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On Well-Defined Solvability Of A Boundary Value Problem For An Elliptic Differential Equation In Hilbert Space

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Abstract. *In the paper, on a semi-axis a well-defined solvability of a boundary value problem for a second order elliptic operator- differential equation with discontinuous coefficient was established. The estimates of the norms of intermediate derivates having close relation with solvability conditions were precised. Note that the principal part of the equation under consideration contains a normal operator and the boundary condition has some abstract operator.*

Let H be a separable Hilbert space, A a normal operator with completely continuous inverse A^{-1} and its spectrum be contained in the angular sector

$$S_\varepsilon = \{\lambda : |\arg \lambda| \leq \varepsilon\},\ 0 \leq \varepsilon < \pi/2.$$

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormed system of eigen vectors of the operator A i.e.

$$Ae_n = \lambda_n e_n,\ (e_n, e_m) = \delta_{n,m} = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases}\ \lambda_n \in S_\varepsilon.$$

Then the operator A is represented in the form $A = UC$, where

$$Cx = \sum_{n=1}^{\infty} \mu_n (x, e_n) e_n,\ x \in D(A) = D(C),\ \mu_n = |\lambda_n|, 0 < \mu_0 \leq \mu_1 \leq \mu_2 \leq ...,$$

and

$$Uy = \sum_{n=1}^{\infty} e^{i\varphi_n} (y, e_n) e_n,\ y \in H,\ \varphi_n = \arg \lambda_n,\ |\varphi_n| \leq \varepsilon.$$

Obviously, C is a positive-definite self-adjoint operator ($C = C^* \geq cE, c > 0, E$ is a unique operator), and U is a unitary operator.

Denote by H_γ a Hilbert space $H_\gamma = D(A^\gamma)$ with the norm $\|x\|_\gamma = \|C^\gamma x\|$. For $\gamma = 0$ we asume $H_0 = H$.

Let $L_2(R_+; H)$ be a Hilbert space of vector-functions $f(t)$, determined in $R_+ = (0, +\infty)$ almost everywhere, with the values in H, with the finite norm

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|f(t)\|^2 dt \right)^{1/2}.$$

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Further, following the monograph [1] define the Hilbert space

$$W_2^2(R_+;H) = \{u : u'' \in L_2(R_+;H),\ A^2u \in L_2(R_+;H)\}$$

with the norm

$$\|u\|_{W_2^2(R_+;H)} = \left(\|u''\|^2_{L_2(R_+;H)} + \left\|A^2u\right\|^2_{L_2(R_+;H)}\right)^{1/2}.$$

Denote by $L(X,Y)$ a space of linear bounded operators acting from the Hilbert space X to the Hilbert space Y.

Suppose that the linear operator $T \in L\left(H_{1/2}, H_{3/2}\right)$ and consider the subspace

$$W_{2,T}^2(R_+;H) = \{u : u \in W_2^2(R_+;H),\ u(0) = Tu'(0)\}$$

of the space $W_2^2(R_+;H)$. From the traces theorem it follows that $W_{2,T}^2(R_+;H)$ is a complete Hilbert space.

Consider in the Hilbert space H the boundary value problem

$$P(d/dt)\,u(t) = -u''(t) + \rho(t)A^2u(t) + A_1u'(t) + A_2u(t) = f(t),\ t \in R_+, \tag{1}$$

$$u(0) = Tu'(0), \tag{2}$$

where the operator coefficients satisfy the following conditions:

1) A is a normal operator with completely inverse A^{-1} and its spectrum is contained in the angular sector

$$S_\varepsilon = \{\lambda : |\arg\lambda| \le \varepsilon\},\ 0 \le \varepsilon < \pi/2;$$

2) $\rho(t)$ is number function such that

$$\rho(t) = \begin{cases} \alpha^2,\ 0 < t < 1, \\ \beta^2,\ 1 < t < \infty, \end{cases} \alpha, \beta > 0,$$

in what follows for simplicity we assume $\alpha < \beta$;

3) $T \in L\left(H_{1/2}, H_{3/2}\right)$;

4) $B_1 = A_1A^{-1}, B_2 = A_2A^{-2}$ are bounded operators in H.

