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Application Of The Contour Integral Method To The Solution Of The Problem For An Equation Belonging To Standard Classification

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Abstract. *The paper is devoted to application of the M.L.Rasulov's contour integral method to the mixed problem stated for higher order equation not belonging to Petrovsky type. Under the accepted conditions it was proved that in the complex λ plane we can select such a domain R_δ that the solution of the mixed problem for all $\lambda \in R_\delta$ from this domain may be shown in the from of integral on the contour located entirely on R_δ and whose infinitely distant parts coincide with the rays $arg\ \lambda = \frac{\pi}{2} + \delta$ passing from the origin of coordinate $arg\ \lambda = \frac{3\pi}{4} - \delta$.*

It is known that longitudinal and lateral oscillations of homogeneous elastic-viscous bars are described by the equations of the form

$$\frac{\partial^2 \upsilon}{\partial t^2} + a\frac{\partial^{2p}\upsilon}{\partial x^{2p}} + b\frac{\partial^{2p+1}\upsilon}{\partial x^{2p}\partial t} = f(x,t)$$

that don't belong to standard classification. Up to now the problems of longitudinal ($p = 1$) [1] and lateral ($p = 2$) oscillations [2] of elastico-viscous bars were considered to whose solutions mainly the residue method was applied. Then the problem for a fourth order weak parabolic equation [3] and for a general wave equation [4] were considered, to whose solutions different methods including M.L.Rasulov's contour integral method was applied. The goal of the paper is to prove that the above mentioned methods may be applied to the solution of the problem for an equation not belonging to I.G. Petrovsky standard classification.

Thus, we consider a problem of finding the solution of the equation

$$c\frac{\partial^2 \upsilon}{\partial t^2} = a\frac{\partial^4 \upsilon}{\partial x^4} + b\frac{\partial^3 \upsilon}{\partial x^2 \partial t} \tag{1}$$

satisfying the boundary conditions

$$\upsilon(0,t) = \upsilon_x(0,t) - \upsilon_x(1,t) = \upsilon_{xx}(0,t) - \upsilon_t(1,t)$$
$$= \upsilon_{xxx}(0,t) + \upsilon_{xt}(1,t) = 0, \quad for\ \ t > 0 \tag{2}$$

and initial conditions

$$\upsilon_t^{(k)}(x,0) = \Phi_k(x) \ \ (k = 0,1), \ \ x \in (0,1), \tag{3}$$

where a, b, c are constant numbers, while $\Phi_k(x)$ $(k = 0,1)$ are sufficiently smooth functions.

It is assumed that the following conditions are fulfilled:

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1^0 The coefficients a, b, c are such that equation (1) does not belong to the Petrovsky standard classification, i.e. the roots of the characteristic equation

$$cv^2 + bv - a = 0$$

are not real, are not pure imaginary and not $\mathrm{Re}v < 0$.

2^0 The initial functions $\varPhi_k(x)$ $(k = 0, 1)$ have continuous derivatives to $(3-2k)-$ th $(k = 0, 1)$ order, for $x \in (0, 1)$ furthermore,

$$\varPhi_0(0) = \varPhi_0'(0) = \varPhi_0''(0) = 0.$$

By the scheme of M.L.Rasulov's [5] contour integral method, to the mixed problem (1)-(3) we assign a complex parameter dependent boundary value problem called a spectral problem that was investigated in the paper [6].

In the present paper we will construct the strong solution of problem (1)-(3).

Similar to what has been done in the papers [3]-[5], by the immediate check we prove the

Theorem. *Subject to conditions 1^0 , 2^0 , the problem (1)-(3) has a unique solution $\upsilon(x, t)$ having continuous derivatives to fourth order with respect to x for $x \in (0, 1)$, to second order with respect to t for $t > 0,\ x \in (0, 1)$, to firist ordet for $t \geq 0,\ x \in (0, 1]$, to third order with respect to x for $x \in (0, 1]$ and $t > 0$ that is represented in the from*

$$\upsilon(x, t) = \frac{1}{\pi\sqrt{-1}} \int\limits_S \lambda e^{\lambda^2 t} y(x, \lambda) d\lambda, \tag{4}$$

where S is an infinitely open contour entirely situated in the sector of a complex plane defined as follows:

$$R_\delta = \left\{ \lambda : |\lambda| > R, \frac{\pi}{2} + \frac{\delta}{2} < \arg\lambda < \frac{3\pi}{4} - \frac{\delta}{2} \right\}, \tag{5}$$

moreover R is a rather large positive number, furthermore the infinitely distant parts of the contour, S coincide with the continuation of the rays (fig.1)

$$\arg\lambda = \frac{3\pi}{4} - \delta \ \text{ and } \ \arg\lambda = \frac{\pi}{2} + \delta,$$

finally, $y(x, \lambda)$ is the solution of the spectral problem that corresponds to the mixed problem (1)-(3) determined by the formula

$$y(x, \lambda) = \int\limits_0^1 G(x, \xi, \lambda) F(\xi, \lambda) d\xi \tag{6}$$

