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On Solvability Of A Boundary Value Problem For Second Order Operator-Differential Equations In The Space Of Smooth Vector-Functions

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Abstract. *In the pap er we find sufficient conditions that provide the existence and uniqueness of a boundary value problem for a class of operator differential equations of second order, that contain a normal operator in the principal part, in the space of smooth vector-functions. These conditions are expressed by the properties of operator coefficients. We also obtain the estimations of the norms of intermediate derivatives in the space of smooth vector-functions.*

Keywords. Hilbert space, operator-differential equation, boundary value problem, smooth solution.

Consider in separable Hilbert space the boundary value problem

$$-u''(t) + A^2u(t) + A_1u'(t) + A_2u(t) = f(t),\ t \in R_+ = (0, \infty) \tag{1}$$

$$u'(0) = 0, \tag{2}$$

where $u(t),\ f(t)$ are the functions determined in $R_+ = (0, \infty)$ almost everywhere, with the values from H, the derivatives are understood in the sense of distributions theory [1], and operator coefficients of equation (1) satisfy the condition:

1) A is a normal operator with bounded inverse whose spectrum is contained in the angular sector

$$S_\varepsilon = \{\lambda : |\arg\lambda| \le \varepsilon,\ 0 \le \varepsilon < \pi/2\};$$

2) The operators $A_j \in L\left(H_j, H\right) \cap L(H_{j+1}.H_1),\ j = 1, 2.$

Here $L(X, Y)$ is the space of bounded operators acting from the space X to the space Y, and the space $H_\gamma = D(C^\gamma),\ \gamma \ge 0\ (H_0 = H)$ with the norm $\|x\|_\gamma = \|C^\gamma x\|$, where the operator C is a positive-definite self-adjoint operator from the representation of the operator $A = UC$, and U is unitary operator.

Let $L_2(R_+; H)$ be a Hilbert space of all functions $f(t)$ determined in R_+ almost everywhere, measurable, with the values in H for which

$$\|f\|_{L_2(R_+:H)} = \left(\int_0^\infty \|f(t)\|^2\, dt\right)^{1/2}.$$

Following the monograph [1], we introduce the following Hilbert space

$$W_2^m(R_+; H) = \left\{u : u^{(m)} \in L_2(R_+; H),\ C^{(m)}u \in L_2(R_+; H)\right\},\ m = 1, 2, 3$$

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with the norm

$$\|u\|_{W_2^m(R_+;H)} = \left(\left\| u^{(m)} \right\|^2_{L_2(R_+;H)} + \left\| C^m u \right\|^2_{L_2(R_+;H)} \right)^{1/2}.$$

In connection with studying the problem (1), (2), consider the Hilbert space

$$\overset{\circ}{W}{}_2^3(R_+;H) = \left\{ u : u \in W_2^3(R_+ : H),\ u'(0) = 0 \right\}.$$

The spaces $L_2(R;H)$ and $W_2^m(R;H)$ are similary determined for $R = (-\infty, \infty)$.

Definition 1. *If for* $f(t) \in W_2^1(R;H)$ there *exists* $u(t) \in W_2^3(R_+;H)$ *that satisfies equation (1) in* R_+ *identically, boundary condition (2) in the sense of convergence*

$$\underset{t \to +0}{litm} \left\| u'(t) \right\|_{3/2} = 0$$

and estimation

$$\|u\|_{W_2^3(R_+;H)} \le const \, \|f\|_{W_2^1(R_+;H)},$$

we say that problem (1), (2) is well-defined solvable in $W_2^1(R_+;H)$. *Note that for* $f(t) \in L_2(R_+;H)$, $u \in W_2^2(R_+;H)$ *this problem was studied by many authors when* A *is a self-adjoint operator (see e.i. [2-4]), and when* A *is a normal operator in the paper [5].*

Note that the existence of smooth solutions of boundary value problems when A is a self-adjoint operator, $f \in W_2^1(R_+;H)$ while $u \in W_2^3(R_+;H)$ was investigated in the papers [6-7], and for $f \in W_2^2(R_+;H)$, $u \in W_2^4(R_+;H)$ in the paper [8]. On the whole axis, the well-posedness of the equation (1) was studied in (9).