Define the following operators acting from $W_{2,T}^2(R_+;H)$ to $L_2(R_+;H)$:

$$P_0u = -u'' + \rho(t)A^2u,\ P_1u = A_1u' + A^2u,\ u \in W_{2,T}^2(R_+;H).$$

From the inequality $\|P_0u\|^2_{L_2(R_+;H)} \le 2\max(1,\beta^4)\,\|u\|^2_{W_2^2(R_+;H)}$ and condition 4) subject to the theorem on intermediate derivatives it follows that the both operators P_0 and P_1 are linear bounded operators acting from $W_{2,T}^2(R_+;H)$ to $L_2(R_+;H)$, i.e. $P_0, P_1 \in L(W_{2,T}^2(R_+;H), L_2(R_+;H))$ [9].

The following theorem is easily proved.

Theorem 1 [9]. *Let conditions 1)-3) be fulfilled, and the operator*

$$L = \left(E - \frac{\beta-\alpha}{\beta+\alpha}\right)e^{-2\alpha A} + AT\left(E + \frac{\beta-\alpha}{\beta+\alpha}\right)e^{-2\alpha A},$$

be invertible in the space $H_{1/2}$. Then the operator P_0 realizes an isomorphism between the spaces $W_{2,T}^2(R_+;H)$ and $L_2(R_+;H)$.

Now we prove the following theorem on the estimation of the norms of operators of intermediate derivatives

Theorem 2. *Let conditions 1)-3),be fulfilled* $\mathrm{Re}UAT \ge 0$ *and* $\mathrm{Re}CT \ge 0$ *in $H_{1/2}$. Then for any $u \in W_{2,T}^2(R_+;H)$ the following inequalities hold:*

$$\left\|A^2u\right\|_{L_2(R_+;H)} \le c_0(\varepsilon)\,\|P_0u\|_{L_2(R_+;H)}, \tag{3}$$

$$\left\| Au' \right\|_{L_2(R_+;H)} \le c_1(\varepsilon) \left\| P_0 u \right\|_{L_2(R_+;H)}, \tag{4}$$

where

$$c_0(\varepsilon) = \begin{cases} \frac{1}{\alpha^2}, & 0 \le \varepsilon \le \pi/4, \\ \frac{1}{\sqrt{2}\alpha^2} \frac{1}{\cos\varepsilon}, & \pi/4 \le \varepsilon < \pi/2, \end{cases}$$

$$c_1(\varepsilon) = \frac{1}{2\alpha} \frac{1}{\cos\varepsilon}, 0 \le \varepsilon < \pi/2.$$

Proof. Let $u \in W^2_{2,T}(R_+;H)$ .Then from the equality

$$P_0 u = -u'' + \rho(t) A^2 u$$

we get

$$\left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(R_+;H)} = \left\| -\rho^{-1/2} u'' + \rho^{1/2} A^2 u \right\|^2_{L_2(R_+;H)}$$

$$= \left\| \rho^{-1/2} u'' \right\|^2_{L_2(R_+;H)} + \left\| \rho^{1/2} A^2 u \right\|^2_{L_2(R_+;H)} - 2\mathrm{Re}(u'', A^2 u)_{L_2(R_+;H)}. \tag{5}$$

On the other hand , integrating by parts and using spectral expansion of the operator A we have:

$$-\mathrm{Re}(u'', A^2 u)_{L_2(R_+;H)} = -2\mathrm{Re} \int_0^\infty (u'', A^2 u) dt$$

$$= \left(C^{1/2} u'(0), U^2 C^{3/2} u(0) \right) + 2\mathrm{Re} \int_0^\infty (A^* u', Au') dt$$

$$= (u'(0), UATu'(0))_{1/2} + 2\cos 2\varepsilon (Au', Au')_{L_2(R_+;H)} \ge 2\cos 2\varepsilon \left\| Au' \right\|^2_{L_2(R_+;H)}. \tag{6}$$

