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Generalized fixed-point theorems. Applications

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Abstract. *In this work, the existence of the fixed points of the mappings does independent of their smoothness, of the single-value or multi-value using a new geometrical approach is studied. Here, the fixed-point theorems are proved, which generalize the fixed-point theorems of Brouwer and Schauder, and also Kakutani, in some sense. This approach is based on the idea of the Poincare article [1] and the geometry of the image of mappings and is independent of the topological properties of spaces, which allows studying mappings acting in vector spaces. We studied the solvability of the nonlinear equations and inclusions by applying the obtained general results. Here some auxiliary results are obtained, also.*

Keywords. nonlinear mappings, fixed-point theorem, nonlinear equations, nonlinear inclusion, convex sets.

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1 Introduction

The aim of this work is to study the existence of a fixed-point of mappings, and also the existence of solutions to the problems with such mappings under more general conditions, i.e. without the smoothness of the examined mappings and of any compactness. Namely, here shows that the existence of the fixed-point of the nonlinear mapping (in single-valued and multi-valued cases) can study also under certain geometrical conditions. This approach allows for studying nonlinear equations and inclusions with mappings f acting in vector topological spaces (VTS).

Well-known theorem Cauchy on the "mean" value of the continuous functions acting in R^1 one can assume is one of the first fixed-point theorems. Then J. Hadamard showed, that the mappings acting in $1-$dimension vector spaces also possess this property on a connected subset if its image is a connected subset. Later will be formulated the mentioned theorems in the form of a fixed-point theorem, which shows that these results can reckon as the fixed-point theorems in the $1-$dimension case. But results of such type doesn't possible already in the $2-$dimensional case. In [1] H. Poincare proved the existence of the fixed-point for the continuous mapping in the 2-dimension case but under sufficiently severe constraints.

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It needs to be noted in still earlier (1907) H. Poincare proposed to prove the general results on the existence of the fixed-point of the continuous mappings and stated the importance of such type results. One can suppose the question posed H. Poincare many mathematics began to study the existence of the fixed point of the mappings. One can suppose the question posed by H. Poincare many mathematics began to study the existence of the fixed point of the mappings. This problem, in the beginning, was solved in various variants: L. E. J. Brouwer proved the well-known Brouwer fixed-point theorem for the finite-dimensional cases, and T. Banach proved the well-known Banach fixed-point theorem for the contractions operator acting on the Banach space. For brevity, we won't cite other results relative to this problem (for more see, e.g. [2] - [11], [13], [18], [28], [32], [35], [38], etc.). In the infinite-dimensional case with solving the problem of the existence of the fixed-point of mappings were engaged many authors (see, e.g. [2] - [10], [20], [28], [31], [32], [36], [39] and their references) used different approaches. J. P. Schauder generalized the Brouwer theorem to the infinite-dimensional case and later this result was generalized in the joint work by J. Leray and J. Schauder called the Leray-Schauder theorem. There exist some generalization of the Banach theorem in the case when the operator is nonexpansive. It should be noted the fixed-point theorems of Schauder and Kakutani had certain generalizations. In this work, in the other sense the generalizations of these and also the fixed-point theorems of Brouwer, and Kakutani have been obtained.

So, here proved the theorems that generalize, in some sense the Brouwer and Schauder fixed-point theorems, and also results of such type in the multi-valued cases, moreover are proved some new fixed-point theorems. All obtained here results are based on the geometry of the image of the examined mappings. The used here approach connected, in some sense, with the geometrical method. One can reckon our approach is based on the generalization of the above-mentioned theorem Cauchy and on the properties of the convex sets. In this work is generalized also the lemma called the "acute-angled lemma" which is the variant of the Brouwer theorem. It needs note that this lemma successfully was applied under investigations of the nonlinear differential equations and inequations (see, e.g. [14, 15, 30, 32, 36, 38, 40, 43], etc.).

The obtained in this work general results, in particular, are applicable to the study of the solvability of the nonlinear equations and inclusions in VTS (and also to the investigations of the boundary value problems for nonlinear equations and inclusions). Here also obtained sufficient conditions at which images of the examined mappings can be the convex subsets. We would note that the essentiality of this condition below will be shown.

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So, here considered a nonlinear mapping $f : D(f) \subseteq X \longrightarrow Y$, where X and Y are the VTS and is investigated the question: under which conditions does a given $y \in Y$ belong to the image $f(G)$ of some subset $G \subseteq D(f)$? In the case when $f : D(f) \subseteq X \longrightarrow X$ is investigated the question: under which conditions does the mapping f has a fixed point?

It is clear these questions are equivalent to the questions on the solvability of the equations $f(x) = y$ or there existing such $\widetilde{x}$ that $f(\widetilde{x}) = \widetilde{x}$, and also the inclusion of $f(x) \ni y$ or there existing $\widetilde{x}$ that $\widetilde{x} \in f(\widetilde{x})$ depending on the single-valued or multi-valued mapping f.

For the study of these questions have been used the special approach based on the geometrical structure of the image $f(G)$ of the given subset $G \subseteq D(f)$ of the examined mapping, that a priori independent of any smoothness of the examined mapping f. Therefore, this approach one can call a geometrical approach.

Now we will lead the simple variant of the main fixed-point theorem of this work in the case of the Hilbert space (for brevity) and one general proposition.

Theorem 1.1 *(Fixed-point theorem) Let X be a Hilbert space, $B_r^X(0) \subset X$ is the closed ball and the mapping f acting in X be such that $f\left(B_r^X(0)\right) \subseteq B_r^X(0)$. Then if the image $f_1\left(B_r^X(0)\right) \subset X$ be a open (or closed) convex set then there exists such $x_0 \in B_r^X(0)$ that $f(x_0) = x_0$ (or $f(x_0) \ni x_0$ if f is the multi-valued), where $f_1(x) \equiv x - f(x)$ for $\forall x \in B_r^X(0)$.*

Proposition 1.1 *Let X, Y be $LVTS$, and $f : D(f) \subseteq X \longrightarrow Y$ is the single-valued mapping. Let the image $f(G)$ of some subset $G \subseteq D(f)$ is connected open or closed body[1] in Y. Then for each fixed element $y \in int\ f(G)$, there exists such subset of G on which for the mapping $f_1(x) = f(x) - y$ the conditions of Theorem 3.1 are fulfills, in the other words, in this case, $y \in Y$ contains to $f(G)$ iff there exists such subset of G on which for the mapping $f_1(x) = f(x) - y$ one of the conditions of Theorem 3.1 are fulfills.*

This article is constructed as follows. In the beginning the above-posed questions in multi-dimensional (Section 2) and infinite-dimensional (Section 3) cases are investigated. The case of the reflexive Banach spaces is studied separately in Section 4. Section 5 the nonlinear equations and inclusions in Banach spaces by application of the obtained general results are investigated. Section 6 some sufficient conditions for fulfillment the conditions of the theorems are obtained, and Section 7 leads the concrete examples of problems.

2 Some generalization of the Brouwer fixed-point theorem and its corollaries

In the beginning, we will provide some known results that are necessary for the studies (see, [8,9,13,16–19], [21–27], [35]).

We denote by $B_r^{R^n}(x_0)$ of the closed ball in R^n, $n \geq 1$ and by $S_r^{R^n}(x_0)$ of the sphere (boundary of the ball $B_r^{R^n}(x_0)$) a center $x_0 \in R^n$ and radius $r > 0$.

Theorem 2.1 *(see, [3], [15]) Let f acted in R^n and for some $r > 0$ on the closed ball $B_r^{R^n}(0) \subseteq D(f)$ satisfies conditions: 1) f is continuous; 2) the inequality $\langle f(x), x\rangle \geq 0$ for any $x \in S_r^{R^n}(0)$. Then there exists, at least, one element $x_1 \in B_r^{R^n}(0)$ such that $f(x_1) = 0$. (For the proof see, [3], [15].)*

Well-known that the closed ball $B_r^{R^n}(0)$ is homeomorphic to the closed convex absorbing subset and these are replaceable. Below will be shown that this theorem is the generalization to the finite-dimension case of one theorem Cauchy in R^1. Moreover, this theorem is one of the answers to the posed question in the case when mapping is continuous, and the space is R^n.

[1] We would note that sufficient conditions for the image $f(G)$ will be body are provided in this article.

Theorem 2.2 *(see, [21]) Any two non-intersecting nonempty convex sets of linear space can separate if either one of these has a nonempty interior or they be subsets of finite-dimension space.*

Theorem 2.3 *(see, [22]) If M is the closed convex subset of the locally convex linear topological space X and $x_0 \notin M$ then there exists a nonzero linear continuous functional $x_0^* \in X^*$ that separates M and x_0, i.e. will be found constants $c > 0$, $c_0 > 0$ such that*

$$\sup \{ \langle x, x_0^* \rangle |\ x \in M \} \leq c_0 - c < c_0 = \langle x_0, x_0^* \rangle , \tag{2.1}$$

where $\langle \circ, \circ \rangle$ is the dual form for the pair (X, X^*).

Theorem 2.4 *(see, [35]) Let X be a real vector topological space, M be an open convex subset in X, N is a convex subset in X and $M \cap N = \varnothing$. Then there exists such linear continuous functional x_0^* on X, and also a real number $\alpha \in R^1$ that $\langle x, x_0^* \rangle \geq \alpha$ for $\forall x \in M$ and $\langle x, x_0^* \rangle < \alpha$ for $\forall x \in N$.*

We will reduce some concepts from the vector spaces theory (see, [5,6,11,13,25,35]) below will use. Let X be a vector space, $L \subset X$ is called a linear manifold in X (or affine subspace of X), if L is the certain shift of some subspace X_0 of X, i.e. there exists $x_0 \in X$ such that $L = X_0 + x_0$; $L \subset X$ is called a hyperplane in X if L is the maximal affine subspace of X, differently from X, i.e. there exists such nonzero continuous linear functional $x_0^* \in X^*$ and a number $\alpha \in R^1$ that $L \equiv \{ x \in X|\ \langle x, x_0^* \rangle = \alpha \}$. If $M \subset X$ is a convex set and the generated over M space is the affine subspace of X then the totality interior elements of M from X are called relatively interior elements of M and denoted by riM. A convex set K from the vector space X is called a convex cone a vertex on zero of X, if K is invariants relative to all homothety, i.e. for $\forall \alpha \in R_+^1$ the $x \in K \Longrightarrow \alpha x \in K$ fulfills, if in addition, $0 \in K \subset X$ then K is called a pointed cone.