(see [6]). As it has been already said, the given mixed problem was studied in [6]. In this connection, here we cite some formulas that we will need in further calculations. More exactly,

$$G(x, \xi, \lambda) = g_0(x, \xi, \lambda) + \sum_{k,m=1}^{4} A_{km}(\xi, \lambda) y_m(x, \lambda), \tag{7}$$

where

$$g_0(x, \xi, \lambda) = \begin{cases} \sum\limits_{k=1}^{2} \frac{W_{4k}(\xi,\lambda)}{W(\xi,\lambda)} y_k(x, \lambda), & for\ \ 0 \leq \xi < x \leq 1, \\ -\sum\limits_{k=3}^{4} \frac{W_{4k}(\xi,\lambda)}{W(\xi,\lambda)} y_k(x, \lambda), & for\ \ 0 \leq \xi < x \leq 1 \end{cases} \tag{8}$$

moreover $y_k(x, \lambda) = e^{\lambda\nu_k x}$ $(k = \overline{1, 4})$ is a fundamental system of particular solutions of the equation of the spectral problem, $W(\xi, \lambda)$ is the Wronskian determinant of these solutions, $W_{4k}(\xi, \lambda)$ is the cofactor of the elements of the fourth row, of the k-th column $(k = \overline{1, 4})$ of the determinant $W(\xi, \lambda)$,ν_k $(k = \overline{1, 4})$ are the roots of the characteristic equation

$$a\nu^4 + b\nu^2 - c = 0. \tag{3'}$$

Finally, the coefficients $A_{km}(\xi,\lambda)$ are determined as follows:

$$A_{km}(\xi,\lambda)=\frac{\Delta_{km}(\lambda)}{\Delta(\lambda)}l_k(g_0(x,\xi,\lambda)),(k,m=\overline{1,4}) \tag{9}$$

where $\Delta(\lambda)$ is a characteristic determinant, $\Delta_{km}(\lambda)$ $(k,m=\overline{1,4})$ is the cofactor of the elements of the k-th row, of the m-th column of the determinant $\Delta(\lambda)$, $l_k(g_0(x,\xi,\lambda)),(k,m=\overline{1,4})$ are the expressions obtained by applying boundary operators to the function $g_0(x,\xi,\lambda)$ as a function of x. In what follows, the function $F(x,\lambda)$ is determined by the formula

$$F(x,\lambda)=-c\Phi_1(x)+b\Phi_0''(x)-c\lambda^2\Phi_0''(x). \tag{10}$$

For proving the theorem, we choose the coefficients of equation (1) so that the condition 1^0 is fulfilled. For example, if we choose as in [6], i.e. $a=1,b=1-2i,c=2i$ the roots will be $\pm(1+i),\ \pm i$. We enumerate them so that the following inequalities be fulfilled:

$$\mathrm{Re}\lambda\nu_1\le\mathrm{Re}\lambda\nu_2\le0\le\mathrm{Re}\lambda\nu_3\le\mathrm{Re}\lambda\nu_4. \tag{11}$$

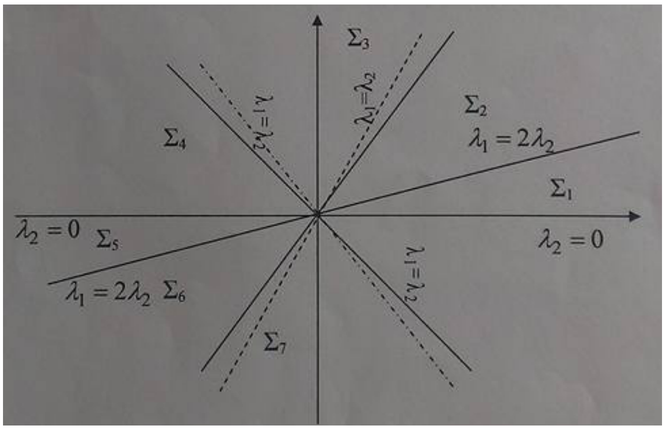

Fig.1.

