On Solvability Of A Boundary Value Problem For Second Order Operator-Differential Equations In The Space Of Smooth Vector-Functions

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Abstract. In the paper we find sufficient conditions that provide the existence and uniqueness of a boundary value problem for a class of operator differential equations of second order, that contain a normal operator in the principal part, in the space of smooth vector-functions. These conditions are expressed by the properties of operator coefficients. We also obtain the estimations of the norms of intermediate derivatives in the space of smooth vector-functions.

Keywords. Hilbert space, operator-differential equation, boundary value problem, smooth solution.

Consider in separable Hilbert space the boundary value problem

\[ -u''(t) + A^2 u(t) + A_1 u'(t) + A_2 u(t) = f(t), \quad t \in \mathbb{R}_+ = (0, \infty) \]

\[ u'(0) = 0, \]

where \( u(t), f(t) \) are the functions determined in \( \mathbb{R}_+ = (0, \infty) \) almost everywhere, with the values from \( H \), the derivatives are understood in the sense of distributions theory \([1]\), and operator coefficients of equation (1) satisfy the condition:

1) \( A \) is a normal operator with bounded inverse whose spectrum is contained in the angular sector

\[ S_\varepsilon = \{ \lambda : |\arg \lambda| \leq \varepsilon, \quad 0 \leq \varepsilon < \pi/2 \}; \]

2) The operators \( A_j \in L(H_j, H) \cap L(H_{j+1}, H_1), \quad j = 1, 2. \)

Here \( L(X, Y) \) is the space of bounded operators acting from the space \( X \) to the space \( Y \), and the space \( H_\gamma = D(C^\gamma), \quad \gamma \geq 0 \) \( (H_0 = H) \) with the norm \( \|x\|_\gamma = \|C^\gamma x\| \), where the operator \( C \) is a positive-definite self-adjoint operator from the representation of the operator \( A = UC \), and \( U \) is unitary operator.

Let \( L_2(R_+; H) \) be a Hilbert space of all functions \( f(t) \) determined in \( R_+ \) almost everywhere, measurable, with the values in \( H \) for which

\[ \|f\|_{L_2(R_+; H)} = \left( \int_0^\infty \|f(t)\|^2 \, dt \right)^{1/2}. \]

Following the monograph \([1]\), we introduce the following Hilbert space

\[ W^m_2(R_+; H) = \left\{ u : u^{(m)} \in L_2(R_+; H), \quad C^{(m)} u \in L_2(R_+; H) \right\}, \quad m = 1, 2, 3 \]

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with the norm
\[ \|u\|_{W^m_2(R_+; H)} = \left( \|u^{(m)}\|^2_{L_2(R_+; H)} + \|C^m u\|^2_{L_2(R_+; H)} \right)^{1/2}. \]

In connection with studying the problem (1), (2), consider the Hilbert space
\[ W^2_0(R_+; H) = \{ u : u \in W^2_2(R_+; H), u'(0) = 0 \}. \]

The spaces \( L_2(R; H) \) and \( W^m_2(R; H) \) are similarly determined for \( R = (-\infty, \infty) \).

**Definition 1.** If for \( f(t) \in W^1_2(R; H) \) there exists \( u(t) \in W^2_2(R_+; H) \) that satisfies equation (1) in \( R_+ \) identically, boundary condition (2) in the sense of convergence
\[ \lim_{t \to +0} \|u'(t)\|_{3/2} = 0 \]
and estimation
\[ \|u\|_{W^2_2(R_+; H)} \leq \text{const} \|f\|_{W^2_2(R_+; H)}, \]
we say that problem (1), (2) is well-defined solvable in \( W^2_2(R_+; H) \). Note that for \( f(t) \in L_2(R_+; H) \), \( u \in W^2_2(R_+; H) \) this problem was studied by many authors when \( A \) is a self-adjoint operator (see e.g. [2-4]), and when \( A \) is a normal operator in the paper [5].

Note that the existence of smooth solutions of boundary value problems when \( A \) is a self-adjoint operator, \( f \in W^1_2(R_+; H) \) while \( u \in W^2_2(R_+; H) \) was investigated in the papers [6-7], and for \( f \in W^1_2(R_+; H) \), \( u \in W^2_2(R_+; H) \) in the paper [8]. On the whole axis, the well-posedness of the equation (1) was studied in (9).