Let R^n $(n \geq 1)$ $n-$dimension Euclid space, f be nonlinear mapping acting in R^n, $B_r^{R^n}(x_0) \subset R^n$ be a closed ball a center x_0 and a radius $r > 0$ and $S_r^{R^n}(x_0)$ be its boundary, denote by $D(f) \subseteq R^n$ the domain of f.

Theorem 2.5 *Let the subset $G \subset R^n$ belong to $D(f)$ and the following conditions are fulfilled: 1) $f(G)$ be a convex subset R^n; 2) there exists such subspace X of R^n the dimension k $(0 < k \leq n)$ that for any $x_0 \in S_1^{R^n}(0) \cap X$ there exists such $x_1 \in G \cap X$ that*

$$\{ \langle y, x_0 \rangle \mid y \in f(x_1) \cap X \} \cap R_+^1 \neq \varnothing, \quad R_+^1 = (0, \infty) , \tag{2.2}$$

holds (here the R_+^1 can be substituted by R_-^1). Then $0 \in f(G)$, i.e. $\exists \widehat{x} \in G$ satisfying $0 \in f(\widehat{x})$ (if f is single-valued then $f(\widehat{x}) = 0$).

Remark 2.1 We will formulate this result for R^1 in the following form.

Let the mapping f acts in $R^1$and the image $f(G)$ of some bounded subset $G \subset R^1$ is the connected subset of R^1. If there exist such points $a, b \in G$ $(a < b)$ that the inequations $(f(a) - C) \cdot (-1) \geq 0$ and $(f(b) - C) \cdot (1) \geq 0$ fulfilled then a point $C \in R^1$ belongs to $f(G)$ (and consequently there is such point $c \in G$ that $C \in f(c)$).

The proof follows from the connectivity of $f(G)$. This result is generalized to the results of Cauchy and Hadamard as one-dimension vector space is equivalent to R^1.

Before the proof of Theorem 2.5, we will prove some particular variants of the theorem, which have an independent interest. In beginning, we bring a simple variant.

Lemma 2.1 *Let for some $r > 0$ the image $f\left(B_r^{R^n}(0)\right)$ of the ball $B_r^{R^n}(0)$ is closed (or opened) convex set and is fulfilled the inequation $\{ \langle y, x \rangle \mid y \in f(x) \} \cap (0, \infty) \neq \varnothing$ for $\forall x \in S_r^{R^n}(0)$ then $0 \in f\left(B_r^{R^n}(0)\right)$, i.e. $\exists x_1 \in B_r^{R^n}(0)$ such that $0 \in f(x_1)$. (if f single-valued then $f(x_1) = 0$).*

Proof. We will provide from the inverse. Let $0 \notin f\left(B_r^{R^n}(0)\right) \equiv M$. According to condition M is closed or opened convex set in R^n then due to the separation theorem of convex subsets there exists a linear bounded functional $\overline{x} \in R^n$ separating of M and zero according to the conditions of the lemma. Since $B_r^{R^n}(0)$ is the absorbing set of R^n, therefore one can assume that functional $\overline{x}$ belongs to $S_r^{R^n}(0)$. Whence we get $\langle y, \overline{x}\rangle < 0$ for $\forall y \in f(\overline{x})$ that contradicts the condition of the lemma. Lemma proved.

Analogously is proved the following result.

Lemma 2.2 *Let $f\left(B_r^{R^n}(0)\right)$ be the convex set and on the sphere $S_r^{R^n}(0)$ fulfills the inequation $\{\langle y, x\rangle \mid y \in f(x)\} \cap (0, \infty) \neq \varnothing$ for $\forall x \in S_r^{R^n}(0)$ then $0 \in f\left(B_r^{R^n}(0)\right)$.*

Lemma 2.3 *Let $f\left(B_r^{R^n}(x_0)\right)$ be the convex set and there exists mapping g acting in R^n such that $g\left(S_r^{R^n}(x_0)\right)$ is the boundary of an absorbing subset of R^n. Then if for $\forall x \in S_r^{R^n}(x_0)$ the expression*

$$\{\langle y, z\rangle \mid \forall y \in f(x),\ \forall z \in g(x)\} \cap (0, \infty) \neq \varnothing \tag{2.3}$$

holds then $0 \in f\left(B_r^{R^n}(x_0)\right)$.

Remark 2.2 It is clear that if f is single-valued then expression (2.3) one can rewrite as

$$\{\langle f(x), z\rangle \mid\ \forall z \in g(x)\} \cap (0, \infty) \neq \varnothing.$$

Proof. (Lemma 2.3) As the above lemma, the proof we lead from the inverse. Let $0 \notin f\left(B_r^{R^n}(x_0)\right)$ then repeating of previous argue we get there is such point (a linear bounded functional) $z_0 \in S_1^{R^n}(0) \subset R^n$ that $\langle y, z_0\rangle \leq 0$ for $\forall y \in f(x)$ and $\forall x \in B_r^{R^n}(x_0)$ due to the condition on convexity of $f\left(B_r^{R^n}(x_0)\right)$ and Theorem 2.2 on separation, which contradicts the condition of the lemma. Consequently, Lemma proved.

Corollary 2.1 *Let the mapping f act in R^n and for a subspace X dimension $k \leq n$ the $f\left(B_r^{R^n}(x_0)\right) \cap X$ is a convex set and there exists mapping g acting in R^n such that $g\left(S_r^{R^n}(x_0) \cap X\right)$ is the boundary of an absorbing subset of X. Then if for $\forall x \in S_r^{R^n}(x_0) \cap X$ the expression (3) is fulfilled then $0 \in f\left(B_r^{R^n}(x_0)\right)$.*

The proof follows from the proof of the Lemma 2.3, more exactly, the above reasoning enough to conduct for the convex set $f\left(B_r^{R^n}(x_0)\right) \cap X$ (that is a subset of $f\left(B_r^{R^n}(x_0)\right)$) in the subspace X, since X is also the $k-$dimensional space.

Proof. (of Theorem 2.5) As seen from the above proofs of the Lemmas the selection of the subset from the domain of the examined mappings isn't essential since it one can select due to the preimage of the convexity of $f\left(B_r^{R^n}(x_0)\right) \cap X$, here is essential the convexity of the image of the subset from the domain.

So, if X is R^n then the proof follows from Lemmas 2.1 and 2.2. Therefore, let $X = R^k$, $1 \leq k < n$. Assume $0 \notin f(G)$ and the affine space generated over the convex set $f(G)$ is the hyperplane $L \subset R^n$ and show that under condition 2 of the theorem L couldn't be the hyperplane different of the subspace of R^n. The affine space L isn't a subspace of R^n according to the assumption $0 \notin L$ then there exists $x_0 \in f(G)$ and subspace X_0, $\dim X_0 < n$ such that $L = X_0 + x_0$. Consequently, X_0 is the subspace generated over $f(G) - x_0$. Since $0 \notin L$ there is such element z of $S_1^{R^n}(0)$ that $\langle y, z\rangle < 0$ for $\forall y \in L$, hence for $\forall y \in f(G)$ as $f(G) \subset L$, due to the separation theorems. But this is contradict to condition 2 of Theorem 2.5 as there is such $y_0 \in f(G)$ that $\langle y_0, z\rangle > 0$ according to condition 2. Whence, we get that L is the subspace of X_0 since $X_0 \subseteq X \subset R^n$. Moreover, $L = X_0 = X$ is the $k-$dimension subspace of R^n due to condition 2 as it fulfills for all $x \in S_1^{R^k}(0)$. Thus, we arrive at the case considered in Lemma 2.3. Then we get $0 \in f(G)$ due to this lemma.

From Theorem 2.5 follows the correctness of the following result.

Corollary 2.2 *Let the acting in R^n mapping f is such that the subset M of the image $\Re(f)$satisfies the following conditions:*

i) M is the convex set; ii) There exists such $k-$dimension subspace $X \subseteq R^n$ the $k: 1 \leq k \leq n$ that for $\forall z \in S_1^{R^n}(0) \cap X$ there exists $y \in M$ such that $\langle z, y\rangle > 0$ or ($\langle z, y\rangle < 0$). Then $0 \in \Re(f)$.

It is clear that one can the above inequation rewrite in the form

$$\left\{\langle z, y\rangle \mid \forall z \in S_1^{R^n}(0) \cap X,\ \exists x \in D(f),\ \text{for some } y \in M \cap f(x)\right\} \cap (0, \infty) \neq \varnothing. \tag{2.4}$$

Theorem 2.6 *(Fixed-point Theorem) Let f act in R^n and the ball $B_r^{R^n}(x_0)$ center on x_0 and the radius $r > 0$ belongs to the domain of f. Let $f\left(B_r^{R^n}(x_0)\right) \subseteq B_r^{R^n}(x_0)$ and for some subspace R^k, $k: 1 \leq k \leq n$, takes place $f\left(B_r^{R^n}(x_0) \cap R^k\right) \subset B_r^{R^n}(x_0) \cap R^k$. Let f_1 be the mapping defined in the form $f_1(x) \equiv I(x) - f(x)$ for $\forall x \in B_r^{R^n}(x_0)$. If the image $f_1\left(B_r^{R^n}(x_0)\right)$ is the convex subset then the mapping f has a fixed point in $B_r^{R^n}(x_0)$.*

The proof immediately follows from the above-mentioned results, therefore here it doesn't will be provided. It is enough to note that the fulfillment of the conditions of Theorem 2.5 for the mapping f_1 on the ball $B_r^{R^n}(x_0) \cap R^k$ implies fromTheorem 2.6 due to its conditions.