In the present paper we find the conditions on the coefficients of equation (1) that provide well-defined solvability of problem (1), (2) in $W_2^1(R_+;H)$. Denote

$$P_0 u = -u'' + A^2 u,\ P_1 u = A_1 u' + A_2 u,\ u \in \overset{\circ}{W}{}_2^3(R_+;H).$$

At first we prove the following lemma.

Lemma 1. *Let conditions 1) be fulfilled. Then for* $u \in \overset{\circ}{W}{}_2^3(R_+;H)$ *it holds the inequality*

$$\|P_0 u\|^2_{W_2^1(R_+;H)} \ge \|u\|^2_{W_2^3(R_+;H)} + (1 + 2\cos 2\varepsilon) \left\| A u' \right\|^2_{W_2^1(R_+;H)} \tag{3}$$

Proof. Let $u \in \overset{\circ}{W}{}_2^3(R_+;H)$ $(u'(0) = 0)$. Then

$$\|P_0 u\|^2_{W_2^1(R_+;H)} = \left\| -u''' + A^2 u' \right\|^2_{L_2(R_+;H)} + \left\| -Cu'' + CA^2 u \right\|^2_{L_2(R_+;H)}$$

$$= \left\| u''' \right\|^2_{L_2(R_+;H)} + \left\| A^2 u' \right\|^2_{L_2(R_+;H)} - 2\mathrm{Re}(u''', A^2 u')_{L_2(R_+;H)} + \left\| C u'' \right\|^2_{L_2(R_+;H)}$$

$$+ \left\| CA^2 u \right\|^2_{L_2(R_+;H)} - 2\mathrm{Re}(Cu'', CA^2 u)_{L_2(R_+;H)}. \tag{4}$$

After integrating by parts, we get

$$-2\mathrm{Re} \left(u''', A^2 u' \right)_{L_2(R_+;H)}$$

$$= 2\mathrm{Re} \left(A^* u'', A u'' \right)_{l_2(R_+;H)} \ge 2\cos 2\varepsilon \left\| C u'' \right\|^2_{L_2(R_+;H)} \tag{5}$$

and

$$-2\mathrm{Re} \left(Cu'', CA^2 u \right)_{L_2(R_+;H)}$$

$$= 2\mathrm{Re} \left(C^2 u', A^2 u' \right)_{l_2(R_+;H)} \ge 2\cos 2\varepsilon \left\| C^2 u' \right\|^2_{L_2(R_+;H)}. \tag{6}$$

Taking into account inequalities (5) and (6) in equality (4), we complete the proof of the lemma.

Now estimate the intermediate derivatives in the space $W_2^1(R_+;H)$. It holds the following

Theorem 1. *Let condition 1) be fulfilled. Then for* $u \in \overset{\circ}{W}{}_2^3(R_+;H)$ *it holds the inequality*

$$\left\| A^{2-j} u^{(j)} \right\|_{W_2^1(R_+;H)} \le c_j(\varepsilon) \left\| P_0 u \right\|_{W_2^1(R_+;H)}, \quad j = 0,1 \tag{7}$$

where

$$c_0(\varepsilon) = \begin{cases} 1, & 0 \le \varepsilon \le \pi/4 \\ \dfrac{1}{\sqrt{2\cos\varepsilon}}, & \pi/4 \le \varepsilon < \pi/2 \end{cases}$$

$$c_1(\varepsilon) = \frac{1}{2\cos\varepsilon}, \quad 0 \le \varepsilon < \pi/2.$$

Proof. Let $j = 1$. Then for $u \in \overset{\circ}{W}{}_2^3(R_+;H)$ we have:

$$\left\| Au' \right\|^2_{W_2^1(R_+;H)} = \left\| Cu' \right\|^2_{W_2^1(R_+;H)} = \left\| C^2 u' \right\|^2_{L_2(R_+;H)} + \left\| Cu'' \right\|^2_{L_2(R_+;H)}$$

$$= \left(C^2 u', C^2 u' \right)_{L_2(R_+;H)} + \left(Cu'', Cu'' \right)_{L_2(R_+;H)}.$$

