. b-frame properties of the system obtained by elimination of arbitrary element from uncountable b-frame. In Section 4, compact Noeherian perturbation and stability of uncountable b-frame are studied. The obtained results are the generalizations of previously known results for uncountable frames in non-separable Hilbert spaces and for frames in tensor products of Hilbert spaces (see [12,34]).

## 2 Uncountable b-Besselian systems

Let $X$, $H$ be non-separable Hilbert spaces with scalar products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_H$, respectively; $E$ be a Banach space with the norm $\|\cdot\|_E$; $L(X,E)$ be a Banach space of bounded linear operators from $X$ to $E$ with $L(X,X) = L(X)$; $\ker T$ and $R_T$ be a kernel and an image of the operator $T$, respectively; $T^*$ be an operator conjugate to $T$; $I_X$ be an identity operator in $X$; $\overline{A}$ be a closure of the set $A \subset X$ in $X$; $I$ be an uncountable set of indices, $I^a$ be a totality of at most countable subsets $\omega \subset I$, and $I_0$ be a totality of finite subsets $J \subset I$.

Consider a bilinear mapping $b : X \times E \to H$ satisfying the following condition: there exists $M > 0$ such that for $\forall (x,f) \in X \times E$

$$\|b(x,f)\|_H \le M \|x\|_X \|f\|_E .$$

Define the mapping $\omega_b : H \times E \to X$ as follows:

$$(\omega_b(h,f),x)_X = (h, b(x,f))_H, \forall (h,f) \in H \times E, \forall x \in X.$$

It is not difficult to show that the mapping $\omega_b$ is linear and continuous with respect to $h$ and satisfies the relation

$$\|\omega_b(h,f)\|_X \le M \|h\|_H \|f\|_E , \forall (h,f) \in H \times E.$$

Let $E_0 \subset E$ be some set. By $span_b(E_0)$ we denote the totality of all possible finite sums of the form $\sum_{\alpha\in J} b(x_\alpha, f_\alpha)$, $J \in I_0$.

**Definition 2.1** *([30]) A system $\{f_\alpha\}_{\alpha\in I} \subset E$ is called b-complete in $H$ if $\overline{span}_b \{f_\alpha\}_{\alpha\in I} = H$.*

The following criterion for b-completeness of system is true.

**Lemma 2.1** *A system $\{f_\alpha\}_{\alpha\in I} \subset E$ is b-complete in $H$ if and only if, for $h \in H$, from $\omega_b(h, f_\alpha) = 0$ it follows that $h = 0$ for $\forall\alpha \in I$.*

**Proof. Necessity.** Let $\overline{span}_b \{f_\alpha\}_{\alpha\in I} = H$ and $h \in H$ be such that $\omega_b(h, f_\alpha) = 0$ for $\forall\alpha \in I$. Then for $\forall x \in X$ we have

$$0 = (x, \omega_b(h, f_\alpha))_X = (b(x, f_\alpha), h)_H, \forall\alpha \in I.$$

Hence it follows that for $\forall z \in H$,$(z, h)_H = 0$. Then $(h, h)_H = 0$, i.e. $h = 0$.

**Sufficiency.** Assume the contrary, i.e. let $\overline{span}_b \{f_\alpha\}_{\alpha\in I} \neq H$. Then there exists $h_0 \in H$ such that for $\forall z \in \overline{span}_b(\{f_\alpha\}_{\alpha\in I}$ the equality $(z, h_0)_H = 0$ holds. Consequently, $\forall x \in X$

$$(x, \omega_b(h_0, f_\alpha))_X = (b(x, f_\alpha), h_0)_H = 0, \forall\alpha \in I.$$

So $\omega_b(h_0, f_\alpha) = 0$, $\forall\alpha \in I$, and from the condition we obtain $h_0 = 0$. This contradicts $h_0 \notin \overline{span}_b \{f_\alpha\}_{\alpha\in I}$. The obtained contradiction shows the validity of the relation $\overline{span}_b \{f_\alpha\}_{\alpha\in I} = H$. The lemma is proved.

Denote by $l_2(X)$ the Hilbert space of systems $\bar{x} = \{x_\alpha\}_{\alpha\in I} \subset X$ such that $I_{\bar{x}} = \{\alpha \in I : x_\alpha \neq 0\} \in I^a$ and

$$\sum_{\alpha\in I} \|x_\alpha\|_X^2 < +\infty,$$

with scalar product

$$(\bar{x}, \bar{y})_{l_2(X)} = \sum_{\alpha\in I} (x_\alpha, y_\alpha)_X.$$

The following concept is a generalization of uncountable Besselian systems.

**Definition 2.2** *A system $\{f_\alpha\}_{\alpha\in I} \subset E$ is called uncountable b-Besselian in $H$ if $\exists B > 0$: $\forall\omega \in I^a$ and $\forall h \in H$*

$$\sum_{\alpha\in\omega} \|\omega_b(h, f_\alpha)\|_X^2 \leq B \|h\|_H^2 . \tag{2.1}$$

*The constant b is called a boundary of the uncountable b-Besselian system $\{f_\alpha\}_{\alpha\in I}$.*

From the definition of uncountable b-Besselian system $\{f_\alpha\}_{\alpha\in I}$ it follows that for $\forall h \in H$ the set $I(h) = \{\alpha \in I : \omega_b(h, f_\alpha) \neq 0\}$ belongs to $I^a$. In fact, denote $I_n(h) = \left\{\alpha \in I : \|\omega_b(h, f_\alpha)\|_X > \frac{1}{n}\right\}$. It is clear that $I(h) = \bigcup_{n=1}^{\infty} I_n(h)$. From 2.1 it follows $I_n(h) \in I_0$. Consequently, $I(h) \in I^a$.

The following criterion for uncountable b-Besselness of system is true.