Remark 2.3 Theorem 2.6 is the Fixed-point Theorem, which doesn't such its smoothness, some compactness, single-value, or multi-value conditions onto mapping f that is usually assumed. Consequently, this theorem can be considered as the generalization of such type theorems in the above sense.

Now we provide an example that shows the rigor of inequation in condition 2 of the above results is essential. For simplicity assume $n = 2$, i.e. mapping f act in $R^2$and the image of ball $B_r^{R^2}(0) \subset D(f)$ ($r > 0$) is

$$f\left(B_r^{R^2}(0)\right) = B_r^{R^2}(0) \cap \left\{x = (x_1, x_2) \in R^2 \mid,\ x_2 > 0\right\} \cup \left[\left(\frac{r}{2}, 0\right), (r, 0)\right]. \tag{2.5}$$

It isn't difficult to see that for $\forall z \in S_1^{R^2}(0)$ there exists an $y \in f\left(B_r^{R^2}(0)\right)$ such that $\langle z, y\rangle \geq 0$ but $0 \notin f\left(B_r^{R^2}(0)\right)$. In particular, the mapping $f: B_r^{R^2}(0) \longrightarrow R^2$ of such type one can define in the following way

$$f(x_1, x_2) = \begin{cases} (y_1, y_2) = (x_1, -x_2), & -r \leq x_1 \leq r,\ -r \leq x_2 < 0 \\ (y_1, y_2) = (x_1, x_2), & -r \leq x_1 \leq r,\ 0 < x_2 \leq r \\ (y_1, y_2) = \left(\frac{x_1}{2} + r, 0\right), & -r \leq x_1 \leq 0,\ x_2 = 0 \\ (y_1, y_2) = \left(\frac{r}{2} + x_1, 0\right), & 0 < x_1 < \frac{r}{2},\ x_2 = 0 \\ (y_1, y_2) = (x_1, 0), & \frac{r}{2} \leq x_1 \leq r,\ x_2 = 0 \end{cases}. \tag{2.6}$$

The essentialness of the condition the image $f(G)$, (for $G \subseteq D(f)$) of the examined mapping is the convex set is obviously. Indeed, if assume that in the above example we have $f\left(B_r^{R^2}(0)\right) = B_r^{R^2}(0) \setminus B_{r/2}^{R^2}(0)$ and $f(0) = k$ then condition 2 satisfies but $B_r^{R^2}(0) \setminus B_{r/2}^{R^2}(0) \cup \{k\}$ isn't a convex set and obviously $0 \notin B_r^{R^2}(0) \setminus B_{r/2}^{R^2}(0) \cup \{k\}$.

It is clear that Lemma 2.1 is the generalization of the "acute-angle" lemma (see, e.g. [3, 12, 15, 16, 40, 43, 44]) in the above-mentioned sense. Consequently, in the above-mentioned sense, Theorem 2.6 is the generalization of the Brauwer fixed-point theorem (see, e.g. [3, 6, 15, 32, 34]) (since it is equivalent to "acute-angle" lemma), and also the Kakutani fixed-point theorem (see, e.g. [13, 16, 19, 26]) for the multi-valued case.

3 Generalization of Theorem 2.5 to infinite-dimension cases and their corollaries

In this section, we will generalize the results of the previous section to more general spaces. The possibility of such generalizations is due to the convexity concept being independent of the dimension and the topology of the space. In the beginning, we will generalize the results to the case of the Hausdorff VTS. In this section, we will generalize the results of the previous section to the general spaces.

Let $f : D(f) \subseteq X \longrightarrow Y$ be a nonlinear mapping.

Let X and Y be locally convex VTS ($LVTS$), X^* and Y^* be their dual spaces, respectively. Denote by ∂U (∂U^*) the boundary of a convex closed bounded absorbing set U (U^*) in the appropriate space (see, [33])[2]. In particular, if X be a Banach space then ball $B_r^X(0)$ is the absorbing set in X, and the sphere $S_r^X(0)$ is the relevant boundary.

Theorem 3.1 *Let $f : D(f) \subseteq X \longrightarrow Y$ be some mapping and a set $G \subseteq D(f)$ be such that the image $f(G)$ one of the following conditions satisfies:*

i) $f(G)$ is the convex set with a nonempty interior in Y; ii) $f(G)$ is the open (closed) convex set in Y. Then if for any linear continuous functional $y^* \in \partial U^* \subset Y^*$ there exists $x \in G$ such that the following inequation

$\{\langle y^*, y\rangle \mid y \in f(x)\} \cap (0, \infty) \neq \varnothing$ *in the case i);* $\{\langle y^*, y\rangle \mid y \in f(x)\} \cap [0, \infty) \neq \varnothing$ *in the case ii) holds then* $0 \in f(G)$.

Proof. The proof we bring from the inverse. Let $0 \notin f(G)$ then there exists such linear continuous functional $y_0^* \in \partial U^*$ that $\langle y_0^*, y\rangle \leq 0$ for any $y \in f(G)$ in case i), due to convexity of $f(G)$ according to the separation theorems of the convex sets in the $LVTS$ (see, e.g. [15–17, 19]). We get, by the analogously reasoning to the proof of the theorem in the previous section, that in case ii) there exists such linear continuous functional $y_0^* \in \partial U^*$ that $\langle y_0^*, y\rangle < 0$ for any $y \in f(G)$ according to the separation theorems in the$LVTS$. But leading the analogous reasoning to the above-mentioned we arrive at results, which contradicted the conditions of this theorem. Thus, we obtain that must be only the inclusion of $0 \in f(G)$ due to the obtained contradiction.

It should be noted that such type result is correct and also for the Hausdorff VTS.

Consider the case when X and Y be vector spaces (VS) and $f : D(f) \subseteq X \longrightarrow Y$ be some mapping.

Theorem 3.2 *Let $f : D(f) \subseteq X \longrightarrow Y$ be a mapping, the image $f(G)$ of set $G \subseteq D(f)$ be a convex subset in Y, and there exist such subspace Y_1 that for any subspace Y_0 with $co\dim_{Y_1} Y_0 = 1$ the following expressions fulfill*

$$f(G) \cap (Y_1)^+_{Y_0} \neq \varnothing \;\&\; f(G) \cap (Y_1)^-_{Y_0} \neq \varnothing. \tag{3.1}$$

Then zero of the space Y ($0 \in Y$) belongs to $f(G) \subseteq Y$, i.e. there exists such $x_0 \in G$ that $0 \in f(x_0)$.

Proof. Let $0 \notin f(G)$. In conditions of the Theorem, there exist the subspace $Y_2 \subseteq Y_1$ relative to which the set $f(G)$ is the convex set with a nonempty $C-$interior according to the convexity of $f(G)$ (see, [22]). Moreover is sufficient to choose the affine space generated over $f(G)$ that will be a subspace of Y due to conditions of the Theorem on the $f(G)$.[3]

[2] This is chosen for simplicity. In reality, one can choose the boundary of the closed balanced absorbing set.

[3] Notes. Any hyperplane L of vector space Y is equivalent to some subspace Y_0 of Y with $co\dim_Y Y_0 = 1$.
Any hyperplane L of Y separates this space into two half-space, which can be denoted as Y_L^+ and Y_L^-.
An element y_0 of the subset $U \subset Y$ called $C-$interior element if for $\forall y \in U$ there exists $\varepsilon > 0$ such that for all $\delta : 0 < |\delta| < \varepsilon$ the inclusion $y_0 + \delta y \in U$ holds.

Consequently, without loss of generality can be accounted that $Y_2 \equiv Y_1$. For simplicity, in the beginning, assume $Y_1 = Y$. Then there exists such subspace $Y_3 \subset Y$ that relative to which $f(G)$ belongs to one of half-spaces $Y^+_{Y_3}$ or $Y^-_{Y_3}$ (see, [23,24]). Moreover, $f(G) \cap Y_3 = \varnothing$ according to the separation theorems of the convex subsets in VS (see, [21]). But this contradicts condition (3.1), which shows the correctness of the state of the theorem in the case when $Y_1 = Y$.

Let now $Y_1 \subset Y$ and denote the mapping $f_0(x) = f(x) \cap Y_1$ for $\forall x \in G$ then $f_0(G) = f(G) \cap Y_1$. Clear that $f_0(G)$ is a convex set in Y_1. Consequently, one can repeat the above reasoning for the mapping $f_0(x)$ and the space Y_1 as independent VS from Y that again will give the same result as the previous, which contradicts condition (3.1).

Thus we obtain the correctness of the state of the Theorem 3.2.

From theorems mentioned above flow out the correctness of the following fixed-point theorem.

Corollary 3.1 *(Fixed-Point Theorem) Let the mapping f acts into the space X that is (a) VS or (b) $LVTS$ and the convex subset $G \subseteq D(f)$. Let $f : G \longrightarrow G$ and denote by $f_1 : G \longrightarrow G$ the mapping $f_1 = Id - f$ ($f_1(x) = Id\, x - f(x)$ for $\forall x \in G$). Assume the mapping f_1 on the set G satisfies condition (3.1), in case (a); the condition of the Theorem 3.1, in case (b). Then the mapping f in the set G has a fixed point, i.e. there exists $x_0 \in G$ such that $x_0 \in f(x_0)$ (if f is a single-value mapping then $f(x_0) = x_0$).*

The proof is obvious. If to compare this result with the Schauder and Fan-Kakutani fixed-point theorems (see, e.g. [13,17,22,23,39]), then can be seen this result is generalized to these theorems, in the above-mentioned sense.