After integrating by parts, we have

$$\left(C^2 u', C^2 u' \right)_{L_2(R_+;H)} = -\left(C^3 u, Cu'' \right)_{L_2(R_+;H)}$$

$$\le \frac{1}{2} \left(\left\| C^3 u \right\|^2_{L_2(R_+;H)} + \left\| Cu'' \right\|^2_{L_2(R_+;H)} \right)$$

and

$$\left(Cu'', Cu'' \right)_{L_2(R_+;H)}$$

$$= -\left(C^2 u', u''' \right)_{L_2(R_+;H)} \le \frac{1}{2} \left(\left\| u''' \right\|^2_{L_2(R_+;H)} + \left\| C^2 u' \right\|^2_{L_2(R_+;H)} \right).$$

Consequently,

$$\left\| Au' \right\|^2_{W_2^1(R_+;H)} \le \frac{1}{2} \left\| u \right\|^2_{W_2^3(R_+;H)} + \frac{1}{2} \left\| Cu' \right\|^2_{W_2^1(R_+;H)}$$

$$= \frac{1}{2} \left\| u \right\|^2_{W_2^3(R_+;H)} + \frac{1}{2} \left\| Au' \right\|^2_{W_2^1(R_+;H)}.$$

Thus,

$$\left\| Au' \right\|^2_{W_2^1(R_+;H)} \le \left\| u \right\|^2_{W_2^3(R_+;H)}.$$

Using lemma 1, we get

$$\left\| Au' \right\|^2_{W_2^1(R_+;H)} \le \left\| P_0 u \right\|^2_{W_2^1(R_+;H)} - (1 + 2\cos 2\varepsilon) \left\| Au' \right\|^2_{W_2^1(R_+;H)}.$$

Consequently,

$$2(1 + \cos 2\varepsilon) \left\| Au' \right\|^2_{W_2^1(R_+;H)} \le \left\| P_0 u \right\|^2_{W_2^1(R_+;H)}$$

i.e.

$$\left\| Au' \right\|_{W_2^1(R_+;H)} \le \frac{1}{2\cos\varepsilon} \left\| P_0 u \right\|_{W_2^1(R_+;H)}, \quad 0 \le \varepsilon < \pi/2.$$

Now prove the inequality in the case when $j = 0$. Obviously,

$$\left\| A^2 u \right\|^2_{W_2^1(R_+;H)} = \left\| C^2 u \right\|_{W_2^1(R_+;H)} = \left\| C^2 u' \right\|^2_{L_2(R_+;H)} + \left\| C^3 u \right\|^2_{L_2(R_+;H)}$$

and

$$\left\| u'' \right\|^2_{W_2^1(R_+;H)} = \left\| Cu'' \right\|^2_{L_2(R_+;H)} + \left\| u''' \right\|^2_{L_2(R_+;H)}.$$

Then

$$\left\|A^2u\right\|^2_{W_2^1(R_+;H)}+\left\|u''\right\|^2_{W_2^1(R_+;H)}=\left(\left\|u'''\right\|^2_{L_2(R_+;H)}+\left\|C^3u\right\|^2_{L_2(R_+;H)}\right)$$

$$+\left(\left\|Cu''\right\|^2_{L_2(R_+;H)}+\left\|C^2u'\right\|^2_{L_2(R_+;H)}\right)=\left\|u\right\|^2_{W_2^3(R_+;H)}+\left\|Au'\right\|^2_{W_2^1(R_+;H)}.$$

Using lemma1, we get

$$\left\|A^2u\right\|^2_{W_2^1(R_+;H)}+\left\|u''\right\|^2_{W_2^1(R_+;H)}\le\left\|P_0u\right\|^2_{W_2^1(R_+;H)}-2\cos 2\varepsilon\left\|Au'\right\|^2_{W_2^1(R_+;H)}.$$

Let $0\le\varepsilon\le\pi/4$ $(\cos 2\varepsilon\ge 0)$. Then we get

$$\left\|A^2u\right\|^2_{W_2^1(R_+;H)}+\left\|u''\right\|^2_{W_2^1(R_+;H)}\le\left\|P_0u\right\|^2_{W_2^1(R_+;H)}$$

i.e.