**Theorem 2.1** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$. For the system $\{f_\alpha\}_{\alpha\in I}$ to be uncountable b-Besselian in $H$ with the boundary $b$, it is necessary and sufficient that the operator $T : l_2(X) \to H$ defined by*

$$T(\bar{x}) = \sum_{\alpha\in I} b(x_\alpha, f_\alpha), \tag{2.2}$$

*is bounded linear and $\|T\| \leq \sqrt{B}$.*

**Proof. Necessity.** Let the system $\{f_\alpha\}_{\alpha\in I}$ be uncountable b-Besselian in $H$ with the boundary b. Consider an arbitrary set $\omega \in I^a$. From the inequality

$$\sum_{\alpha\in\omega} \|\omega_b(h, f_\alpha)\|_X^2 \leq B \|h\|_H^2, \forall h \in H,$$

it follows that the series $\sum_{\alpha\in\omega} b(x_\alpha, f_\alpha)$ is convergent, the operator $T_\omega : l_2(X) \to H$ is defined by the formula

$$T_\omega(\bar{x}) = \sum_{\alpha\in\omega} b(x_\alpha, f_\alpha),$$

and $\|T_\omega\| \leq \sqrt{B}$. Then the series $\sum_{\alpha\in I} b(x_\alpha, f_\alpha)$ is convergent and

$$\|T(\bar{x})\|_H = \left\|\sum_{\alpha\in I} b(x_\alpha, f_\alpha)\right\|_H = \|T_{I_{\bar{x}}}(\bar{x})\|_H$$

$$\leq \sup_{\omega\in I^a} \|T_\omega(\bar{x})\|_H \leq \sqrt{B} \|\bar{x}\|_{l_2(X)}.$$

**Sufficiency.** Let the operator $T : l_2(X) \to H$ defined by 2.2 be bounded. Let's find the operator $T^*$. For $\forall x \in X$ and $\forall h \in H$ we have

$$(T(\bar{x}), h)_H = (\sum_{\alpha\in I} b(x_\alpha, f_\alpha), h)_H = \sum_{\alpha\in I}(b(x_\alpha, f_\alpha), h)_H = \sum_{\alpha\in I}(x_\alpha, \omega_b(h, f_\alpha))_X.$$

Let $T^*h = \bar{y} = \{y_\alpha\}_{\alpha\in I}$. Then

$$(T(\bar{x}), h)_H = (\bar{x}, T^*h)_{l_2(X)} = \sum_{\alpha\in I}(x_\alpha, y_\alpha)_X.$$

Comparing these expressions, we have

$$\sum_{\alpha\in I}(x_\alpha, \omega_b(h, f_\alpha))_X = \sum_{\alpha\in I}(x_\alpha, y_\alpha)_X.$$

Hence it follows $\omega_b(h, f_\alpha) = y_\alpha$ for $\forall\alpha \in I$. Consequently,

$$T^*h = \{\omega_b(h, f_\alpha)\}_{\alpha\in I}, \forall h \in H, \tag{2.3}$$

and

$$\sum_{\alpha\in I} \|\omega_b(h, f_\alpha)\|_X^2 = \|T^*h\|_{l_2(X)} \leq \|T\|^2 \|h\|_H^2 \leq B \|h\|_H^2,$$

i.e. $\{f_\alpha\}_{\alpha\in I}$ is uncountable b-Besselian in $H$. The theorem is proved.

The next theorem is a sufficient condition for a system to be uncountable b-Besselian.

**Theorem 2.2** *Let the system $\{f_\alpha\}_{\alpha\in I} \subset E$ be such that the series $\sum_{\alpha\in I} b(x_\alpha, f_\alpha)$ is convergent for $\forall\bar{x} = \{x_\alpha\}_{\alpha\in I} \in l_2(X)$. Then there exists $B > 0$ such that*

$$\left\|\sum_{\alpha\in I} b(x_\alpha, f_\alpha)\right\|_H \leq B \|\bar{x}\|_{l_2(X)}, \forall\bar{x} = \{x_\alpha\}_{\alpha\in I} \in l_2(X).$$

**Proof.** Consider an arbitrary set $\omega \in I^a$. As the series $\sum_{\alpha\in\omega} b(x_\alpha, f_\alpha)$ is convergent, the operator $T_\omega(\bar{x}) = \sum_{\alpha\in\omega} b(x_\alpha, f_\alpha)$ is bounded. Besides, for $\forall h \in H$ the series $\sum_{\alpha\in\omega} \|\omega_b(h, f_\alpha)\|_X^2$ is convergent. Consequently, for $\forall h \in H$ we have $I(h) = \{\alpha \in I : \omega_b(h, f_\alpha) \neq 0\} \in I^a$. Let's show the boundedness of the operator $T$ defined by 2.2. We have

$$\|T(\bar{x})\|_H = \left\|\sum_{\alpha\in I} b(x_\alpha, f_\alpha)\right\|_H = \sup_{\|h\|=1}\left|\left(\sum_{\alpha\in I} b(x_\alpha, f_\alpha), h\right)_H\right|$$

$$= \sup_{\|h\|=1}\left|\sum_{\alpha\in I}(b(x_\alpha, f_\alpha), h)_H\right| = \sup_{\|h\|=1}\left|\sum_{\alpha\in I}(x_\alpha, \omega_b(h, f_\alpha))_X\right|$$

$$\leq \sup_{\|h\|=1}\sum_{\alpha\in I}\|x_\alpha\|_X\|\omega_b(h, f_\alpha)\|_X \leq B\|\bar{x}\|_{l_2(X)},$$

where $B = \sup\limits_{\|h\|=1}\left(\sum_{\alpha\in I}\|\omega_b(h, f_\alpha)\|_X^2\right)^{\frac{1}{2}}$. It remains to show the validity of $B < +\infty$. Assume the contrary, i.e. let $B = +\infty$. Then $\exists h_n \in H$: $\|h_n\|_H = 1$ and

$$\left(\sum_{\alpha\in I}\|\omega_b(h_n, f_\alpha)\|_X^2\right)^{\frac{1}{2}} > n.$$

Let $\omega_0 = \bigcup_{n=1}^{\infty} I(h_n)$. Clearly, $\omega_0 \in I^a$. Consequently, there exists $B_0 > 0$ such that

$$\|T_{\omega_0}(\bar{x})\|_H \leq B_0\|\bar{x}\|_{l_2(X)}, \forall\bar{x} = \{x_\alpha\}_{\alpha\in I} \in l_2(X).$$

Let $n > B_0$. Denote

$$\bar{x}^{(n)} = \left\{x_\alpha^{(n)}\right\}_{\alpha\in I} : x_\alpha^{(n)} = \begin{cases}\omega_b(h_n, f_\alpha), \alpha \in I(h_n)\\ 0, \alpha \notin I(h_n)\end{cases}.$$