It isn't difficult to see the correctness of the following result.

Corollary 3.2 *Let the mapping f acting from $LVTS$ X to $LVTS$ Y on some subset $G \subseteq D(f)$ satisfies the following condition: there exists such subspace Y_0 of Y ($Y_0 \subseteq Y$) that $f(G) \cap Y_0$ is a convex set, moreover, either (a) is open (or closed), or with the nonempty interior with respect to Y_0. Then if for each $y^* \in \partial U^* \cap Y_0^* \subseteq Y^*$ there exists such element $x \in G$ that the expression*

$$(a)\ \{\langle y, y^*\rangle \mid y \in f(x) \cap Y_0\} \cap [0, \infty) \neq \varnothing;$$

$$(b)\ \{\langle y, y^*\rangle \mid y \in f(x) \cap Y_0\} \cap (0, \infty) \neq \varnothing$$

holds. Then $0 \in Y$ belongs to $f(G)$, i.e. $0 \in f(G)$.

Now we will reduce examples showing the essentialness of the conditions of the above-proved theorems. Let X be a reflexive Banach space and $Y = X^*$, i.e. dual space to the X. Assume $f : D(f) \subseteq X \longrightarrow X^*$ and $B^X_{r_0}(x_0) \subseteq D(f)$, moreover, $f\left(B^X_{r_0}(x_0)\right) = B^{X^*}_{r_0}(x^*_0) \subset X^*$, where $r_0 > 0$ and the centers $x_0 \in X, x^*_0 \in X^*$ of these balls such that $\|x_0\|_X, \|x^*_0\|_{X^*} > r_0$. Let the mapping f is such as the duality mapping between the dual spaces, more exactly, for $\forall x \in B^X_{r_0}(x_0)$ fulfill the ensue expressions $f(x) = x^* \in B^{X^*}_{r_0}(x^*_0)$ and $\langle f(x), x\rangle = \langle x^*, x\rangle = \|x\|_X \cdot \|x^*\|_{X^*} > 0$. Condition 1 is fulfilled but condition 2 isn't fulfilled since there exists such $\widehat{x} \in S^X_1(0)$ that no exists $\widetilde{x} \in B^X_{r_0}(x_0)$ satisfying the inequality $\langle f(\widetilde{x}), \widehat{x}\rangle \geq 0$ consequently, the claims of these theorems don't fulfill. The essentialness of the convexity of the image is obvious.

We will prove one result (for simplicity, in the case of the single-valued mappings) in the case when the space Y is $LVTS$, which is sufficient, in some sense for fulfilling the conditions of Theorem 3.1.

Proposition 3.1 *Let X, Y be $LVTS$, and $f : D(f) \subseteq X \longrightarrow Y$ is the single-valued mapping. Let the image $f(G)$ of some subset $G \subseteq D(f)$ is connected open or closed body in Y. Then for each fixed element $y \in int\ f(G)$, there exists such subset of G on which for the mapping $f_1(x) = f(x) - y$ the conditions of Theorem 3.1 are fulfilled.*

Proof. Indeed, for each point $y \in f(G)$ there exists an open or closed convex neighborhood $V(y) \subseteq f(G)$ that contains this point accordingly to the conditions of the proposition. Then the mapping f_1 is enough to consider on the subset G_1 that is the preimage of $V(y)$, i.e. $f^{-1}(V(y)) \equiv G_1 \subseteq G$, consequently, $f_1 : G_1 \longrightarrow V(y)$.

We won't consider the more general cases.

4 On mappings acted in reflexive Banach spaces

In this section, we will investigate mappings acted from one Banach space to another. It is clear, the results obtained in Section 3 true also in this case. We will study this case separately since well-known that the geometry of the reflexive Banach spaces was studied sufficiently complete that allows proving more exact results. Therefore, these results are more applicable for studying the detail of various problems.

So, we assume spaces X and Y are the reflexive Banach spaces with the strongly convex norms jointly with their dual spaces in the whole of this section (see, e.g. [7], [15], [22], [37], [38], etc.). As well-known, each reflexive Banach space can be renormalized in such a way this space and its dual will the strongly convex spaces (see, [37], and also [7], [15]). Consequently, in what follows, without loss of generality, we will account that all of the examined reflexive Banach spaces are strongly convex spaces.

Let X be strongly convex reflexive Banach space jointly with its dual space X^*. For simplicity in the beginning assume $Y = X^*$ and the mapping f acts from X to X^*. Thus the main result of this section will be formulated as follows.

Theorem 4.1 *Let $f : D(f) \subseteq X \longrightarrow X^*$ be some mapping and $G \subseteq D(f)$. Assume $f(G)$ be a convex subset in X^* and there exists such subspace $X_0^* \subseteq X^*$ that belongs to the affine space $X^*_{f(G)}$ generating over $f(G)$ and either $co\dim_{X^*} X_0^* \geq 1$ or $0 \in X^*_{f(G)}$; Moreover, there exists such $X_1 \subseteq X$ that $X_0^* \subseteq X_1^*$, $co\dim_{X_1} X_0 \geq 0$ and for $\forall x_0 \in S_1^X(0) \cap X_1$ the inequation*

$$\{\langle x^*, x_0\rangle \mid \exists x \in G \ \&\ x^* \in f(x) \cap X_1^*\} \cap (0, \infty) \neq \varnothing \tag{4.1}$$

holds. Then $0 \in f(G)$, i.e. $\exists x_1 \in G \Longrightarrow 0 \in f(x_1)$ (if the mapping f is single-value then $f(x_1) = 0$).

For the proof of this theorem previously need to prove some auxiliary results are necessary. We start with a simple variant of Theorem 4.1.

Lemma 4.1 *Let $f : D(f) \subseteq X \longrightarrow X^*$ be a mapping and $G \subseteq D(f)$. Assume $f(G)$ be a convex subset in X^* and there exists such subspace $X_0^* \subseteq X^*$ that belongs to the affine space $X^*_{f(G)}$ generated over $f(G)$ and $co\dim_{X^*} X_0^* \geq 1$. Then if for $\forall x_0 \in S_1^X(0)$ there exists $x \in G$ such that $\{\langle f(x), x_0\rangle\} \cap (0, \infty) \neq \varnothing$ then $0 \in f(G)$.*[4]

Proof. We would note if here the condition of the case (i) of Theorem 3.1 is fulfilled then the proof ensues from Theorem 3.1. Assuming this condition isn't fulfilled then we will use the following result that will be proved later.

[4] In what follow we will denote for briefness by $\{\langle f(x), x_0\rangle\}$ the set $\{\langle y, x_0\rangle \mid y \in f(x)\}$, where $(y, x_0) \in (X^*, X)$.

Lemma 4.2 *Let $f : D(f) \subseteq X \longrightarrow Y$ be a mapping and $G \subseteq D(f) \subseteq X$, where the spaces X, Y be $LVTS$. Let $U^* \subset Y^*$ be a closed bounded balanced absorbing set in Y^* with the boundary ∂U^*. Then if for $\forall y^* \in \partial U^*$ there exists $x \in G$ such that $\{\langle f(x), y^* \rangle\} \cap (0, \infty) \neq \varnothing$ fulfills then the affine space $Y_{f(G)}$ generated over $f(G)$, at least, is everywhere dense affine subspace in the space Y.*

Proof. Continuation of the proof of Lemma 4.1. According to Lemma 4.2 under the condition of Lemma 4.1 $X^*_{f(G)}$ is, at least, everywhere dense linear subspace in X^*. Let X_1^* be a subspace of X^* that belongs to $X^*_{f(G)}$. Then $f(G) \cap X_1^*$ is the convex set with nonempty interiors in X_1^* (see, e.g. [5], [17], [21], [22]). As X is the reflexive Banach space then X_1^* also is the reflexive space. Obviously that under the conditions of Lemma 4.1 on X_1^* for $\forall y^* \in \partial U^* \cap X_1^*$ there exists $x \in G$ such that $\{\langle f(x) \cap X_1^*, y^* \rangle\} \cap (0, \infty) \neq \varnothing$ fulfills (for the proof see, [32]). Thus, we get that with respect to X_1^* for the examined mapping all conditions of the Theorem 3.1 are fulfilled, therefore using this theorem the correctness of the claim of Lemma 4.1 is obtained.

It remains to prove of the Lemma 4.2.

Proof. (of the Lemma 4.2) The proof we lead from the inverse. Let $Y_{f(G)}$ isn't the everywhere dense affine subspace in the space Y. Denote the closure of $Y_{f(G)}$ by $Y_1 \equiv \overline{Y_{f(G)}}^{Y}$. According to the assumption $Y_1 \subset Y$ and $Y_1 \neq Y$,and also that is the closed convex affine subspace. Then there exists $y^* \in Y^*$ such that $\langle y^*, Y_1 \rangle < 0$, which contradicts the condition of Lemma 4.2.

Now we provide one result using that one can to prove the Lemma 4.1 in another way.

Proposition 4.1 *Let X be a strongly convex reflexive Banach space jointly with its dual space X^*, and $\Omega \subset X$ be a bounded convex set. Then if each one-dimension subspace L of X intersects Ω, i.e. $L \cap \Omega \neq \varnothing$, then either $0 \in \Omega$ or Ω is such a convex body that zero of X belongs to the closure of Ω, i.e. $0 \in \overline{\Omega}$.*

It is clear that under the condition of this result the affine space generated over Ω contains a linear subspace of X.

Lemma 4.3 *Let the conditions of the Theorem 4.1 be fulfilled with the following distinction, in which the expression (4.1) is fulfilled in the following form: Let $X_1 \subset X$ be a subspace and $co \dim_{X_1} X_{f(G)} \geq 0$. Then if for $\forall x_0 \in S_1^X(0) \cap X_1$ there exists an $x \in G$ such that $\{\langle f(x) \cap X_1^*, x_0 \rangle\} \cap (0, \infty) \neq \varnothing$ holds then $0 \in f(G)$.*

The proof of this lemma analogously to the proof of the Lemma 4.1, therefore we don't provide it.