$$\left\|A^2u\right\|_{W_2^1(R_+;H)}\le\left\|P_0u\right\|_{W_2^1(R_+;H)}.$$

Consequently, for $0\le\varepsilon<\pi/4$

$$\left\|A^2u\right\|_{W_2^1(R_+;H)}\le c_0(\varepsilon)\left\|P_0u\right\|_{W_2^1(R_+;H)}=\left\|P_0u\right\|_{W_2^1(R_+;H)}.$$

Now suppose that $\pi/4\le\varepsilon\le\pi/2$ $(\cos 2\varepsilon\le 0)$. Then using inequality (7) for $j=1$ we get

$$\left\|A^2u\right\|^2_{W_2^1(R_+;H)}+\left\|u''\right\|^2_{W_2^1(R_+;H)}\le\left\|P_0u\right\|^2_{W_2^1(R_+;H)}-2\cos 2\varepsilon\left\|Au'\right\|^2_{W_2^1(R_+;H)}$$

$$=\left\|P_0u\right\|^2_{W_2^1(R_+;H)}-2\cos 2\varepsilon\frac{1}{4\cos^2\varepsilon}\left\|P_0u\right\|^2_{W_2^1(R_+;H)}$$

$$=\left(1-\frac{\cos 2\varepsilon}{2\cos^2\varepsilon}\right)\left\|P_0u\right\|^2_{W_2^1(R_+;H)}$$

$$=\frac{2\cos^2\varepsilon-\cos 2\varepsilon}{2\cos^2\varepsilon}\left\|P_0u\right\|^2_{W_2^1(R_+;H)}=\frac{1}{2\cos^2\varepsilon}\left\|P_0u\right\|^2_{W_2^1(R_+;H)}.$$

Consequently, for $\pi/4\le\varepsilon\le\pi/2$ we have

$$\left\|A^2u\right\|_{W_2^1(R_+;H)}\le\frac{1}{\sqrt{2}\cos\varepsilon}\left\|P_0u\right\|_{W_2^1(R_+;H)}.$$

Thus, for $0\le\varepsilon<\pi/2$

$$\left\|A^2u\right\|_{W_2^1(R_+;H)}\le c_0(\varepsilon)\left\|P_0u\right\|_{W_2^1(R_+;H)}.$$

The theorem is proved.

Theorem 2. *The operator P_0 isomorphically maps the space $\overset{\circ}{W}{}_2^3(R_+;H)$ onto $W_2^1(R_+;H)$.*

Proof. As the equation $-u''(t)+A^2u(t)=0$ has the general solution $u_0(t)$ from $W_2^3(R_+;H)$ in the form $u_0(t)=e^{-tA}\varphi$, $\varphi\in H_{5/2}$, then from the condition $u'(0)=0$ it follows $-A\varphi=0$ i.e. $\varphi=0$. Consequently, for $u_0(t)=0$ i.e. $KerP_0=\{0\}$. Show that the equation $P_0u=f$ has a solution for any $f\in W_2^1(R_+;H)$. As is known [1], if $f\in W_2^1(R_+;H)$, then we can continue it in the right semi-axis so that the continuation $f_1(t)\in W_2^1(R;H)$ and $\|f_1\|_{W_2^1(R_+;H)}\le const\,\|f\|_{W_2^1(R;H)}$. Denote by $\widehat{f}_1(\lambda)$ the Fourier transformation of the function $f_1(t)$ and construct the vector-function

$$u_1(t)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}(\lambda^2E+A^2)^{-1}\widehat{f}_1(\lambda)e^{i\lambda t}d\lambda,\ \ t\in R.$$

Show that $u_1(t) \in W_2^3(R;H)$. By the Plancherel theorem

$$\|u_1\|^2_{W_2^3(R_+;H)} = \left\|C^3 u_1\right\|^2_{L_2(R;H)} + \|u_1'''\|^2_{L_2(R;H)}$$

$$= \left\|C^3 \widehat{u}_1(\lambda)\right\|^2 + \left\|\lambda^3 \widehat{u}_1(\lambda)\right\|^2_{L_2(R;H)}$$

$$= \left\|C^3(\lambda^2 E + A^2)^{-1}\widehat{f}_1(\lambda)\right\|^2_{L_2(R;H)} + \left\|\lambda^3(\lambda^2 E + A^2)^{-1}\widehat{f}_1(\lambda)\right\|^2_{L_2(R;H)}$$

$$\leq \sup_{\lambda\in R}\left\|C^2(\lambda^2 E + A^2)^{-1}\right\| \left\|C\widehat{f}_1(\lambda)\right\|^2_{L_2(R;H)}$$

$$+\sup_{\lambda\in R}\left\|\lambda^2(\lambda^2 E + A^2)^{-1}\right\| \left\|\lambda\widehat{f}_1(\lambda)\right\|_{L_2(R;H)}.$$