Then

$$\left\|\bar{x}^{(n)}\right\|_{l_2(X)} = \left(\sum_{\alpha\in I}\|\omega_b(h_n, f_\alpha)\|_X^2\right)^{\frac{1}{2}} > n.$$

We have

$$(T_{\omega_0}(\bar{x}^{(n)}), h_n)_H = (\sum_{\alpha\in\omega_0} b(x_\alpha^{(n)}, f_\alpha), h_n)_H = \sum_{\alpha\in\omega_0}(b(x_\alpha^{(n)}, f_\alpha), h_n)_H$$

$$= \sum_{\alpha\in\omega_0}(x_\alpha^{(n)}, \omega_b(h_n, f_\alpha))_X = \sum_{\alpha\in\omega_0}\|\omega_b(h_n, f_\alpha)\|_X^2 = \left\|\bar{x}^{(n)}\right\|_{l_2(X)}^2.$$

On the other hand,

$$\left\|\bar{x}^{(n)}\right\|_{l_2(X)}^2 = (T_{\omega_0}(\bar{x}^{(n)}), h_n)_H \leq \left\|T_{\omega_0}(\bar{x}^{(n)})\right\|_H \leq B_0\left\|\bar{x}^{(n)}\right\|_{l_2(X)}.$$

Hence it follows $\|\bar{x}^{(n)}\|_{l_2(X)} \leq B_0$. Then $n < \|\bar{x}^{(n)}\|_{l_2(X)} \leq B_0$. We arrived at a contradiction. So $B < +\infty$. The theorem is proved.

The next theorem shows that the uncountable Besselness on the whole space follows from the uncountable Besselness on the everywhere dense subspace.

**Theorem 2.3** *Let $V$ be an everywhere dense set of $H$ and $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-Besselian system in $V$ with the boundary b. Then $\{f_\alpha\}_{\alpha\in I}$ is uncountable b-Besselian in $H$ with the boundary b.*

**Proof.** Consider $\forall h \in H$. As $V$ is everywhere dense in $H$, we have $\exists h_n \in H$: $\lim_{n\to\infty} h_n = h$. We also have $I(h) \subset \bigcup_{n=1}^{\infty} I(h_n)$. In fact, if $\alpha \notin \bigcup_{n=1}^{\infty} I(h_n)$, then $\alpha \notin I(h_n)$, i.e. $\omega_b(h_n, f_\alpha) = 0$, $\forall n \in N$. Then $\omega_b(h, f_\alpha) = \lim_{n\to\infty} \omega_b(h_n, f_\alpha) = 0$, i.e. $\alpha \notin I(h)$. Let $I(h) = \{\alpha_k\}_{k\in N}$ and $m \in N$ be a fixed number. By 2.1 we have

$$\sum_{k=1}^{m} \|\omega_b(h_n, f_{\alpha_k})\|_X^2 \le B\,\|h_n\|_H^2 . \tag{2.4}$$

Passing to the limit in 2.4 as $n \to \infty$, we obtain

$$\sum_{k=1}^{m} \|\omega_b(h, f_{\alpha_k})\|_X^2 \le B\,\|h\|_H^2 . \tag{2.5}$$

Now let's pass to the limit in 2.5 as $m \to \infty$. Then we obtain

$$\sum_{k=1}^{\infty} \|\omega_b(h, f_{\alpha_k})\|_X^2 \le B\,\|h\|_H^2 ,$$

i.e. $\{f_\alpha\}_{\alpha\in I} \subset E$ is uncountable b-Besselian in $H$ with the boundary b. The theorem is proved.

## 3 Uncountable b-frame

Let $\{H_\alpha\}_{\alpha\in I}$ be a system of Hilbert spaces. Let's state an uncountable generalization of the concept of $g$-frame in non-separable Hilbert spaces.

**Definition 3.1** *A system $\{\Lambda_\alpha \in L(H, H_\alpha)\}_{\alpha\in I}$ is called an uncountable g-frame in $H$ if $\forall h \in H$ $I_\Lambda(h) = \{\alpha \in I : \Lambda_\alpha(h) \neq 0\} \in I^a$ and $\exists A, B > 0$: $\forall h \in H$*

$$A\,\|h\|_H^2 \le \sum_{\alpha\in I} \|\Lambda_\alpha(h)\|_{H_\alpha}^2 \le B\,\|h\|_H^2 .$$

*The constants $A$ and b are called the lower and upper bounds of the g-frame $\{\Lambda_\alpha\}_{\alpha\in I}$, respectively.*

The next concept is a generalization of the concept of uncountable frames in non-separable Hilbert spaces (see [34]).

**Definition 3.2** *A system $\{f_\alpha\}_{\alpha\in I} \subset E$ is called an uncountable b-frame in $H$ if $\forall h \in H$ $I(h) = \{\alpha \in I : \omega_b(h, f_\alpha) \neq 0\} \in I^a$ and $\exists A, B > 0$: $\forall h \in H$*

$$A\,\|h\|_H^2 \le \sum_{\alpha\in I} \|\omega_b(h, f_\alpha)\|_X^2 \le B\,\|h\|_H^2 . \tag{3.1}$$

*The constants $A$ and b are called the lower and upper bounds of the b-frame $\{f_\alpha\}_{\alpha\in I}$, respectively.*

The following criterion for uncountable b-frameness of system is true.

**Theorem 3.1** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$. For the system $\{f_\alpha\}_{\alpha\in I}$ to be uncountable b-frame in $H$, it is necessary and sufficient that the operator $T : l_2(X) \to H$ defined by 2.2 is bounded surjective.*

**Proof. Necessity.** Let the system $\{f_\alpha\}_{\alpha\in I}$ be uncountable b-frame in $H$ with the bounds $A$ and b. Then it is clear that $\{f_\alpha\}_{\alpha\in I}$ is uncountable b-Besselian in $H$. Therefore, by Theorem 1, the operator $T(\bar{x}) = \sum_{\alpha\in I} b(x_\alpha, f_\alpha)$ is bounded and $\|T\| \leq \sqrt{B}$. From 2.3 it follows that $T^*h = \{\omega_b(h, f_\alpha)\}_{\alpha\in I}$, $\forall h \in H$. Then from 3.1 we obtain

$$A\|h\|_H^2 \leq \sum_{\alpha\in I} \|\omega_b(h, f_\alpha)\|_X^2 = \|T^*(h)\|_{l_2(X)}^2.$$

Consequently, $T$ is a surjective operator.