Proof. (of the Theorem 4.1) This proof follows from lemmas 4.1 and 4.3. Indeed Lemma 4.1 shows correctness of the Theorem 4.1 from one side, and Lemma 4.3 shows correctness of the Theorem 4.1 from in another side. Consequently, the Theorem 4.1 complete proved.

The correctness of the following statements immediately ensues from the above-mentioned results.

Corollary 4.1 *(Fixed-Point Theorem) Let f be a mapping acting in the reflexive Banach space X and $B_r^X(x_0) \subseteq D(f) \subseteq X$ be a closed ball. Assume f map $B_r^X(x_0)$ into itself, where $r > 0$ is a number. Let f_1 be a mapping defined as $f_1(x) \equiv x - f(x)$ for $\forall x \in B_r^X(x_0)$. Then if the image $f_1\left(B_r^X(x_0)\right)$ of the ball $B_r^X(x_0)$ is a convex set either open (closed) or $f_1\left(B_r^X(x_0)\right) \subset int\ B_r^X(x_0)$ and $int\ f_1\left(B_r^X(x_0)\right) \neq \varnothing$ then the mapping f has a fixed point in the ball $B_r^X(x_0)$.*

Corollary 4.2 *(Fixed-Point Theorem) Let f be a mapping acting in the reflexive Banach space X and $B_r^X(x_0) \subseteq D(f) \subseteq X$ be a closed ball. Assume f map $B_r^X(x_0)$ into itself, where $r > 0$ is a number. Let f_1 be a mapping defined as $f_1(x) \equiv x - f(x)$ for $\forall x \in B_r^X(x_0)$. Then if the image $f_1\left(B_r^X(x_0)\right)$ of the ball $B_r^X(x_0)$ is a convex set and $f_1\left(B_r^X(x_0)\right) \subset int\, B_r^X(x_0)$ moreover, the affine space generated over $f_1\left(B_r^X(x_0)\right)$ contains some subspace of X then the mapping f has a fixed point in the ball $B_r^X(x_0)$.*

Remark 4.1 Since each closed convex body is homeomorphic to the closed ball, then for the closed convex body of the Banach space we can analogously prove the above-cited corollaries.

5 On solvability of the nonlinear equations and inclusions

In the beginning, we will provide the results, which in some sense are corollaries of results from the above sections, therefore we will lead them in the simplified variants.

Let X and Y be $LVTS$, and f be a mapping acting from X to Y.

Theorem 5.1 *Let $y \in Y$ be an element. Let there exist such subset $G \subseteq D(f) \subseteq X$ that $f(G)$ is a convex subset of Y satisfying the condition i) or ii) of Theorem 3.1. Then if for $\forall y^* \in \partial U^* \subset Y^*$ there exists such $x \in G$ that fulfills the corresponding inequation*

$$i)\ \langle f(x) - y, y^* \rangle \cap (0, \infty); \quad or \quad \langle f(x) - y, y^* \rangle \cap [0, \infty)$$

then $y \in f(G)$, i.e. $\exists x_1 \in G \Longrightarrow y \in f(x_1)$ ($f(x_1) = y$).

For the proof is sufficient to note that if by f_1 to denote the mapping $f_1(\cdot) \equiv f(\cdot) - y$, then it isn't difficult to see that mapping $f_1 : G \longrightarrow Y$ satisfies on subset G all conditions of the Theorem 3.1, therefore, and its claim fulfills also.

Remark 5.1 We should be noted the condition "for $\forall y^* \in \partial U^* \subset Y^*$ there exists such $x \in G$ that" is a relation between ∂U^* and a subset G_0 of G, therefore one can denote it as a mapping g acting from X to Y^* such that for each $y^* \in \partial U^* \Longrightarrow g(y^*) = x \in G_0$, moreover, $g(\partial U^*) = G_0$, or the inverse $g^{-1}(G_0) = \partial U^*$. Unlike the general case for the concrete problem is necessary to seek the above-mentioned mapping. In what follows in this section, we will use mappings of such type.

Corollary 5.1 *Let the mapping f act from Banach space X to the Banach space Y, where (iii) $Y \equiv X^*$ or (iv) $Y \equiv X \equiv X^{**}$. Then the equation (the inclusion) $f(x) = y$ ($f(x) \ni y$) is solvable for any $y \in Y$ if fulfilling conditions:*

1) There exists such subset $G \subseteq D(f) \subseteq X$ that $f(G)$ is a convex subset with a nonempty interior in Y;

2) For $\forall x \in \partial U^* \subset Y^*$ takes place the expression (iii) $\langle f(x) - y, x \rangle \cap (0, \infty) \neq \varnothing$; (iv) $\langle f(x) - y, J(x) \rangle \cap (0, \infty) \neq \varnothing$, where J is the duality operator: $J : X \longrightarrow X^*$.

Now we will prove certain results that generalize the well-known theorems of articles [15], [32], [40], etc. Here the generalization is understood in the sense that on a subset from the domain of an examined mapping that is fulfilled the needed conditions can be such

(a) a mapping is without any smoothness conditions;

(b) a subset from the domain of mapping doesn't have any conditions.

So, let Y be a semi-reflexive $LVTS$ (see, e.g. [5]), S be a weakly complete "reflexive" $pn-$space (see, [32], [44], [45] and references in these), X be a separable VTS, moreover $X \subset S$ and is everywhere dense in S (or Y and S be semi-reflexive $LVTS$), X_m be an $m-$dimension linear subspace of X, generated over first m elements from total systems of X.

3) *Let $f : S \longrightarrow Y$ be a bounded* [5] *and weakly closed mapping.*[6] Let $G \subset X$ be a such neighborhood of zero of space X that each subset $G \cap X_m = G_m$ $(m = 1, 2, ..)$ is the closed neighborhood of zero of X and $f(G_m)$ is the closed convex subset in Y.

Theorem 5.2 *Let the condition 3) is fulfilled and $A : X \longrightarrow Y^*$ be a linear continuous operator. Then each $y \in Y$ belongs to the subset $f(G) - \ker A^*$ in other words $y + z \in f(G)$ i.e. there exists $x_0 \in G$ such that $f(x_0) = y + z$ if operator $A : X \longrightarrow Y^*$ such that the inequation*

$$\langle f(x) - y, Ax \rangle \geq 0, \quad x \in \partial G_m,\ m = 1, 2, ... \tag{5.1}$$

holds for each m, where $z \in \ker A^* \subset Y$.

For the proof sufficiently noted it is led to applying the Galerkin method, which usually is applied in such type cases (see, [15], [40], [45], etc), however here instead of "acute-angle lemma" is used its generalization, which has been proved in Section 2.

Remark 5.2 In the multivalued case of the mapping the weakly closedness of the mapping be understood as: if a sequence $\{x_\alpha\}$ from $D(f)$ weakly converges to $x \in D(f)$ and correspondent sequence $\{f(x_\alpha)\}$ weakly converges to the subset $¥ \subset Y$ then $¥ = f(x)$. Consequently, the claim of Theorem 5.2, in this case, will be as each $y \in Y$ belongs to the subset $f(G) - \ker A^*$ in other words $y + z \in f(G)$ i.e. there exist $x_0 \in G$ such that $y + z \in f(x_0)$ if operator $A : X \longrightarrow Y^*$ such that the inequation (5.1) holds for each m, where $z \in \ker A^* \subset Y$.

Now consider equations (also inclusions) in the Banach space.

Let Y and S be spaces such as above, X be a separable reflexive Banach space that is everywhere dense in S, and $f : S \longrightarrow Y$ is the bounded mapping.

Consider the following conditions.

c) Let there exists $r_0 > 0$ such that for any closed ball $B_r^X(0) \subset X$ $(r > r_0 > 0)$ there exists such neighborhood G_r of $0 \in S$ that $B_r^X(0) \subseteq G_r$, $G_r \cap B_{r_1}^X(0) \subseteq B_{r_1}^X(0)$ and $f(G_r)$ be open (or closed) convex set in Y, where $r_1(r) \geq r$ (r_1 dependent only on r);

d) There exists such linear bounded operator $A : X \longrightarrow Y^*$ that the following relation

$$\langle f(x), Ax \rangle \geq \varphi([x]_S)\ [x]_S \ \ \&\ \varphi([x]_S) \nearrow \infty \ \text{ at } \ [x]_S \nearrow \infty,$$

holds, where $\varphi : R_+^1 \longrightarrow R^1$ is the continuous function and $[x]_S$ is the $p-$norm on S;

e) There exists such nonlinear operator $g : X \subseteq S \longrightarrow Y^*$ that $\frac{g(x)}{\|g(x)\|_{Y^*}} = \widehat{g}(x)$, and also $\widehat{g}(X) = S_1^{Y^*}(0)$, moreover, on X the following relation

$$\langle f(x), \widehat{g}(x) \rangle \geq \varphi([x]_S)\ [x]_S \ \ \&\ \varphi([x]_S) \nearrow \infty \ \text{ at } \ [x]_S \nearrow \infty,$$

holds, where φ and $[x]_S$ are same as in condition d).

(In the case $f : X \longrightarrow Y$ in the above relation instead $p-$norm is necessary to take $\|x\|_X$.)

Theorem 5.3 *Let conditions c) and d) fulfill. Then for any $y \in Y$ satisfying condition*

$$\sup \left\{ \frac{\langle y, Ax \rangle}{[x]_S} \mid x \in X \right\} < \infty \tag{5.2}$$

there exist such x_0 and $y_0 \in \ker A^* \cap Y$ that $f(x_0) = y + y_0$.