In the present paper we find the conditions on the coefficients of equation (1) that provide well-defined solvability of problem (1), (2) in \( W^2_2(R_+; H) \). Denote
\[ P_0u = -u'' + A^2u, \quad P_1u = A_1u' + A_2u, \quad u \in \tilde{W}^3_2(R_+; H). \]

At first we prove the following lemma.

**Lemma 1.** Let conditions 1) be fulfilled. Then for \( u \in \tilde{W}^3_2(R_+; H) \) it holds the inequality
\[ \|P_0u\|^2_{W^2_2(R_+; H)} \geq \|u\|^2_{W^2_2(R_+; H)} + (1 + 2 \cos 2\varepsilon) \|Au\|^2_{W^2_2(R_+; H)} \] \hspace{1cm} (3)

**Proof.** Let \( u \in \tilde{W}^3_2(R_+; H) \) (\( u'(0) = 0 \)). Then
\[ \|P_0u\|^2_{W^2_2(R_+; H)} = \left\| -u''' + A^2u' \right\|^2_{L_2(R_+; H)} + \left\| -Cu''' + C^2A^2u \right\|^2_{L_2(R_+; H)} \]
\[ = \left\| u''' \right\|^2_{L_2(R_+; H)} + \left\| A^2u' \right\|^2_{L_2(R_+; H)} - 2\Re(u'''A^2u')_{L_2(R_+; H)} + \left\| Cu''' \right\|^2_{L_2(R_+; H)} + \left\| C^2A^2u \right\|^2_{L_2(R_+; H)} - 2\Re(Cu'''C^2A^2u)_{L_2(R_+; H)}. \] \hspace{1cm} (4)

After integrating by parts, we get
\[ -2\Re(u'''A^2u')_{L_2(R_+; H)} = 2\Re(A^*u'''A^2u')_{L_2(R_+; H)} \geq 2 \cos 2\varepsilon \left\| Cu'' \right\|^2_{L_2(R_+; H)} \] \hspace{1cm} (5)
and
\[ -2\Re(Cu'''C^2A^2u)_{L_2(R_+; H)} = 2\Re(C^2u''A^2u')_{L_2(R_+; H)} \geq 2 \cos 2\varepsilon \left\| C^2u' \right\|^2_{L_2(R_+; H)}. \] \hspace{1cm} (6)

Taking into account inequalities (5) and (6) in equality (4), we complete the proof of the lemma.
Now estimate the intermediate derivatives in the space $W^2_2(R^+;H)$. It holds the following

**Theorem 1.** Let condition 1) be fulfilled. Then for $u \in \tilde{W}^3_2(R^+;H)$ it holds the inequality

\[
\left\| A^{2-j}u^{(j)} \right\|_{W^2_2(R^+;H)} \leq c_j(\varepsilon) \| P_0 u \|_{W^2_2(R^+;H)}, \quad j = 0, 1
\]

where

\[
c_0(\varepsilon) = \begin{cases} 1, & 0 \leq \varepsilon \leq \pi/4 \\ \frac{1}{\sqrt{2 \cos \varepsilon}}, & \pi/4 \leq \varepsilon < \pi/2 \\ 0, & 0 \leq \varepsilon < \pi/2. \end{cases}
\]

\[
c_1(\varepsilon) = \frac{1}{2 \cos \varepsilon}, \quad 0 \leq \varepsilon < \pi/2.
\]

**Proof.** Let $j = 1$. Then for $u \in \tilde{W}^3_2(R^+;H)$ we have:

\[
\| Au' \|_{W^2_2(R^+;H)}^2 = \| Cu' \|_{W^2_2(R^+;H)}^2 = \| C^2 u' \|_{L^2(R^+;H)}^2 + \| Cu'' \|_{L^2(R^+;H)}^2
\]

\[
= \left( C^2 u', C^2 u' \right)_{L^2(R^+;H)} + (Cu'', Cu'')_{L^2(R^+;H)}.
\]