**Sufficiency.** Let $T$ be a bounded surjective operator. By Theorem 1, from the boundedness of $T$ it follows that the system $\{f_\alpha\}_{\alpha\in I}$ is uncountable b-Besselian in $H$, and the surjectivity of $T$ implies that there exists $c > 0$ such that

$$\|T^*(h)\|_{l_2(X)} \geq c\|h\|_H, \forall h \in H.$$

Thus,

$$c^2\|h\|_H^2 \leq \|T^*(h)\|_{l_2(X)}^2 \leq \|T\|^2\|h\|_H^2,$$

i.e. $\{f_\alpha\}_{\alpha\in I}$ is an uncountable b-frame in $H$. The theorem is proved.

Let $\{f_\alpha\}_{\alpha\in I}$ be an uncountable b-frame in $H$ and the operator $T$ be defined by 2.2. Consider the operator $S = TT^*$. Clearly, $S \in L(H)$, and for $\forall h \in H$ we have

$$S(h) = TT^*(h) = \sum_{\alpha\in I} b(\omega_b(h, f_\alpha), f_\alpha). \tag{3.2}$$

We will call the operator $S$ an uncountable b-frame operator for $\{f_\alpha\}_{\alpha\in I}$. Theorem below establishes some properties of b-frame operator.

**Theorem 3.2** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $H$ with the bounds $A$ and b. The following properties are true:*
*1) $S$ is a self-adjoint positive operator and $AI_H \leq S \leq BI_H$;*
*2) $S$ is a boundedly invertible operator and $B^{-1}I_H \leq S^{-1} \leq A^{-1}I_H$;*
*3) the system $\{S_{f_\alpha} \in L(H, X)\}_{\alpha\in I}$ is an uncountable g-frame in $H$ with the bounds $B^{-1}$ and $A^{-1}$, where the operator $S_{f_\alpha}$ is defined by the formula $S_{f_\alpha}(h) = \omega_b(S^{-1}h, f_\alpha)$.*

**Proof.** We have

$$S^* = (TT^*)^* = TT^* = S.$$

On the other hand, from 3.2 we obtain

$$(S(h), h)_H = \sum_{\alpha\in I}(b(\omega_b(h, f_\alpha), f_\alpha), h)_H = \sum_{\alpha\in I} \|\omega_b(h, f_\alpha)\|_X^2. \tag{3.3}$$

It follows that $(S(h), h)_H \geq 0$.

Using 3.1 and 3.3, we obtain

$$A\|h\|_H^2 \leq \sum_{\alpha\in I} \|\omega_b(h, f_\alpha)\|_X^2 = (S(h), h)_H \leq \|S(h)\|_H \|h\|_H, \forall h \in H,$$

or $A\|h\|_H \leq \|S(h)\|_H$. Then $KerS = \{0\}$ and $R_S$ is closed. If $R_S \neq H$, then there exists $h_0 \in H\backslash\{0\}$ such that $(S(h), h_0)_H = 0$, $\forall h \in H$. Consequently, $(S(h_0), h_0)_H = 0$. From this relation and 3.3 it follows $\omega_b(h_0, f_\alpha) = 0, \forall\alpha \in I$. Then, by 3.1, we obtain $h_0 = 0$. The obtained contradiction shows that $R_S = H$. Thus, by Banach's homeomorphism theorem, the operator $S$ is boundedly invertible. From 3.1, for $\forall h \in H$ we obtain

$$A \left\|S^{-1}h\right\|_H^2 \leq (S^{-1}(h), h)_H \leq \left\|S^{-1}(h)\right\|_H \|h\|_H ,$$

or

$$\left\|S^{-1}(h)\right\|_H \leq A^{-1} \|h\|_H , \forall h \in H.$$

Therefore,

$$(S^{-1}(h), h)_H \leq A^{-1} \|h\|_H^2 , \forall h \in H. \tag{3.4}$$

Also, for $\forall h \in H$ we have

$$\|h\|_H^4 = ((S^{-1}(Sh), h)_H)^2 \leq (S^{-1}(Sh), Sh)_H (S^{-1}(h), h)_H$$

$$= (Sh, h)_H (S^{-1}(h), h)_H \leq B \|h\|_H^2 (S^{-1}(h), h)_H$$

or

$$B^{-1} \|h\|_H^2 \leq (S^{-1}(h), h)_H), \forall h \in H. \tag{3.5}$$

From 3.4 and 3.5 it follows that $B^{-1}I_H \leq S^{-1} \leq A^{-1}I_H$.

Finally, for $\forall h \in H$ we have

$$\sum_{\alpha\in I} \|S_{f_\alpha}(h)\|_X^2 = \sum_{\alpha\in I} \left\|\omega_b(S^{-1}h, f_\alpha)\right\|_X^2 = (S^{-1}h, h)_H.$$

Consequently, from 3.4 and 3.5 we obtain

$$B^{-1} \|h\|_H^2 \leq \sum_{\alpha\in I} \|S_{f_\alpha}(h)\|_X^2 \leq A^{-1} \|h\|_H^2 , \forall h \in H.$$

Thus, by 2.5, the system $\{S_{f_\alpha}\}_{\alpha\in I}$ forms an uncountable $g$-frame for $H$ with the bounds $B^{-1}$ and $A^{-1}$. The theorem is proved.

Similar to the expansion of every element with respect to the frame elements, for $\forall h \in H$, by 3.2, the following representation holds:

$$h = SS^{-1}(h) = \sum_{\alpha\in I} b(\omega_b(S^{-1}h, f_\alpha), f_\alpha) = \sum_{\alpha\in I} b(S_{f_\alpha}(h), f_\alpha). \tag{3.6}$$

**Definition 3.3** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $H$. The system $\{S_{f_\alpha}\}_{\alpha\in I}$ will be called an uncountable dual g-frame of the b-frame $\{f_\alpha\}_{\alpha\in I}$ in $H$.*

The following theorem is true for the coefficients of expansion with respect to the elements of a b-frame.