Since for each $\lambda \in R$

$$\left\|C^2(\lambda^2 E + A^2)^{-1}\right\| = \sup_{\substack{\mu\geq\mu_0\\ |\varphi|\leq\varepsilon}}\left|\mu^2(\lambda^2 + \mu^2 e^{2i\varphi})^{-1}\right|$$

$$= \sup_{\substack{\mu\geq\mu_0\\ |\varphi|\leq\varepsilon}}\left|\mu^2(\lambda^4 + \mu^4 + 2\lambda^2\mu^2\cos 2\varphi)^{-1/2}\right|$$

$$\leq \sup_{\mu\geq\mu_0}\left|\mu^2(\lambda^4 + \mu^4 + 2\lambda^2\mu^2\cos 2\varepsilon)^{-1/2}\right|.$$

Obviously, for $\cos 2\varepsilon \geq 0$ $(0 \leq \varepsilon \leq \pi/4)$

$$\left\|C^2(\lambda^2 E + A^2)^{-1}\right\| \leq \sup_{\mu\geq\mu_0}\left|\mu^2(\lambda^4 + \mu^4)^{-1/2}\right| \leq 1,$$

and for $\cos 2\varepsilon \leq 0$ the function $(\pi/4 \leq \varepsilon < \pi/2)$

$$\sup_{\mu\geq\mu_0}\left|\mu^2(\lambda^4 + \mu^4 + 2\lambda^2\mu^2\cos 2\varepsilon)^{-1/2}\right|$$

$$\leq \sup_{\tau\geq 0}\left|\left(\tau^4 + 1 + 2\tau^2\cos 2\varepsilon\right)^{-1/2}\right| \leq \frac{1}{\sin 2\varepsilon},$$

therefore $\sup\left\|C^2\left(\lambda^2 E + A^2\right)^{-1}\right\| \leq K_0(\varepsilon)$, where $K_0(\varepsilon) = 1$ for $0 \leq \varepsilon \leq \pi/4$ and $K_0(\varepsilon) = (\sin 2\varepsilon)^{-1}$ for $\pi/4 \leq \varepsilon < \pi/2$.

Similarly we get

$$\sup_{\lambda}\left|\lambda^2(\lambda^2 E + A^2)^{-1}\right| \leq K_2(\varepsilon) = const.$$

Consequently, $u_1 \in W_2^3(R;H)$ and $\|u_1\|_{W_2^3(R;H)} \leq const\,\|f\|_{W_2^1(R;H)} \leq$
$\leq const\,\|f\|_{W_2^1(R_+;H)}$.

Obviously, $u_1(t)$ satisfies the equation $-u''(t) + A^2u(t) = f(t)$ for $t \in R_+$ almost everywhere. But the both hand sides of the equation are continuous functions, then $u_1(t)$ satisfy the given equation identically in R_+. Contraction $u_1(t)$ on $[0,\infty)$ belongs to the space $W_2^3(R_+;H)$. Denote this continuation by $\xi_1(t) \in W_2^3(R_+;H)$. Hence, from the theorem on traces it follows $\xi_1^{(j)}(0) \in H_{3-j-1/2}$, $j = 0,1,2$ moreover $\left\|\xi_1^{(j)}(0)\right\|_{3-j-1/2} \leq const\,\|f\|_{W_2^1(R_+;H)}$, $j = \overline{0,2}$. Now we will look for the solution of the equation $P_0u = f$ in the form

$$u(t) = \xi_1(t) + e^{-tA}\varphi,$$

where $\varphi \in H_{5/2}$ is a still unknown vector. From condition $u'(0) = 0$ it follows that $-A\varphi + \xi_1'(0) = 0$ i.e. $\varphi = A^{-1}\xi_1'(0) \in H_{5/2}$. Thus,

$$u(t) = \xi_1(t) + A^{-1}e^{-tA}\xi_1'(0) \in \overset{\circ}{W}{}_2^3(R_+;H).$$

On the other hand,

$$\|u_1\|_{W_2^3(R_+;H)} \leq \|\xi_1\|_{W_2^3(R_+;H)} + \left\|A^{-1}e^{-tA}\xi'(0)\right\|_{W_2^3(R_+;H)}$$

$$\leq \|u_1\|_{W_2^3(R_+;H)} + const\left\|\xi'(0)\right\|_{3/2} \leq const\,\|f\|_{W_2^1(R_+;H)}\,.$$

The theorem is proved.