After integrating by parts, we have

\[
\left( C^2 u', C^2 u' \right)_{L^2(R^+;H)} = \left( C^2 u, Cu'' \right)_{L^2(R^+;H)} \leq \frac{1}{2} \left( \| C^2 u \|_{L^2(R^+;H)}^2 + \| Cu'' \|_{L^2(R^+;H)}^2 \right)
\]

and

\[
\left( Cu'', Cu'' \right)_{L^2(R^+;H)} = \left( C^2 u', u''' \right)_{L^2(R^+;H)} \leq \frac{1}{2} \left( \| u''' \|_{L^2(R^+;H)}^2 + \| C^2 u' \|_{L^2(R^+;H)}^2 \right).
\]

Consequently,

\[
\| Au' \|_{W^2_2(R^+;H)}^2 \leq \frac{1}{2} \| u \|_{W^2_2(R^+;H)}^2 + \frac{1}{2} \| Cu'' \|_{W^2_2(R^+;H)}^2
\]

\[
= \frac{1}{2} \| u \|_{W^2_2(R^+;H)}^2 + \frac{1}{2} \| Au' \|_{W^2_2(R^+;H)}^2.
\]

Thus,

\[
\| Au' \|_{W^2_2(R^+;H)}^2 \leq \| u \|_{W^2_2(R^+;H)}^2.
\]

Using lemma 1, we get

\[
\| Au' \|_{W^2_2(R^+;H)}^2 \leq \| P_0 u \|_{W^2_2(R^+;H)}^2 - (1 + 2 \cos 2\varepsilon) \| Au' \|_{W^2_2(R^+;H)}^2.
\]

Consequently,

\[
2(1 + \cos 2\varepsilon) \| Au' \|_{W^2_2(R^+;H)}^2 \leq \| P_0 u \|_{W^2_2(R^+;H)}^2,
\]

i.e.

\[
\| Au' \|_{W^2_2(R^+;H)}^2 \leq \frac{1}{2 \cos \varepsilon} \| P_0 u \|_{W^2_2(R^+;H)}^2, \quad 0 \leq \varepsilon < \pi/2.
\]

Now prove the inequality in the case when $j = 0$. Obviously,

\[
\| A^2 u \|_{W^2_2(R^+;H)}^2 = \| C^2 u \|_{W^2_2(R^+;H)}^2 = \| C^2 u' \|_{L^2(R^+;H)}^2 + \| Cu'' \|_{L^2(R^+;H)}^2
\]

and

\[
\| u'' \|_{W^2_2(R^+;H)}^2 = \| Cu'' \|_{L^2(R^+;H)}^2 + \| u''' \|_{L^2(R^+;H)}^2.
\]
Theorem 2. The operator \( P_0 \) isomorphically maps the space \( W^2_2(R_+; H) \) onto \( W^{\frac{3}{2}}_2(R_+; H) \).

Proof. As the equation \(-u''(t) + A^2 u(t) = 0\) has the general solution \( u_0(t) \) from \( W^2_2(R_+; H) \) in the form \( u_0(t) = e^{-tA}\varphi, \varphi \in H_{\frac{3}{2}} \), then from the condition \( u'(0) = 0 \) it follows \(-A\varphi = 0 \) i.e. \( \varphi = 0 \). Consequently, for \( u_0(t) = 0 \) i.e. \( kerP_0 = \{0\} \). Show that the equation \( P_0 u = f \) has a solution for any \( f \in W^2_2(R_+; H) \). As is known [1], if \( f \in W^2_2(R_+; H) \), then we can continue it in the right semi-axis so that the continuation \( f_1(t) \in W^2_2(R; H) \) and \( \|f_1\|_{W^2_2(R; H)} \leq const \| f \|_{W^2_2(R; H)} \). Denote by \( f_1(\lambda) \) the Fourier transformation of the function \( f_1(t) \) and construct the vector-function

\[
u_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\lambda^2 E + A^2)^{-1} f_1(\lambda) e^{i\lambda t} d\lambda, \quad t \in R.
\]
Show that \( u_1(t) \in W_2^2(R; H) \). By the Plancherel theorem