**Theorem 3.3** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $H$, $h \in H$ have an expansion $h = \sum_{\alpha\in I} b(x_\alpha, f_\alpha)$, $\bar{x} = \{x_\alpha\}_{\alpha\in I} \in l_2(X)$, and $\{S_{f_\alpha}\}_{\alpha\in I}$ be an uncountable dual g-frame of $\{f_\alpha\}_{\alpha\in I}$. Then*

$$\left\|\{x_\alpha\}_{\alpha\in I}\right\|_{l_2(X)}^2 = \left\|\{S_{f_\alpha}(h)\}_{\alpha\in I}\right\|_{l_2(X)}^2 + \left\|\{x_\alpha - S_{f_\alpha}(h)\}_{\alpha\in I}\right\|_{l_2(X)}^2 . \tag{3.7}$$

**Proof.** From 3.6 we obtain

$$(S^{-1}h, h) = \sum_{\alpha\in I} \|S_{f_\alpha}(h)\|_X^2 = \left\|\{S_{f_\alpha}(h)\}_{\alpha\in I}\right\|_{l_2(X)}^2 .$$

We also have

$$(h, S^{-1}h)_H = (\sum_{\alpha\in I} b(x_\alpha, f_\alpha), S^{-1}h)_H = \sum_{\alpha\in I}(x_\alpha, \omega_b(S^{-1}h, f_\alpha))_H$$

$$= \sum_{\alpha\in I}(x_\alpha, S_{f_\alpha}(h))_H = (\bar{x}, \{S_{f_\alpha}(h)\}_{\alpha\in I})_{l_2(X)}.$$

Consequently,

$$0 = (\bar{x} - \{S_{f_\alpha}(h)\}_{\alpha\in I}, \{S_{f_\alpha}(h)\}_{\alpha\in I})_{l_2(X)}.$$

Therefore,

$$\left\|\{x_\alpha\}_{\alpha\in I}\right\|_{l_2(X)}^2 = \left\|\{x_\alpha - S_{f_\alpha}(h) + S_{f_\alpha}(h)\}_{\alpha\in I}\right\|_{l_2(X)}^2$$

$$= \left\|\{x_\alpha - S_{f_\alpha}(h)\}_{\alpha\in I}\right\|_{l_2(X)}^2 + \left\|\{S_{f_\alpha}(h)\}_{\alpha\in I}\right\|_{l_2(X)}^2 .$$

The theorem is proved.

**Corollary 3.1** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $H$ and $\{S_{f_\alpha}\}_{\alpha\in I}$ be an uncountable dual g-frame of $\{f_\alpha\}_{\alpha\in I}$. Then $\forall h \in H$*

$$\left\|\{S_{f_\alpha}(h)\}_{\alpha\in I}\right\|_{l_2(X)} = \inf\left\{\left\|\{x_\alpha\}_{\alpha\in I}\right\|_{l_2(X)} : h = \sum_{\alpha\in I} b(x_\alpha, f_\alpha), \{x_\alpha\}_{\alpha\in I} \in l_2(X)\right\}.$$

**Proof.** The proof follows immediately from the equality 3.7. The corollary is proved.

**Corollary 3.2** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $H$, $\beta \in I$ and $\{S_{f_\alpha}\}_{\alpha\in I}$ be an uncountable dual g-frame of $\{f_\alpha\}_{\alpha\in I}$. Then $\forall x \in X$ the relation*

$$\sum_{\alpha\neq\beta} \|S_{f_\alpha} b(x, f_\beta)\|_X^2 = \frac{\|x\|_X^2 - \|(I_X - S_\beta)x\|_X^2 - \|S_\beta x\|_X^2}{2} \tag{3.8}$$

*holds, where the operator $S_\beta$ is defined by $S_\beta x = S_{f_\beta} b(x, f_\beta)$.*

**Proof.** From 3.6 we obtain

$$b(x, f_\beta) = \sum_{\alpha\in I} b(S_{f_\alpha}(b(x, f_\beta)), f_\alpha).$$

We also have

$$b(x, f_\beta) = \sum_{\alpha\in I} b(\delta_{\alpha\beta} x, f_\alpha).$$

Then, by 3.7, we obtain

$$\|x\|_X^2 = \left\|\{S_{f_\alpha}(b(x, f_\beta))\}_{\alpha\in I}\right\|_{l_2(X)}^2 + \left\|\{\delta_{\alpha\beta} x - S_{f_\alpha}(b(x, f_\beta))\}_{\alpha\in I}\right\|_{l_2(X)}^2$$

$$= \sum_{\alpha\neq\beta} \|S_{f_\alpha} b(x, f_\beta)\|_X^2 + \|S_\beta(x)\|_X^2 + \sum_{\alpha\neq\beta} \|S_{f_\alpha} b(x, f_\beta)\|_X^2 + \|(I_X - S_\beta)(x)\|_X^2 .$$

Hence we get the validity of 3.8. The corollary is proved.

Now let's consider the uncountable b-frameness of the system obtained by elimination of arbitrary element from the uncountable b-frame.

**Theorem 3.4** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $H$, $\beta \in I$, $\{S_{f_\alpha}\}_{\alpha\in I}$ be an uncountable dual g-frame of $\{f_\alpha\}_{\alpha\in I}$, and $S_\beta x = S_{f_\beta} b(x, f_\beta)$. Then,*
*1) if $\ker(I_X - S_\beta) \neq \{0\}$, then the system $\{f_\alpha\}_{\alpha\neq\beta}$ is not b-complete in $H$;*
*2) if $\ker(I_X - S_\beta) = \{0\}$ and $(I_X - S_\beta)^{-1} \in L(X)$, then $\{f_\alpha\}_{\alpha\neq\beta}$ forms an uncountable b-frame for $H$.*

**Proof.** 1) Let $x \in \ker(I_X - S_\beta)$ and $x \neq 0$. Then $S_\beta(x) = x$ and from 3.8 we obtain

$$\sum_{\alpha\neq\beta} \|S_{f_\alpha} b(x, f_\beta)\|_X^2 = 0.$$

It follows that $0 = S_{f_\alpha} b(x, f_\beta) = \omega_b(S^{-1}b(x, f_\beta), f_\alpha), \forall\alpha \in I\setminus\{\beta\}$. As $S^{-1}b(x, f_\beta) \neq 0$, by Lemma 1, the system $\{f_\alpha\}_{\alpha\neq\beta}$ is not b-complete in $H$.