[5] Let X and Y be $LVTS$ and f be a mapping acting from X to Y. A mapping f is called bounded mapping if the image of each bounded subset of X is a bounded subset of Y.

[6] Let X and Y be $LVTS$ and f be a mapping acting from X to Y. A mapping f be called weakly closed mapping if a sequence $\{x_\alpha\}$ from $D(f)$ weakly converges to $x \in D(f)$ and sequence $\{f(x_\alpha)\}$ converges to $y \in Y$ then $y = f(x)$.

Proof. For the proof is sufficient to note that according to condition d) there exists ball $B_{r_0}^X(0)$ such that on $S_{r_0}^X(0)$ inequation $|\langle y, Ax\rangle| \leq \varphi([x]_S)[x]_S$ holds due to (5.2). Therefore, one can use condition c). According to condition c) there exist such neighborhood G_{r_0} of 0 and ball $B_{r_{01}}^X(0)$ that on $S_{r_{01}}^X(0)$ inequation $\langle f(x) - y, Ax\rangle \geq 0$ holds. Consequently, conditions of Theorem 5.1 satisfy then its claim satisfies also.

Theorem 5.4 *Let conditions c) and e) fulfill. Then for any $y \in Y$ satisfying condition*

$$\sup\left\{\frac{\langle y, \widehat{g}(x)\rangle}{[x]_S} \mid x \in X\right\} < \infty \tag{5.3}$$

there exists such x_0 that $f(x_0) = y$.

The proof of this theorem is analogous to the above-proved.

Remark 5.3 We should be noted one can formulate and prove the theorems of the previous type also in the case, when Y, X are Banach spaces moreover, Y is reflexive space, and $f : X \longrightarrow Y$ is the bounded mapping. But theirs we don't will adduce here, since the formulate of these theorems, and also their proofs are analogous to the above-mentioned theorems.

We provide now one result for the equation with the odd operator.

Theorem 5.5 *Let f acts from reflexive Banach space X to its dual space X^* and is the single-value odd operator. Assume there exists such closed balanced convex neighborhood $G \subset X$ that $f(G)$ is a convex closed subset of X^*. Then there exists such subset $G_1 \subseteq G$ and a subspace $X_0^* \subseteq X^*$ that for each $x^* \in X_0^*$ satisfying of inequation $\|x^*\|_{X^*} \leq \|f(x)\|_{X^*}$, $\forall x \in G_1$ the equation $f(x) = x^*$ is solvable in G.*

Proof. At the condition of the theorem $0 \in f(G)$. Consequently, an affine space X_0^* generated over $f(G)$ is the subspace of X^*, and $f(G)$ is the convex closed body in the subspace X_0^* due to the convexity of the image $f(G)$. Whence, without loss of generality, one can suppose that $X_0^* \equiv X^*$. Since X and X^* are the reflexive Banach spaces, one can assume these spaces are strongly convex, and the dual mapping J: $X \underset{J^{-1}}{\overset{J}{\longleftrightarrow}} X^*$ is the monotone bijection (one-to-one, see, e.g. [31], [40]).

As $f(G)$ is the closed convex set there exists a subset G_1 of G for which takes place the relations $f(G_1) = \partial f(G)$, $f^{-1}(\partial f(G)) \supseteq G_1$. Another hand using the dual mapping J one can determine a subset $G_0 = J^{-1}(\partial f(G))$. Then there exists such one-to-one mapping f_0 acting in X that $f_0 : G_1 \longleftrightarrow G_0$ and for each $x \in G_1$ fulfill the equation

$$\langle f(x), f_0(x)\rangle = \left\langle f(x), J^{-1}(f(x))\right\rangle = \|f(x)\|_{X^*}\left\|J^{-1}(f(x))\right\|_X .$$

The set $\partial f(G)$ is the boundary of the closed convex absorbing subset of X^*, therefore, the subset $G_0 = J^{-1}(\partial f(G))$ also will be a boundary of the closed absorbing neighborhood of zero of X according to the condition of the theorem. Then using of Theorem 5.1, we obtain the solvability of the equation $f(x) = y$ for any $y \in X^*$ satisfying the inequality $\left\langle f(x) - y, J^{-1}(f(x))\right\rangle \geq 0$. Consequently, the equation $f(x) = y$ is solvable for any $y \in X^*$ satisfying the inequality

$$\left\langle y, J^{-1}(f(x))\right\rangle \leq \left\langle f(x), J^{-1}(f(x))\right\rangle = \|f(x)\|_{X^*}\left\|J^{-1}(f(x))\right\|_X$$

due to the above reasoning.

Note 5.1 As the solvability of the inclusions in the Banach space can study an analogous way, using the appropriate theorems, therefore we won't provide results of such type.

6 Some sufficient conditions for the convexity of the image of mappings

Let X, Y be the VTS, and f be a mapping acting from X to Y.

Lemma 6.1 *Let the mapping f, acting from $LVTS\ X$ to $LVTS\ Y$, some subsets from $D(f) \subseteq X$ translated to connected subsets of Y. Let there exist such subsets $G_1 \subset G \subseteq D(f)$ that $f(G_1)$, $f(G)$ be connected subsets with the nonempty interiors of Y and such convex subset M of Y that inequations $f(G_1) \subseteq M \subseteq f(G)$ hold. Then there exists such subset $G_2 \subset G$ that $f(G_2) = M$ (in the case when the mapping f is multi-valued then one can determine the mapping f_1 by restricting the mapping f as $f_1(x) = f(x) \cap M$ for each $x \in G_2$).*

The proof is obvious.

Proposition 6.1 *Let the mapping f, acting from $LVTS\ X$ to $LVTS\ Y$, be locally homeomorphic from $D(f) \subseteq X$ on $\Re(f) \subseteq Y$. Then if the $D(f)$ contains a subset with a nonempty interior then the mapping f fulfills the condition of Lemma 6.1.*

Lemma 6.2 *Let X be reflexive Banach space, and f acting from X to its dual space X^*, be a potential mapping with a convex potential F (i.e. the Gateaux derivative is $\partial F = f$). Then the image $f(G)$ be a convex subset of X^* of each convex subset G, generated by the functional F, where a subset G, generated by potential is understood as $G = \{x \in X \mid F(x) \leq C, \}$, and C be a constant.*

Proof. The mapping f is the Gateaux derivative of a differentiable convex functional F, i.e. $\partial F(x) = f(x)$ for each $x \in int\ D(f)$ according to the condition of the lemma. Well-known, the dual functional F^* to F also is convex functional (see, [12], [16], [19], [25], [26], [27]). Moreover, the following relations

$$(i)\ \forall x \in domF \Longrightarrow f(x) \in domF^*; \quad (ii)\ \forall x^* \in \partial F(x) \Longleftrightarrow x \in \partial F^*(x^*) \tag{6.1}$$

are fulfilled, where ∂F is the subdifferential of the functional F.

In this case, the subdifferential ∂F is the Gateaux derivative of F. Then, if the subset $G \subset int\ dom\ F$ is the convex closed subset in X then the corresponding subset $G^* \subset int\ domF^*$ will be convex closed subset in X^* (see, [16], [19]). Whence follows, if $G \subset X$ is the convex subset then the image $f(G) = G^* \subset X^*$ is also the convex subset.

Whence implies the correctness of the following result.

Corollary 6.1 *Let X be reflexive Banach space, and f acts from X to its dual space X^*, then if the mapping f is the subdifferential of a convex functional then the claim of the Lemma 6.2 is true.*

Now we will lead one result for the case when X and Y are the spaces of functions, and the mappings are concrete chosen.

Lemma 6.3 *Let a bounded mapping f acting in R^1 the connected subset translate to a connected subset and satisfy the inequalities*

$$f(\xi) \cdot \xi \geq c_0 |\xi|^{p+1} - \widehat{c}_0; \quad |f(\xi)| \leq c_1 |\xi|^p + \widehat{c}_1, \quad \forall \xi \in R^1,$$

for some constants, c_0, $c_1 > 0$, $\widehat{c}_0, \widehat{c}_1 \geq 0$, $p > 1$.

Assume $A : W_0^{k,p_0}(\Omega) \longrightarrow L_{p_0}(\Omega)$ is the linear continuous operator satisfying the inequation

$$c_2 \|u\|_{W_0^{k,p_0}} - \widehat{c}_2 \leq \|A(u)\|_{L_{p_0}} \leq c_3 \|u\|_{W_0^{k,p_0}} + \widehat{c}_3,$$

for $\forall u\in W_{0}^{k,p_{0}}\left(\Omega\right)$, *where* $\Omega\subset R^{n}$, $n\geq1$, *is the bounded domain with the sufficiently smooth boundary* $\partial\Omega$, *and* c_{2},$c_{3}>0$, $\widehat{c}_{2},\widehat{c}_{3}\geq0$, $p_{0}\geq p\cdot p_{1}$, $p,p_{1}>1$, $k\geq0$ *are constants.*

Then for each ball $B_{r}^{W_{0}^{k,p_{0}}}\left(0\right)\subset W_{0}^{k,p_{0}}\left(\Omega\right)$, $r\geq r_{0}>0$ *there exists such subset* $G_{r}\subset W_{0}^{k,p_{0}}\left(\Omega\right)$ *that* $f_{0}\left(G_{r}\right)\equiv\left(f\circ A\right)\left(G_{r}\right)\equiv f\left(A\left(G_{r}\right)\right)$ *is a convex subset with the nonempty interior of* $L_{p_{1}}\left(\Omega\right)$, *moreover,* $f_{0}\left(B_{r}^{W_{0}^{k,p_{0}}}\left(0\right)\right)\subseteq f_{0}\left(G_{r}\right)\subseteq f_{0}\left(B_{r_{1}}^{W_{0}^{k,p_{0}}}\left(0\right)\right)$ *holds for some* $r_{1}\geq r$, *where* r_{0} *is a constant depending at the above constants.*