By virtue of division of a complex plane in [6] it is divided into eight sectors (fig.1). In this paper we consider the sector $\sum_4$. In this case, we enumerate the roots ν_k $(k=\overline{1,4})$ as follows:

$$\nu_1=(1+i),\ \upsilon_2=i,\ \nu_3=-i,\ \nu_4=-(1+i).$$

Verification shows that if $\lambda\in\sum_4$, the inequalities (11) are fulfilled.

Now prove the theorem. For that we substitute (7) into the right side of (6) and take into account (9). Then we get:

$$\upsilon(x,t)=\frac{1}{\pi\sqrt{-1}}\int\limits_S\lambda e^{\lambda^2t}d\lambda\int\limits_0^1\left\{g_0(x,\xi,\lambda)+\sum_{k,m=1}^{4}A_{km}(\xi,\lambda)y_m(x,\lambda)\right\}F(\xi,\lambda)d\xi. \tag{6'}$$

Now transform the formula $(6')$ in the following way, i.e. represtent the right part in the form of the sum of two addends

$$\upsilon(x,t)=\frac{1}{\pi\sqrt{-1}}\int\limits_S\lambda e^{\lambda^2t}d\lambda\int\limits_0^1g_0(x,\xi,\lambda)F(\xi,\lambda)d\xi$$

$$+\frac{1}{\pi\sqrt{-1}}\int_S \lambda e^{\lambda^2 t} d\lambda \int_0^1 \sum_{k,m=1}^4 A_{km}(\xi,\lambda) y_m(x,\lambda) F(\xi,\lambda) d\xi.$$

Introduce the denotation:

$$\upsilon_1(x,t) = \frac{1}{\pi\sqrt{-1}}\int_S \lambda e^{\lambda^2 t} d\lambda \int_0^1 g_0(x,\xi,\lambda) F(\xi,\lambda) d\xi, \tag{12}$$

$$\upsilon_2(x,t) = \frac{1}{\pi\sqrt{-1}}\int_S \lambda e^{\lambda^2 t} d\lambda \int_0^1 \sum_{k,m=1}^4 A_{km}(\xi,\lambda) y_m(x,\lambda) F(\xi,\lambda) d\xi. \tag{13}$$

By the immediate verification it is easily proved that for the integrand function in the right side of formula (13) it holds the estimation

$$\left|\lambda e^{\lambda^2 t} A_{km}(\xi,\lambda) y_m(x,\lambda)\right| \le \frac{c e^{-\varepsilon|\lambda|(|\lambda| t + |x-\xi|)}}{|\lambda|^2}, \tag{14}$$

valid for all $\lambda \in R_\delta, x,\xi \in (0,1)$ and $t > 0$, whence it follows that

$$\lim_{t\to 0} \upsilon_2(x,t) = 0. \tag{15}$$

In what follows, subject to conditions 1^0 ,2^0 the validity of the expansion formulas

$$\frac{1}{\pi\sqrt{-1}}\int_S \lambda^{2s+1} d\lambda \int_0^1 G(x,\xi,\lambda)\Phi(\xi) d\xi = \begin{cases} 0; & for \quad s=0 \\ \frac{\Phi(x)}{c} & for \quad s=1 \\ -\Phi''(x), & for \quad s=2. \end{cases} \tag{16}$$

is easily proved. Now show that the function determined by formula (6) satisfies equation (1). For that we substitute (6) into the left side of equation (1). By inequality (14), we can introduce the operator of equation (1) under the integral sign , i.e.

$$\left(c\frac{\partial^2}{\partial t^2} - a\frac{\partial^4}{\partial x^4} - b\frac{\partial^3}{\partial x^2 \partial t}\right) \frac{1}{\pi\sqrt{-1}}\int_S \lambda e^{\lambda^2 t} d\lambda \int_0^1 G(x,\xi,\lambda) F(\xi,\lambda) d\xi$$

$$= \frac{1}{\pi\sqrt{-1}}\int_S \lambda e^{\lambda^2 t} d\lambda \int_0^1 \left(c\lambda^4 - a\frac{\partial^4}{\partial x^4} - b\lambda^2\frac{\partial^3}{\partial x^2}\right) G(x,\xi,\lambda) F(\xi,\lambda) d\xi \equiv 0$$

by the Green function property.

In the same way, by the Green function property, allowing for inequality (14) we prove that the function $\upsilon(x,t)$ determined by formula (16) satisfies the boundary conditions as well.