2) For $\forall x \in X$ we have

$$b(x, f_\beta) = \sum_{\alpha\in I} b(S_{f_\alpha}(b(x, f_\beta)), f_\alpha)$$

Hence we obtain

$$b(x, f_\beta) - b(S_{f_\beta} b(x, f_\beta), f_\beta) = \sum_{\alpha\neq\beta} b(S_{f_\alpha}(b(x, f_\beta)), f_\alpha)$$

or

$$b((I_X - S_\beta)x, f_\beta) = \sum_{\alpha\neq\beta} b(S_{f_\alpha}(b(x, f_\beta)), f_\alpha). \tag{3.9}$$

Consider an arbitrary $h \in H$. Let's perform scalar multiplication of both sides of 3.9 by $h \in H$. Then we obtain

$$(b((I_X - S_\beta)x, f_\beta), h)_H = \sum_{\alpha\neq\beta} (b(S_{f_\alpha}(b(x, f_\beta)), f_\alpha), h)_H.$$

This is equivalent to

$$((I_X - S_\beta)x, \omega_b(h, f_\beta))_H = \sum_{\alpha\neq\beta} (S_{f_\alpha} b(x, f_\beta), \omega_b(h, f_\alpha))_H. \tag{3.10}$$

Let $\omega_b(h, f_\beta) \neq 0$. Choose $x = \frac{1}{\|\omega_b(h,f_\beta)\|_X}(I_X - S_\beta)^{-1}\omega_b(h, f_\beta)$. Then from 3.10 we obtain

$$\|\omega_b(h, f_\beta)\|_X = \sum_{\alpha\neq\beta} (S_{f_\alpha} b(x, f_\beta), \omega_b(h, f_\alpha))_H.$$

From here, using Cauchy-Bunyakovsky inequality, we get

$$\|\omega_b(h, f_\beta)\|_X^2 \leq \sum_{\alpha\neq\beta} \|S_{f_\alpha} b(x, f_\beta)\|_X^2 \sum_{\alpha\neq\beta} \|\omega_b(x, f_\beta)\|_X^2$$

$$\leq A^{-1} \|b(x, f_\beta)\|^2 \sum_{\alpha\neq\beta} \|\omega_b(x, f_\beta)\|_X^2$$

$$\leq A^{-1}M^2 \|x\|_X^2 \|f_\beta\|_E^2 \sum_{\alpha\neq\beta} \|\omega_b(x,f_\beta)\|_X^2$$

$$\leq A^{-1}M^2 \left\|(I_X-S_\beta)^{-1}\right\|^2 \|f_\beta\|_E^2 \sum_{\alpha\neq\beta} \|\omega_b(x,f_\beta)\|_X^2$$

$$= C_\beta \sum_{\alpha\neq\beta} \|\omega_b(x,f_\beta)\|_X^2 . \qquad (3.11)$$

From 3.1 and 3.11 we obtain

$$A\|h\|_H^2 \leq \sum_{\alpha\in I} \|\omega_b(x,f_\alpha)\|_X^2 = \|\omega_b(x,f_\beta)\|_X^2 + \sum_{\alpha\neq\beta} \|\omega_b(x,f_\alpha)\|_X^2$$

$$\leq C_\beta \sum_{\alpha\neq\beta} \|\omega_b(x,f_\alpha)\|_X^2 + \sum_{\alpha\neq\beta} \|\omega_b(x,f_\alpha)\|_X^2$$

$$= (1+C_\beta) \sum_{\alpha\neq\beta} \|\omega_b(x,f_\alpha)\|_X^2 . \qquad (3.12)$$

So, taking into account 3.1 and 3.12, for $\forall h \in H$ we have

$$\frac{A}{1+C_\beta}\|h\|_H^2 \leq \sum_{\alpha\neq\beta} \|\omega_b(x,f_\alpha)\|_X^2 \leq \sum_{\alpha\in I} \|\omega_b(x,f_\alpha)\|_X^2 \leq B\|h\|_H^2 .$$

The theorem is proved.

Like in the case of uncountable b-Besselness, the uncountable b-frameness of the system on everywhere dense set implies its uncountable b-frameness on the whole space.

The following theorem is true.

**Theorem 3.5** *Let $V$ be an everywhere dense set of $H$ and the system $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $V$ with the bounds $A$ and $b$. Then $\{f_\alpha\}_{\alpha\in I}$ forms an uncountable b-frame for $H$ with the bounds $A$ and $b$.*

**Proof.** Consider $\forall h \in H$. By Theorem 3, the right-hand side of 3.1 is true for $h \in H$. It remains to show the validity of the left-hand side of 3.1. As $V$ is dense in $H$, $\exists h_n \in H$: $\lim_{n\to\infty} h_n = h$. Let the operator $T$ be defined by 2.2. From the b-frameness of $\{f_\alpha\}_{\alpha\in I}$ in $V$ and the relation 2.3 we obtain

$$A\|h_n\|_H^2 \leq \|T^*h_n\|_{l_2(X)} .$$

Passing here to the limit as $n\to\infty$, we obtain $A\|h\|_H^2 \leq \|T^*h\|_{l_2(X)}$, i.e. the left-hand side of the inequality 3.1 is true for $h \in H$. The theorem is proved.

## 4 Perturbation and stability of uncountable b-frame

In this section, we consider the stability and the compact Noetherian perturbation of an uncountable b-frame in non-separable Hilbert space.

We will need the following theorem.

**Theorem 4.1** *Let the numbers $\lambda_1, \lambda_2 \in [0,1)$ and the linear operator $G : E \to E$ be such that $\|f - Gf\|_E \leq \lambda_1 \|f\|_E + \lambda_2 \|Gf\|_E$, $f \in E$. Then the operator $G$ is boundedly invertible and*

$$\frac{1-\lambda_2}{1+\lambda_1}\|f\|_E \leq \left\|G^{-1}f\right\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|f\|_E , f \in E.$$

Theorem below is true.