Now prove the basic theorem.

Theorem 3. *Let conditions 1) and 2) be fulfilled, and*

$$\alpha(\varepsilon) = c_1(\varepsilon)\max\left(\|A_1\|_{H_1\to H}\,, \|A_1\|_{H_2\to H_1}\right)$$

$$+c_0(\varepsilon)\left(\|A_2\|_{H_2\to H}\,, \|A_1\|_{H_3\to H_1}\right) < 1$$

where $c_0(\varepsilon)$ and $c_1(\varepsilon)$ were determined from theorem 1. Then problem (1), (2) is well-defined solvable in $W_2^1(R_+;H)$.

Proof. Write problem (1), (2) in the form of the equation $Pu = P_0u + P_1u = f$, where $f \in W_2^1(R_+;H)$, $u \in W_2^3(R_+;H)$. By theorem 2, the operator $P_0^{-1} : W_2^1(R_+;H) \to W_2^3(R_+;H)$ is an isomorphism. Assume $u = P_0^{-1}\omega$, then we get the equation $\omega + P_1P_0^{-1}\omega = f$ in $W_2^1(R_+;H)$. On the other hand, for any $\omega \in W_2^1(R_+;H)$ there hold the inequalities

$$\left\|P_1P_0^{-1}\omega\right\|_{W_2^1(R_+;H)} = \|P_1u\|_{W_2^1(R_+;H)} = \left\|A_1u' + A_2u\right\|_{W_2^1(R_+;H)}$$

$$\leq \left\|A_1u'\right\|_{W_2^1(R_+;H)} + \|A_2u\|_{W_2^1(R_+;H)}$$

$$= \left(\left\|A_1u''\right\|^2_{L_2(R_+;H)} + \left\|AA_1u'\right\|^2_{L_2(R_+;H)}\right)^{1/2}$$

$$+\left(\left\|A_2u'\right\|^2_{L_2(R_+;H)} + \|AA_2u\|^2_{L_2(R_+;H)}\right)^{1/2} \leq \max\left(\|A_1\|_{H_1\to H}\,, \|A_1\|_{H_2\to H_1}\right)$$

$$\times\left(\left\|A^2u'\right\|^2_{L_2(R_+;H)} + \left\|Au''\right\|^2_{L_2(R_+;H)}\right)^{1/2} + \max\left(\|A_2\|_{H_2\to H}\,, \|A_2\|_{H_3\to H_1}\right)$$

$$\times\left(\left\|A^2u'\right\|^2_{L_2(R_+;H)} + \left\|A^3u\right\|^2_{L_2(R_+;H)}\right)^{1/2}.$$

Using theorem 1, we get

$$\left\|P_1P_0^{-1}\omega\right\|_{W_2^1(R_+;H)} \leq \left(c_1(\varepsilon)\max\left(\|A_1\|_{H_1\to H}\,, \|A_1\|_{H_2\to H_1}\right)\right.$$

$$\left.+c_0(\varepsilon)\max\left(\|A_2\|_{H_2\to H}\,, \|A_2\|_{H_3\to H_1}\right)\right)\|P_0u\|_{W_2^1(R_+;H)} = \alpha(\varepsilon)\,\|\omega\|_{W_2^1(R_+;H)}\,.$$

Thus, the operator $E + P_1P_0^{-1}$ is invertible in $W_2^1(R_+;H)$. Then

$$u = P_0^{-1}(E + P_1P_0^{-1})^{-1}f$$

and

$$\|u\|_{W_2^3(R_+;H)} \leq const\,\|f\|_{W_2^1(R_+;H)}\,.$$

The theorem is proved.

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