It remains to prove that this functions satisfies the initial conditions as well. Taking into attention (14), prove that the function $\upsilon_1(x,t)$ determined by formula (12) satisfies initial conditions (3). For that we substitute the expression of the function $g_0(x,\xi,\lambda)$ from (8) into the right part of formula (12) and get:

$$\upsilon_1(x,t) = \frac{1}{\pi\sqrt{-1}}\int_S \lambda e^{\lambda^2 t} d\lambda \left\{ \int_0^x \sum_{k=1}^2 \frac{W_{4k}(\xi,\lambda)}{W(\xi,\lambda)} y_k(x,\lambda) F(\xi,\lambda) d\xi \right.$$
$$\left. - \int_x^1 \sum_{k=3}^4 \frac{W_{4k}(\xi,\lambda)}{W(\xi,\lambda)} y_k(x,\lambda) F(\xi,\lambda) d\xi \right\}, \tag{17}$$

where $W(\xi,\lambda)$ is a Wronskian, W_{4k} $(k = \overline{1,4})$ is the cofactor of the elements of the fourth row of the k-th column of this determinant, $y_k(x,\lambda)$ $(k = \overline{1,4})$ is a fundamental system of particular solutions of the equation of the spectral problem [6]. Finally, $F(\xi,\lambda)$ is determined by formula (10). By immediate calculations for the Wronskian and its cofactors of the elements of the fourth row, we get the following formulas:

$$W(\xi,\lambda)=\lambda^6(\nu_2-\nu_1)(\nu_3-\nu_1)(\nu_4-\nu_1)(\nu_3-\nu_2)(\nu_4-\nu_2)(\nu_4-\nu_3)$$
$$W_{41}(\xi,\lambda)=\lambda^3(\nu_2-\nu_3)(\nu_2-\nu_4)(\nu_3-\nu_4)e^{-\lambda\nu_1\xi}$$
$$W_{42}(\xi,\lambda)=\lambda^3(\nu_1-\nu_3)(\nu_1-\nu_4)(\nu_4-\nu_3)e^{-\lambda\nu_2\xi}$$
$$W_{43}(\xi,\lambda)=\lambda^3(\nu_1-\nu_2)(\nu_1-\nu_4)(\nu_2-\nu_4)e^{-\lambda\nu_3\xi}$$
$$W_{44}(\xi,\lambda)=\lambda^3(\nu_2-\nu_1)(\nu_3-\nu_1)(\nu_3-\nu_2)e^{-\lambda\nu_4\xi}.$$

By means of the obtained values $W(\xi,\lambda)$, for the integrand expressions we get:

$$\frac{W_{4k}(\xi,\lambda)}{W(\xi,\lambda)}=\frac{e^{-\lambda\nu_k\xi}}{\lambda^3\prod\limits_{\substack{m=1\\m\neq k}}^{4}(\nu_k-\nu_m)},\ (k=\overline{1,4}). \tag{18}$$

Substituting the expressions $F(\xi,\lambda)$ from (10) and the values $\frac{W_{4k}(\xi,\lambda)}{W(\xi,\lambda)}, (k=\overline{1,4})$ from (18) to the right side of (17) , we get:

$$\upsilon_1(x,t)=-c\int\limits_S e^{\lambda^2 t}d\lambda\int\limits_0^x\left\{\frac{e^{\lambda\nu_1(x-\xi)}}{(\nu_1-\nu_2)(\nu_1-\nu_3)(\nu_1-\nu_4)}\right.$$
$$\left.+\frac{e^{\lambda\nu_2(x-\xi)}}{(\nu_2-\nu_1)(\nu_2-\nu_3)(\nu_2-\nu_4)}\right\}\varPhi_0(\xi)d\xi-\int\limits_x^1\left\{\frac{e^{\lambda\nu_3(x-\xi)}}{(\nu_3-\nu_1)(\nu_3-\nu_2)(\nu_3-\nu_4)}\right.$$
$$\left.+\frac{e^{\lambda\nu_4(x-\xi)}}{(\nu_4-\nu_1)(\nu_4-\nu_2)(\nu_4-\nu_3)}\right\}\varPhi_0(\xi)d\xi-\int\limits_0^x\frac{1}{\lambda^2}\left\{\frac{e^{\lambda\nu_1(x-\xi)}}{(\nu_1-\nu_2)(\nu_1-\nu_3)(\nu_1-\nu_4)}\right.$$
$$\left.+\frac{e^{\lambda\nu_2(x-\xi)}}{(\nu_2-\nu_1)(\nu_2-\nu_3)(\nu_2-\nu_4)}\right\}(c\varPhi_1(\xi)-b\varPhi_0''))d\xi$$
$$-\int\limits_x^1\frac{1}{\lambda^2}\left\{\frac{e^{\lambda\nu_3(x-\xi)}}{(\nu_3-\nu_1)(\nu_3-\nu_2)(\nu_3-\nu_4)}\right.$$
$$\left.+\frac{e^{\lambda\nu_4(x-\xi)}}{(\nu_4-\nu_1)(\nu_4-\nu_2)(\nu_4-\nu_3)}\right\}(c\varPhi_1(\xi)-b\varPhi_0''))d\xi.$$