**Theorem 4.2** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $H$ with the bounds $A$ and $b$ and $\{g_\alpha\}_{\alpha\in I} \subset E$ be some system. Assume that there exist the numbers $\lambda, \mu, \gamma \geq 0$ such that* $\max\left\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\right\} < 1$*, and the condition*

$$\left\|\sum_{\alpha\in J} b(x_\alpha, f_\alpha - g_\alpha)\right\|_H \leq \lambda \left\|\sum_{\alpha\in J} b(x_\alpha, f_\alpha)\right\|_H + \mu \left\|\sum_{\alpha\in J} b(x_\alpha, g_\alpha)\right\|_H + \gamma \left\|\{x_\alpha\}_{\alpha\in J}\right\|_{l_2(X)} \tag{4.1}$$

*holds for $\forall J \in I_0$ and $\{x_\alpha\}_{\alpha\in J} \subset X$. Then the system $\{g_\alpha\}_{\alpha\in I}$ is an uncountable b-frame for $H$ with the bounds*

$$\left(\frac{(1-\lambda)\sqrt{A}-\gamma)}{1+\mu}\right)^2 \quad \text{and} \quad \left(\frac{(1+\lambda)\sqrt{B}+\gamma)}{1-\mu}\right)^2.$$

**Proof.** Using 3.12, we obtain

$$\left\|\sum_{\alpha\in J} b(x_\alpha, g_\alpha)\right\|_H \leq \left\|\sum_{\alpha\in J} b(x_\alpha, f_\alpha)\right\|_H + \left\|\sum_{\alpha\in J} b(x_\alpha, f_\alpha - g_\alpha)\right\|_H$$

$$\leq (1+\lambda) \left\|\sum_{\alpha\in J} b(x_\alpha, f_\alpha)\right\|_H + \mu \left\|\sum_{\alpha\in J} b(x_\alpha, g_\alpha)\right\|_H + \gamma \left\|\{x_\alpha\}_{\alpha\in J}\right\|_{l_2(X)}.$$

After some transformations, we get

$$\left\|\sum_{\alpha\in J} b(x_\alpha, g_\alpha)\right\|_H \leq \frac{1+\lambda}{1-\mu} \left\|\sum_{\alpha\in J} b(x_\alpha, f_\alpha)\right\|_H + \frac{\gamma}{1-\mu} \left\|\{x_\alpha\}_{\alpha\in J}\right\|_{l_2(X)}.$$

It follows that the series $\sum_{\alpha\in I} b(x_\alpha, g_\alpha)$ is convergent for $\forall \bar{x} = \{x_\alpha\}_{\alpha\in I} \in l_2(X)$ and

$$\left\|\sum_{\alpha\in I} b(x_\alpha, g_\alpha)\right\|_H \leq \frac{1+\lambda}{1-\mu} \left\|\sum_{\alpha\in I} b(x_\alpha, f_\alpha)\right\|_H + \frac{\gamma}{1-\mu} \left\|\{x_\alpha\}_{\alpha\in J}\right\|_{l_2(X)}.$$

Consequently, by Theorem 1,

$$\left\|\sum_{\alpha\in I} b(x_\alpha, g_\alpha)\right\|_H \leq \frac{(1+\lambda)\sqrt{B}+\gamma}{1-\mu} \left\|\bar{x}\right\|_{l_2(X)},$$

and

$$\sum_{\alpha\in I} \left\|\omega_b(x, g_\alpha)\right\|_X^2 \leq \left(\frac{(1+\lambda)\sqrt{B}+\gamma}{1-\mu}\right)^2 \left\|h\right\|_H^2.$$

Let $U : l_2(X) \to H$ be defined by $U(\bar{x}) = \sum_{\alpha\in I} b(x_\alpha, g_\alpha)$. Consider the operator $G = UT^*S^{-1}$. We have

$$Gh = UT^*S^{-1}(h) = \sum_{\alpha\in I} b(\omega_b(S^{-1}h, f_\alpha), g_\alpha) = \sum_{\alpha\in I} b(S_{f_\alpha}(h), g_\alpha). \tag{4.2}$$

Taking into account 3.6, 4.1 and 4.2, we obtain

$$\|h-Gh\|_H = \left\|\sum_{\alpha\in I} b(S_{f_\alpha}(h), f_\alpha) - \sum_{\alpha\in I} b(S_{f_\alpha}(h), g_\alpha)\right\|_H$$

$$\leq \lambda \|h\|_H + \mu \|Gh\|_H + \gamma \left\|\{S_{f_\alpha}(h)\}_{\alpha\in I}\right\|_{l_2(X)}$$

$$\leq (\lambda + \frac{\gamma}{\sqrt{A}}) \|h\|_H + \mu \|Gh\|_H .$$

By Theorem 9, the operator $G$ is boundedly invertible and

$$\left\|G^{-1}\right\| \leq \frac{1+\mu}{1-(\lambda+\frac{\gamma}{\sqrt{A}})}.$$

Consequently, for $\forall h \in H$ we get

$$\|h\|_H^4 = ((h,h)_H)^2 = (G(G^{-1}h), h)_H)^2$$

$$= \left(\sum_{\alpha\in I}(b(S_{f_\alpha}(G^{-1}h), g_\alpha), h)_H\right)^2 = \left(\sum_{\alpha\in I}(S_{f_\alpha}(G^{-1}h), \omega_b(h, g_\alpha))_X\right)^2$$

$$\leq \sum_{\alpha\in I} \left\|S_{f_\alpha}(G^{-1}h)\right\|_X^2 \sum_{\alpha\in I} \|\omega_b(h, g_\alpha)\|_X^2 \leq A^{-1} \left\|G^{-1}h\right\|_H^2 \sum_{\alpha\in I} \|\omega_b(h, g_\alpha)\|_X^2$$

$$\leq A^{-1}(\frac{1+\mu}{1-(\lambda+\frac{\gamma}{\sqrt{A}})})^2 \|h\|_H^2 \sum_{\alpha\in I} \|\omega_b(h, g_\alpha)\|_X^2 .$$

Thus,

$$\leq \left(\frac{(1-\lambda)\sqrt{A}-\gamma}{1+\mu}\right)^2 \|h\|_H^2 \leq \sum_{\alpha\in I} \|\omega_b(h, g_\alpha)\|_X^2 .$$

The theorem is proved.

Like in the case of ordinary frames, there is a following compact perturbation of uncountable b-frame.