\[
\| u_1 \|_{W_2^2(R; H)}^2 = \left\| C^3 u_1 \right\|_{L_2(R; H)}^2 + \left\| u''_1 \right\|_{L_2(R; H)}^2 = \left\| C^3 \overline{u}_1 (\lambda) \right\|^2_{L_2(R; H)} + \left\| \lambda^3 \overline{u}_1 (\lambda) \right\|^2_{L_2(R; H)}
\]

\[
= \left\| C^3(\lambda^2 E + A^2)^{-1} f_1 (\lambda) \right\|_{L_2(R; H)}^2 + \left\| \lambda^3(\lambda^2 E + A^2)^{-1} f_1 (\lambda) \right\|^2_{L_2(R; H)}
\]

\[
\leq \sup_{\lambda \in R} \left\| C^2(\lambda^2 E + A^2)^{-1} \right\| \left\| C f_1 (\lambda) \right\|_{L_2(R; H)}^2 + \sup_{\lambda \in R} \left\| \lambda^2(\lambda^2 E + A^2)^{-1} \right\| \left\| \lambda f_1 (\lambda) \right\|_{L_2(R; H)}^2.
\]

Since for each \( \lambda \in R \)

\[
\left\| C^2(\lambda^2 E + A^2)^{-1} \right\| \leq \sup_{\mu \geq \mu_0} \left\| \mu^2(\lambda^4 + \mu^2)\right\| \leq 1,
\]

and for \( \cos 2\varepsilon \leq 0 \) the function \( (\pi/4 \leq \varepsilon < \pi/2) \)

\[
\sup_{\mu \geq \mu_0} \left\| \mu^2(\lambda^4 + \mu^2)\right\| \leq \frac{1}{\sin 2\varepsilon},
\]

therefore

\[
\sup_{\lambda \in R} \left\| \lambda^2(\lambda^2 E + A^2)^{-1} \right\| \leq K_2(\varepsilon), \text{ where } K_0(\varepsilon) = 1 \text{ for } 0 \leq \varepsilon \leq \pi/4 \text{ and } K_0(\varepsilon) = (\sin 2\varepsilon)^{-1} \text{ for } \pi/4 \leq \varepsilon < \pi/2.
\]

Similarly we get

\[
\sup_{\lambda} \left\| \lambda^2(\lambda^2 E + A^2)^{-1} \right\| \leq K_2(\varepsilon) = \text{const}.
\]

Consequently, \( u_1 \in W_2^2(R; H) \) and

\[
\| u_1 \|_{W_2^2(R; H)} \leq \text{const} \| f \|_{W_2^2(R; H)} \leq \text{const} \| f \|_{W_2^2(R; H)}.
\]

Obviously, \( u_1(t) \) satisfies the equation \(-u''(t) + A^2u(t) = f(t)\) for \( t \in R_+ \) almost everywhere. But the both hand sides of the equation are continuous functions, then \( u_1(t) \) satisfy the given equation identically in \( R_+ \). Contraction \( u_1(t) \) on \([0, \infty)\) belongs to the space \( W_2^2(R_+; H) \). Denote this continuation by \( \xi_1(t) \in W_2^2(R_+; H) \). Hence, from the theorem on traces it follows \( \xi_1^{(j)}(0) \in H_{3-j-1/2}, j = 0, 1, 2 \) moreover

\[
\| \xi_1^{(j)}(0) \|_{H_{3-j-1/2}} \leq \text{const} \| f \|_{W_2^2(R_+; H)}, \quad j = 0, 1, 2.
\]

Now we will look for the solution of the equation \( Pu = f \) in the form

\[
u(t) = \xi_1(t) + e^{-tA} \varphi,
\]

where \( \varphi \in H_{5/2} \) is a still unknown vector. From condition \( u'(0) = 0 \) it follows that \(-A \varphi + \xi_1(0) = 0\) i.e. \( \varphi = A^{-1} \xi_1(0) \in H_{5/2} \). Thus,

\[
u(t) = \xi_1(t) + A^{-1} e^{-tA} \xi_1(0) \in W_2^3(R_+; H).
\]
On the other hand, 
\[
\|u_1\|_{W_2^2(R^+; H)} \leq \|\xi_1\|_{W_2^2(R^+; H)} + \left\|A^{-1}e^{-tA}\xi(0)\right\|_{W_2^2(R^+; H)}
\]
\[
\leq \|u_1\|_{W_2^2(R^+; H)} + \text{const} \left\|\xi(0)\right\|_{3/2} \leq \text{const} \|f\|_{W_2^2(R^+; H)}.
\]

The theorem is proved.