Introduce the denotation

$$Q_k(x-\xi,t)=\frac{1}{\pi\sqrt{-1}}\int\limits_S\frac{e^{-\lambda\nu_k(x-\xi)+\lambda^2 t}}{\prod\limits_{\substack{m=1\\m\neq k}}^{4}(\nu_k-\nu_m)}d\lambda,\ (k=\overline{1,4}).$$

By the known formula (see [5], p.307), for the functions $Q_k(x-\xi,t)$ $(k=\overline{1,4})$ we get the following expression

$$Q_k(x-\xi,t)=\frac{1}{\sqrt{\pi t}}\frac{\exp\left(-\frac{\nu_k^2(x-\xi)^2}{4t}\right)}{\prod\limits_{\substack{m=1\\m\neq k}}^{4}(\nu_k-\nu_m)},\ (k=\overline{1,4}). \tag{19}$$

Taking into attention (19) in the right side of the expression for, $\upsilon_1(x,t)$ we get:

$$\upsilon_1(x,t)=\frac{-c}{\sqrt{\pi t}}\int\limits_0^x Q_1(x-\xi,t)\Phi_0(\xi)d\xi-\frac{c}{\sqrt{\pi t}}\int\limits_0^x Q_2(x-\xi,t)\Phi_0(\xi)d\xi$$

$$-\frac{c}{\sqrt{\pi t}}\int\limits_x^1 Q_3(x-\xi,t)\Phi_0(\xi)d\xi-\frac{c}{\sqrt{\pi t}}\int\limits_x^1 Q_4(x-\xi,t)\Phi_0(\xi)d\xi$$

$$-\frac{c}{\pi\sqrt{-1}}\int\limits_S\frac{e^{\lambda^2 t}}{\lambda^2}d\lambda\int\limits_0^1 g_0(x,\xi,\lambda)(c\Phi_1(\xi)-b\Phi_0''))d\xi. \tag{20}$$

In what follows, we introduce the denotation

$$\upsilon_{11}(x,t)=\frac{-c}{\sqrt{\pi t}}\left\{\int\limits_0^x(Q_1(x-\xi,t)+Q_2(x-\xi,t))\Phi_0(\xi)d\xi\right.$$

$$\left.+\int\limits_x^1(Q_3(x-\xi,t)+Q_4(x-\xi,t)\Phi_0(\xi)d\xi\right\} \tag{21}$$

$$\upsilon_{12}(x,t)=-\frac{1}{\pi\sqrt{-1}}\int\limits_S\frac{e^{\lambda^2 t}}{\lambda^2}d\lambda\int\limits_0^1 g_0(x,\xi,\lambda)(c\Phi_1(\xi)-b\Phi_0''))d\xi, \tag{22}$$

and find

$$\upsilon_1(x,t)=\upsilon_{11}(x,t)+\upsilon_{12}(x,t). \tag{20'}$$

Moreover, for the functions $\upsilon_1(x,t)$ and $\upsilon_{12}(x,t)$ it is fulfilled the relation

$$\frac{\partial\upsilon_{12}(x,t)}{\partial t}=-\upsilon_{11}(x,t), \tag{23}$$

changing $\Phi_0(x)$ by $(c\Phi_1(x)-b\Phi_0''(x)$.

Similar to what has been done in the papers [4], [5] and with regard to expansion formula (16), it is easily proved that the function defined by formula (20) satisfies the first initial condition, i.e.

$$\lim_{t\to 0}\upsilon_{11}(x,t)=\Phi_0(x),$$

while allowing for expansion formula (16) and (23) with reserve the function determined by formula (20')will satisfy the second initial condition , i.e.

$$\lim_{t\to 0}\upsilon_1(x,t)=\lim_{t\to 0}(\upsilon_{11}(x,t)+\upsilon_{12}(x,t))=-\Phi_0''(x)+\Phi_1(x)+\Phi_0''(x)=\Phi_1(x).$$

Q.E.D.

What refers the proof of the uniqueness of the obtained one, it is proved as usually by contradiction.

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