**Theorem 4.3** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $H$ with the bounds $A$ and b, $\{g_\alpha\}_{\alpha\in I} \subset E$ be some system and $K \in L(l_2(X), H)$ be a compact operator such that*

$$K(\bar{x}) = \sum_{\alpha\in I} b(x_\alpha, g_\alpha - f_\alpha), \bar{x} = \{x_\alpha\}_{\alpha\in I} \in l_2(X).$$

*Then $\{g_\alpha\}_{\alpha\in I}$ is an uncountable b-frame in $\overline{span_b}\,\{g_\alpha\}_{\alpha\in I}$.*

**Proof.** By Theorem 1, for $\forall \bar{x} = \{x_\alpha\}_{\alpha\in I} \in l_2(X)$ the series $\sum_{\alpha\in I} b(x_\alpha, f_\alpha)$ is convergent, the operator $T(\bar{x}) = \sum_{\alpha\in I} b(x_\alpha, f_\alpha)$ is bounded and $\|T\| \leq \sqrt{B}$. It is clear that the series $\sum_{\alpha\in I} b(x_\alpha, g_\alpha)$ is also convergent. Let $U(\bar{x}) = \sum_{\alpha\in I} b(x_\alpha, g_\alpha)$. Then $U = T + K \in L(l_2(X), H)$. Consequently, by Theorem 1, the system $\{g_\alpha\}_{\alpha\in I}$ is uncountable b-Besselian in $H$. Further, we have

$$UU^* = (T+K)(T^*+K^*) = S(I+W),$$

where $W = S^{-1}(TK^* + KT^* + KK^*)$ is a compact operator. Consequently, $R_{UU^*}$ is closed. Let $h \in \overline{span_b}\{g_\alpha\}_{\alpha\in I}$ and $UU^*h = 0$. Then

$$0 = (UU^*h, h)_H = \sum_{\alpha\in I} \|\omega_b(h, g_\alpha)\|_X^2 .$$

Hence we obtain $\omega_b(h, g_\alpha) = 0$, $\forall\alpha \in I$. Then from Lemma 1 it follows that $h = 0$. Thus, the operator $UU^*$ is boundedly invertible in $\overline{span_b}\{g_\alpha\}_{\alpha\in I}$. Consequently,

$$R_{UU^*} = (\ker UU^*)^\perp = \overline{span_b}\{g_\alpha\}_{\alpha\in I} ,$$

and therefore, $R_U = \overline{span_b}\{g_\alpha\}_{\alpha\in I}$. By Theorem 2, the system $\{g_\alpha\}_{\alpha\in I}$ is an uncountable b-frame in $\overline{span_b}\{g_\alpha\}_{\alpha\in I}$. The theorem is proved.

Let $E_1$ be a Banach space, $H_1$ be a non-separable Hilbert space, and $b_1 : X \times E_1 \to H_1$ be a bounded bilinear mapping.

Next theorem establishes the Noetherian perturbation of an uncountable b-frame.

**Theorem 4.4** *Let $\{f_\alpha\}_{\alpha\in I} \subset E$ be an uncountable b-frame in $H$ with the bounds $A$ and b, $\{g_\alpha\}_{\alpha\in I} \subset E_1$ be some system, and $F \in B(H, H_1)$ be a Noether operator such that for*

$$\forall x \in X F b(x, f_\alpha) = b_1(x, g_\alpha), \alpha \in I.$$

*Then $\{g_\alpha\}_{\alpha\in I}$ is an uncountable $b_1$-frame in $\overline{span_{b_1}}\{g_\alpha\}_{\alpha\in I}$.*

**Proof.** It is not difficult to show that for $\forall h \in H_1$ the relation

$$\omega_{b_1}(h, g_\alpha) = \omega_b(F^*h, f_\alpha), \alpha \in I,$$

is true. So, for $\forall h \in H_1$ we have

$$\{\alpha \in I : \omega_{b_1}(h, f_\alpha) \neq 0\} = \{\alpha \in I : \omega_b(F^*h, f_\alpha) \neq 0\} \in I^a.$$

Then, using 3.1, we obtain

$$\sum_{\alpha\in I} \|\omega_{b_1}(h, g_\alpha)\|_X^2 = \sum_{\alpha\in I} \|\omega_b(F^*h, f_\alpha)\|_X^2 \leq B \|F\|^2 \|h\|_{H_1}^2 , \forall h \in H_1,$$

i.e. the system $\{g_\alpha\}_{\alpha\in I}$ is uncountable $b_1$-Besselian in $H_1$.

Further, the closedness of $R_F$ and the b-completeness of $\{f_\alpha\}_{\alpha\in I}$ in $H$ imply $R_F = \overline{span_{b_1}}\{g_\alpha\}_{\alpha\in I}$. Let $H = \ker F + Z$. Denote by $F_1$ the restriction of the operator $F$ to $Z$. It is clear that $F_1 : Z \to \overline{span_{b_1}}\{g_\alpha\}_{\alpha\in I}$ is a boundedly invertible operator. Consider an arbitrary $h \in \overline{span_{b_1}}\{g_\alpha\}_{\alpha\in I}$. Let $F_1^{-1}h = z$. We have

$$h = F_1 z = Fz = F\left(\sum_{\alpha\in I} b(S_{f_\alpha}(z), f_\alpha)\right) = \sum_{\alpha\in I} b_1(S_{f_\alpha}(z), g_\alpha).$$

Then

$$\|h\|_{H_1}^4 = ((h, h)_{H_1})^2 = ((\sum_{\alpha\in I} b_1(S_{f_\alpha}(z), g_\alpha), h)_{H_1})^2$$

$$= \left(\sum_{\alpha\in I}(b_1(S_{f_\alpha}(z), g_\alpha), h)_{H_1}\right)^2 = \left(\sum_{\alpha\in I}(S_{f_\alpha}(z), \omega_{b_1}(h, g_\alpha))_X\right)^2$$

$$\leq \sum_{\alpha\in I} \|S_{f_\alpha}(z)\|_X^2 \sum_{\alpha\in I} \|\omega_{b_1}(h, g_\alpha)\|_X^2 \leq A^{-1} \|z\|_H^2 \sum_{\alpha\in I} \|\omega_{b_1}(h, g_\alpha)\|_X^2$$

$$= A^{-1}\left\|F_1^{-1}h\right\|_H^2 \sum_{\alpha\in I}\|\omega_{b_1}(h,g_\alpha)\|_X^2 \le A^{-1}\left\|F_1^{-1}\right\|^2\|h\|_{H_1}^2\sum_{\alpha\in I}\|\omega_{b_1}(h,g_\alpha)\|_X^2 .$$

Thus,

$$A\left\|F_1^{-1}\right\|^{-1}\|h\|_{H_1}^2 \le \sum_{\alpha\in I}\|\omega_{b_1}(h,g_\alpha)\|_X^2 .$$

The theorem is proved.

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