Now prove the basic theorem.

**Theorem 3.** Let conditions 1) and 2) be fulfilled, and
\[
\alpha(\varepsilon) = c_1(\varepsilon) \max \left\{\|A_1\|_{H_1 \rightarrow H}, \|A_1\|_{H_2 \rightarrow H}\right\}
\]
\[
+ c_0(\varepsilon) \left(\|A_2\|_{H_2 \rightarrow H}, \|A_1\|_{H_2 \rightarrow H}\right) < 1
\]

where \(c_0(\varepsilon)\) and \(c_1(\varepsilon)\) were determined from theorem 1. Then problem (1), (2) is well-defined solvable in \(W_2^2(R^+; H)\).

**Proof.** Write problem (1), (2) in the form of the equation \(Pu = P_0u + P_1u = f\), where \(f \in W_2^2(R^+; H), u \in W_2^2(R^+; H)\). By theorem 2, the operator \(P_0^{-1} : W_2^2(R^+; H) \rightarrow W_2^2(R^+; H)\) is an isomorphism. Assume \(u = P_0^{-1}\omega\), then we get the equation \(\omega + P_1P_0^{-1}\omega = f\) in \(W_2^2(R^+; H)\). On the other hand, for any \(\omega \in W_2^2(R^+; H)\) there hold the inequalities
\[
\left\|P_1P_0^{-1}\omega\right\|_{W_2^2(R^+; H)} = \|P_1u\|_{W_2^2(R^+; H)} = \|A_1u' + A_2u\|_{W_2^2(R^+; H)}
\]
\[
\leq \|A_1u'\|_{W_2^2(R^+; H)} + \|A_2u\|_{W_2^2(R^+; H)}
\]
\[
= \left(\|A_1u''\|^2_{L_2(R^+; H)} + \|AA_1u'\|^2_{L_2(R^+; H)}\right)^{1/2}
\]
\[
+ \left(\|A_2u''\|^2_{L_2(R^+; H)} + \|AA_2u'\|^2_{L_2(R^+; H)}\right)^{1/2} \leq \max \left(\|A_1\|_{H_1 \rightarrow H}, \|A_1\|_{H_2 \rightarrow H}\right)
\]
\[
\times \left(\|A_1u''\|^2_{L_2(R^+; H)} + \|AA_1u'\|^2_{L_2(R^+; H)}\right)^{1/2} + \max \left(\|A_2\|_{H_2 \rightarrow H}, \|A_2\|_{H_2 \rightarrow H}\right)
\]
\[
\times \left(\|A_2u''\|^2_{L_2(R^+; H)} + \|AA_2u'\|^2_{L_2(R^+; H)}\right)^{1/2}.
\]

Using theorem 1, we get
\[
\left\|P_1P_0^{-1}\omega\right\|_{W_2^2(R^+; H)} \leq \left(\|A_1\|_{H_1 \rightarrow H}, \|A_1\|_{H_2 \rightarrow H}\right)
\]
\[
+ c_0(\varepsilon) \max \left(\|A_2\|_{H_2 \rightarrow H}, \|A_2\|_{H_2 \rightarrow H}\right) \|P_0u\|_{W_2^2(R^+; H)} = \alpha(\varepsilon) \|\omega\|_{W_2^2(R^+; H)}.
\]

Thus, the operator \(E + P_1P_0^{-1}\) is invertible in \(W_2^2(R^+; H)\). Then
\[
u = P_0^{-1}(E + P_1P_0^{-1})^{-1}f
\]
and
\[
\|u\|_{W_2^2(R^+; H)} \leq \text{const} \|f\|_{W_2^2(R^+; H)}.
\]

The theorem is proved.
References