Proof. The proof is sufficient to bring for the mapping $f:L_{p_{0}}\left(\Omega\right)\longrightarrow L_{p_{1}}\left(\Omega\right)$ since it isn't difficult to see the linear operator A is surjective (see, e.g. [5], [15], [19], [26], [29]). Then is enough to consider the subsets of $L_{p_{0}}\left(\Omega\right)$ of the following types

$$M_{r_{2}}\equiv\left\{u\in L_{p_{0}}\left(\Omega\right)\left|\ \left\|f\left(u\right)\right\|_{L_{p_{1}}}\leq r_{2},\ r_{2}>r_{0}\right.\right\}.$$

Whence using the conditions of the lemma and leading some estimations we obtain

$$\bar{c}_{0}\left\|u\right\|_{L_{p_{0}}}^{p}-\widetilde{c}_{0}\leq\left\|f\left(u\right)\right\|_{L_{p_{1}}}\leq\bar{c}_{1}\left\|u\right\|_{L_{p_{0}}}^{p}+\widetilde{c}_{1}.$$

These inequations show that the image $f\left(M_{r_{2}}\right)$ of the above-introduced subset $M_{r_{2}}$ contains some ball and belongs to another ball from $L_{p_{1}}\left(\Omega\right)$. Consequently, there exist such subsets G_{r} that $f_{0}\left(G_{r}\right)$ will be convex subsets with the nonempty interior of $L_{p_{1}}\left(\Omega\right)$.

We will provide some well-known facts from [5] that are necessary for the next result.

Definition 6.1 *(see, [5]) Let* K *be a convex set of the linear space* (LS) X *containing zero as the* $C-$*interior point. If* μ *is the support function of* K, *then the function* τ *defined for all pairs* $x,\ y\in X$ *by the equation*

$$\tau\left(x,y\right)=\lim_{\alpha\searrow+0}\frac{1}{\alpha}\left[\mu\left(x+\alpha y\right)-\mu\left(x\right)\right]$$

called tangential functional of the set K.

If K is the convex subset, as in Definition 6.1 then $\frac{1}{\alpha}\left[\mu\left(x+\alpha y\right)-\mu\left(x\right)\right]$ is a growing function at $\alpha\geq0$, and functional $\tau\left(x,y\right)$ is defined for all pairs $x,\ y\in X$ (see, [5]).

Let K is the subset of the LS X and $x\in K$ be a $C-$boundary point of K then the functional $x^{*}\in X^{*}$ called a tangent to the subset K on the point $x\in K$ if there exists such constant c that $\left\langle x^{*},x\right\rangle=c$ and $\left\langle x^{*},y\right\rangle\leq c$ for $\forall y\in K$.

Theorem 6.1 . *Let* X *be a* VTS *and* $K\subset X$ *be a closed subset of* X, *possessing interior points. Assume the subset* K *possesses the nontrivial tangent functional on all points of an everywhere dense subset of the boundary of* K *then* K *is the convex set.(see, [5])*

For the proof, and also about the correctness of its inverse statement see, [5].

Corollary 6.2 *Let a bounded mapping* f *acting from reflexive Banach space* X *to its dual space* X^{*} *is*

(i) a monotone hemi-continuous coercive operator (see, [15], [43]). Then f *translates a closed convex subset with a nonempty interior of* X *onto a closed convex subset with a nonempty interior of* X^{*};

(ii) a positive homogeneous radially continuous monotone mappings. Then f *translates a convex subset of* X *defined by a functional depending on mapping* f, *for which zero is an interior point, onto a convex subset of* X^{*}.

The proof of the (i) follows from Theorem 6.1, and the proof of the (ii) immediately follows from the presentation of the corresponding functional

$$F(x) \equiv \int_0^1 \langle f(tx), x\rangle \, dt \equiv \frac{1}{\alpha+1} \langle f(x), x\rangle,$$

where α is a exponent of homogeneity.

7 Examples

Example 1. Let X be a reflexive Banach space, J be a duality operator $J : X \longrightarrow X^*$, generated by a strongly monotone growing continuous function (see, e.g. [15])

$$\Phi : R^1_+ \longrightarrow R^1_+, \quad \Phi(0) = 0, \quad \Phi(\tau) \nearrow \infty \text{ at } \tau \nearrow \infty,$$

$\varphi : R^1_+ \longrightarrow R^1_+$ be some mapping translating the connected set to the connected set (i.e. connected mapping) that satisfies the condition: for each interval, $I \subset R^1_+$ there exists $\sup\{\varphi(\tau) \mid \tau \in I\} = \varphi(\tau_I)$, moreover, here maybe $\varphi(\tau_I) = \infty$.

Let X and X^* be strongly convex spaces, $f : X \longrightarrow X^*$ be a mapping having the form as in the following equation

$$f(x) \equiv \varphi(\|x\|_X) \; J(x) = y, \quad y \in X^*. \tag{7.1}$$

Theorem 7.1 *Under the above conditions the equation, (7.1) is solvable for any $y \in X^*$ satisfies the following inequality*

$$\|y\|_{X^*} \leq \sup\{\varphi(r)\,\Phi(r) \mid r \geq 0\} = \varphi(r_0)\,\Phi(r_0).$$

Proof. In the beginning note that according to Lemma 6.2 the image $f\left(B^X_r(0)\right) \subset X^*$ of the ball $B^X_r(0) \subset X$ is a convex subset of X^*. The duality operator J is a bijective mapping due to spaces X and X^* being strongly convexity (see, [5], [15], [23]). Moreover, the image of each ball $B^X_{r_0}(0) \subset X$ is a ball $B^{X^*}_{r_1}(0) \subset X^*$, where $r_1 = \Phi(r_0)$. Whence implies that the mapping f can be represented as

$$f(x) \equiv \varphi(\|x\|_X) \; \Phi(\|x\|_X) \; x^*, \quad \forall x \in B^X_r(0), \tag{7.2}$$

since $J(x) \equiv \Phi(\|x\|_X) \; x^*$, where the functional $x^* \in S^{X^*}_1(0) \subset X^*$ is the functional generating the norm $\|x\|_X$.

Thus the image $f\left(S^X_{r_0}(0)\right)$ of each sphere $S^X_{r_0}(0) \subset X$ is the sphere $S^{X^*}_{r_1}(0) \subset X^*$ under mapping f. Consequently, since $x^* \in S^{X^*}_1(0) \subset X^*$ then from (7.2) implies that the image $f\left(B^X_{r_0}(0)\right)$ of ball $B^X_{r_0}(0) \subset X$ is the ball $B^{X^*}_{r_1}(0) \subset X^*$ with the corresponding radius $r_1 = r_1(r_0)$.

Then using the Theorem 5.1 we get that equation (7.1) is solvable for any $y \in X^*$ satisfying inequation

$$|\langle y, x\rangle| \leq r\varphi(r)\,\Phi(r), \quad \forall x \in S^X_r(0)$$

for some $r > 0$.

In the other words, the equation (7.1) is solvable for any $y \in X^*$ satisfying the following condition: there exists such $r > 0$ that $\|y\|_{X^*} \leq \varphi(r)\,\Phi(r)$.

Example 2. Let $\Omega \subset R^n$ $(n \geq 1)$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider the following problem

$$f(u) \equiv -\Delta u + \psi(u) \ni h(x), \quad x \in \Omega, \quad u \mid_{\partial\Omega} = 0, \tag{7.3}$$

where Δ is the laplacian, and ψ is the mapping acting from $H_0^1 \equiv W_0^{1,2}(\Omega)$ to $L_2(\Omega)$, moreover, can be represented in the form

$$\psi(u) = \{v(x)\}_u = \begin{cases} 1, & x \in \{y \in \Omega \mid u(x) > 0\} \\ \{w\}_u \in L^\infty(\Omega), & x \in \Omega_0 \equiv \{y \in \Omega \mid u(y) = 0\}. \\ -1, & x \in \{y \in \Omega \mid u(x) < 0\} \end{cases}$$

So, ψ be such multi-valued mapping that for each function $u \in H_0^1$ the image $\psi(u)$ is the set of functions $\{v(x)\} \subset L^\infty(\Omega)$ such that $|v(x)| \leq 1$ a.e. on Ω, and also $[-1, 1] \subset \Re(\psi)$. Here $W_0^{1,2}(\Omega)$ is the Sobolev space of the functions $v : \Omega \longrightarrow R^1$, and $W^{-1,2}(\Omega)$ is its dual space (see, [41], [42]).

In the other words, the mapping f acts from H_0^1 to $H^{-1} \equiv W^{-1,2}(\Omega)$, moreover, the inclusion (7.3) is understood in the following sense: $f(u) = -\Delta u + \{v(x)\}_u$ for each function $u \in H_0^1$.

Theorem 7.2 *Let the above conditions on problem (ref) be fulfilled. Then for any $h \in H^{-1}$, the problem (7.3) has solutions $u(x)$, which belongs to the space H_0^1.*

Proof. For the proof enough to show that all conditions of the Theorem 5.1 fulfill for the mapping f. We will show for any $h \in H^{-1}$ there exists such set $G \subset H_0^1$ that $f(G)$ is the convex body and exists such subset ∂G_1 of G that be a boundary of an absorbing subset G_1, on which takes place the inequation

$$\langle f(u) - h, u\rangle > 0, \quad \forall u \in \partial G_1 \subset G \subset H_0^1.$$

Here we will use the Lemma 6.1, namely here enough to choose such balls $B_{r_1}^{H_0^1}(0)$, $B_{r_2}^{H_0^1}(0) \subset H_0^1$ $(0 < r_1 < r_2)$ that satisfy the inequation $f\left(B_{r_1}^{H_0^1}(0)\right) \subseteq M \subseteq f\left(B_{r_2}^{H_0^1}(0)\right)$, where M be a convex subset. We should be noted choosing of balls is dependent on the given $h \in H^{-1}$.

In the beginning, is necessary to show the correctness of some inequalities.

It isn't difficult to see

$$\langle f(u), u\rangle = \langle -\Delta u + \psi(u), u\rangle = \|\nabla u\|_H^2 + \|u\|_{L_1} \geq \|\Delta u\|_{H^{-1}}^2,$$

where $\nabla \equiv \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)$, $\Delta \equiv \nabla^* \circ \nabla$, $H \equiv L_2(\Omega)$. Moreover, there exists a constant $c \geq 1$ such that

$$\langle f(u), u\rangle = \|\nabla u\|_H^2 + \|u\|_{L_1} \leq c\left(\|\Delta u\|_{H^{-1}}^2 + \|\psi(u)\|_{H^{-1}}\right) \tag{7.4}$$

and on each sphere $S_{r_1}^{H_0^1}(0) \subset H_0^1$, $r > 1$ for some constants $c_1, c_2 > 0$ takes place

$$\|\Delta u\|_{H^{-1}}^2 \leq c_1 \|f(u)\| \leq \|\Delta u\|_{H^{-1}}^2 + \|\psi(u)\|_{H^{-1}} \leq c_2 \|\Delta u\|_{H^{-1}}^2. \tag{7.5}$$

For the proof that the image $f\left(B_r^{H_0^1}(0)\right) \subset H^{-1}$ of the ball $B_r^{H_0^1}(0) \subset H_0^1$, $r \geq 1$ be a convex set with the nonempty interior we will use the convexity of the corresponding functional, according to the proof of Lemma 6.2 and the condition on the mapping ψ [7]) one can assume ball $B_r^{H_0^1}(0) \subset H_0^1$ be an effective set of the functional Φ

$$\mathbb{N}_f(\gamma_X, y^*) = \left\{ \langle y, y^* \rangle = \tau \in R^1 \mid y \in f(x),\ x \in \gamma_X(x_1, x_2),\ x_1, x_2 \in G \right\},$$

where $x_1, x_2 \in G$ are some points, and $\gamma_X(x_1, x_2) \subset G$ is the curve connecting of these points, $y^* \in Y^*$ be a linear continuous functional.

Theorem 7.3 *(see, [32] and its references) Let X, Y be linearly connected $LVTS$, and $f : D(f) \subseteq X \longrightarrow Y$ be a mapping, $G \subseteq D(f)$ be a locally connected subset. Then if for any $x_1, x_2 \in G$ there exists such curve $\gamma_X(x_1, x_2)$ that the subset $\mathbb{N}_f(\gamma_X, y^*)$ be connected for each linear continuous functional $y^* \in Y^*$ then $f(G)$ be a connected set.*

See, Soltanov K. N., On the connectivity of sets and the image of discontinuous mappings. On nonlinear mappings. Proc. Inter. Topol. Conf., Baku-87, 1989, v II, 166-173 (Russian) (see, [16], [19]). Since mapping f is the subdifferential of the convex functional

$$\Phi(u) \equiv \frac{1}{2}\|\nabla u\|_{H^n}^2 + \|u\|_{L_1} \equiv \Phi_0(u) + \Phi_1(u)$$

due to well-known results (see, [16], [19]) one can assume ball $B_r^{H_0^1}(0) \subset H_0^1$ be an effective set of the functional Φ. Consequently, it is sufficient to examine an effective set of dual-functional Φ^* of functional Φ. Due to sub-differentiability of the functional Φ, at least on the $int\ dom\ \Phi$, the inclusions $int\ dom\ \Phi \subseteq \Re(\partial\Phi) \subset dom\ \Phi^*$ hold.

As the functional is the sum of functionals Φ_0 and Φ_1 its dual Φ^* is the infimal convolution of functionals Φ_0 and Φ_1 therefore are necessary to define their dual functionals. It is known (see, [30]) under $v^* \in dom\Phi_0^*$ holds $\Phi_0^*(v^*) \equiv \frac{1}{2}\|v^*\|_{H^n}^2$, moreover

$$dom\Phi_0^* \equiv \{v^* \in H^n \mid \|v^*\|_{H^n} \leq r\},$$

and also under $u^* \in dom\Phi_1^* \subset L^\infty(\Omega)$ holds $\Phi_1^*(u^*) \equiv \varkappa\left(u^* \,\middle|\, B_1^{L^\infty}(0)\right)$, where[8] $B_1^{L^\infty}(0)$ is the closed ball radius $r = 1$, centered in zero of $L^\infty(\Omega)$, and $\varkappa(u^* \mid M)$ is the indicator function of the set $M \equiv B_1^{L^\infty}(0) \equiv dom\Phi_1^*$.

Then we get for $h \in dom\Phi^* \subset H^{-1}$

$$\Phi^*(h) \equiv \Phi_0^*(h^*) + \Phi_1^*(u^*)\, h^* + u^* = h$$

holds, and also $dom\Phi^* = dom\Phi_0^* + dom\Phi_1^*$, moreover is known $(\Phi_0 \circ \nabla)^* = \nabla^* \circ \Phi_0^*$ ([19]), in this case $(\Phi_0 \circ \nabla)^*(h^*) = \Phi_0^*(v^*)$, $\nabla^* v^* = h^*$ ([16]).

Thus we obtain

$$\Phi^*(h) \equiv \inf_{u^*, h^*} \left\{ \begin{array}{c} \frac{1}{2}\|v^*\|_{H^n}^2 \mid h^* + u^* = h, \quad h^* = \nabla^* v^*, \\ h^* \in B_1^{H^{-1}}(0) \subset H^{-1}, u^* \in B_1^{L^\infty}(0) \subset L^\infty(\Omega) \end{array} \right\}.$$

Whence imply that $int\ dom\Phi^* \neq \varnothing$, moreover $B_r^{H^{-1}}(0) \subseteq dom\Phi^*$. Consequently, $f\left(B_r^{H_0^1}(0)\right)$ have nonempty interior, and also other condition of Lemma 6.2 fulfills for the mapping f by virtue of inequalities ([16]) and ([17]). Moreover, the $dom\Phi^*$ expands according to the growth of the radius r that proving the correctness of the claim of Theorem 7.2.

[7] Since mapping f is the subdifferential of the convex functional

$$\Phi(u) \equiv \frac{1}{2}\|\nabla u\|_{H^n}^2 + \|u\|_{L_1} \equiv \Phi_0(u) + \Phi_1(u)$$

due to well-known results (see, [16], [19]

[8] $\chi(u^* \mid M) = \{.0 \text{ if } u^* \in M, \infty \text{ if } u^* \notin M\}$

Example 3. Now we will lead with a simple example of the application of one of the fixed-point theorems that were proved in this work.

Let $\Omega \subset R^n$ $(n \geq 1)$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, $H \equiv L_2(\Omega)$ be a Lebesgue space that is the Hilbert space and $f : D(f) \subseteq H \longrightarrow H$ be a nonlinear mapping. Assume the mapping f as acting in H has the representation $f(u) \equiv u - \alpha \|u - u_0\|_H^2 \ (u - u_0)$, where $0 < \alpha \leq 4^{-1}$. We want that $f : B_1^H(u_0) \subset H \longrightarrow B_1^H(u_0)$ holds. Denote by $\widetilde{u} = u - u_0$ for any $u \in B_1^H(u_0)$ then $f(u) \equiv u_0 + \widetilde{u} - \alpha \|\widetilde{u}\|_H^2 \widetilde{u}$. We will show that $\|f(u) - u_0\|_H \leq 1$

$$\|f(u) - u_0\|_H = \left\|u_0 + \widetilde{u} - \alpha \|\widetilde{u}\|_H^2 \widetilde{u} - u_0\right\|_H = \left\|\left(1 - \alpha \|\widetilde{u}\|_H^2\right) \widetilde{u}\right\|_H.$$

Whence the function $\widetilde{u}$ satisfies inequality $\|\widetilde{u}\|_H = \|u - u_0\|_H \leq 1$ by it definition, then

$$\|f(u) - u_0\|_H = \left\|\left(1 - \alpha \|\widetilde{u}\|_H^2\right) \widetilde{u}\right\|_H = \left(1 - \alpha \|\widetilde{u}\|_H^2\right) \|\widetilde{u}\|_H < 1$$

holds, due to the condition on α. Consequently, for any $u \in B_1^H(0)$ is correct the estimation $\|f(u) - u_0\|_H < 1$, in other words, $f\left(B_1^H(u_0)\right) \subset B_1^H(u_0)$ holds.

Thus, if define the mapping $f_1(u) = u - f(u)$ then we get

$$f_1(u) = u - f(u) = \alpha \|u - u_0\|_H^2 (u - u_0) = \alpha \|\widetilde{u}\|_H^2 \widetilde{u}.$$

Consequently, we obtain f_1 is the Gateaux derivative of the convex functional F_1, i.e. $f_1(u) = \partial F_1(u)$, where

$$F_1(u) = \frac{1}{4}\alpha \|u - u_0\|_H^4 = \frac{1}{4}\alpha \|\widetilde{u}\|_H^4.$$

So, the image $f\left(B_1^H(u_0)\right)$ of the mapping f is the convex set with the nonempty interior according to the results of the above section. On other hand, the necessary inequality of the fixed-point theorem is fulfilled since

$$\langle f_1(u), \widetilde{u}\rangle = \left\langle \alpha \|\widetilde{u}\|_H^2 \widetilde{u}, \widetilde{u}\right\rangle = \alpha \|\widetilde{u}\|_H^4 > 0.$$

holds for each $\widetilde{u} \in S_1^H(0)$.

Consequently, the mapping f possesses the fixed-point in $B_1^H(u_0)$, i.e. there exists $u_1 \in B_1^H(u_0)$ such that $f(u_1) = u_1$.

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