Then from equality (5), with regard to inequality (6) it follows

$$\left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(R_+;H)} \ge \left\| \rho^{-1/2} u'' \right\|^2_{L_2(R_+;H)}$$

$$+ \left\| \rho^{1/2} A^2 u'' \right\|^2_{L_2(R_+;H)} + 2\cos 2\varepsilon \left\| Au' \right\|^2_{L_2(R_+;H)}. \tag{7}$$

Further we have:

$$\left\| Au' \right\|^2_{L_2(R_+;H)} = \left\| Cu' \right\|^2_{L_2(R_+;H)} = (Cu', Cu')_{L_2(R_+;H)}$$

$$= \int_0^\infty (Cu', Cu') dt = -(C^{1/2} u'(0), C^{3/2} u(0)) - \int_0^\infty (u'', C^2 u) dt$$

$$= -(C^{1/2} u'(0), C^{1/2} CTu'(0)) - (\rho^{-1/2} u'', \rho^{1/2} C^2 u)_{L_2(R_+;H)}$$

$$= -(u'(0), CTu'(0))_{1/2} - (\rho^{-1/2} u'', \rho^{1/2} C^2 u)_{L_2(R_+;H)}.$$

Hence, using inequality (7) we get:

$$\left\| Au' \right\|^2_{L_2(R_+;H)} = \left\| Cu' \right\|^2_{L_2(R_+;H)} = -\mathrm{Re}(u'(0), CTu'(0))_{1/2}$$

$$-\mathrm{Re}(\rho^{-1/2} u'', \rho^{1/2} C^2 u)_{L_2(R_+;H)} \le -\mathrm{Re}(\rho^{-1/2} u'', \rho^{1/2} C^2 u)_{L_2(R_+;H)}$$

$$\le \left\| \rho^{-1/2} u'' \right\|_{L_2(R_+;H)} \left\| \rho^{1/2} C^2 u \right\|_{L_2(R_+;H)}$$

$$\le \frac{1}{2} \left(\left\| \rho^{-1/2} u'' \right\|^2_{L_2(R_+;H)} + \left\| \rho^{1/2} C^2 u \right\|^2_{L_2(R_+;H)} \right).$$

Then from inequality (7) it follows that

$$\left\|Au'\right\|^2_{L_2(R_+;H)} \leq \frac{1}{2}\left(\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R_+;H)} - 2\cos 2\varepsilon \left\|Au'\right\|^2_{L_2(R_+;H)}\right).$$

Hence we get

$$(1+\cos 2\varepsilon)\left\|Au'\right\|^2_{L_2(R_+;H)} \leq \frac{1}{2}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R_+;H)}$$

or

$$\left\|Au'\right\|_{L_2(R_+;H)} \leq \frac{1}{2\cos\varepsilon}\left\|\rho^{-1/2}P_0u\right\|_{L_2(R_+;H)} \leq \frac{1}{2\cos\varepsilon}\max_t \rho^{-1/2}\left\|P_0u\right\|_{L_2(R_+;H)}$$

$$\leq \frac{1}{2\cos\varepsilon}\frac{1}{\alpha}\left\|P_0u\right\|_{L_2(R_+;H)} = c_1(\varepsilon)\left\|P_0u\right\|_{L_2(R_+;H)},$$

where $c_1(\varepsilon) = \frac{1}{2\cos\varepsilon}\frac{1}{\alpha}$, $0 \leq \varepsilon < \pi/2$. Consequently, inequality (4) is proved.

Prove inequality (3). Let $0 \leq \varepsilon \leq \pi/4$. Then $\cos 2\varepsilon \geq 0$ and from (7) it follows

$$\left\|\rho^{1/2}A^2u\right\|_{L_2(R_+;H)} \leq \left\|\rho^{-1/2}P_0u\right\|_{L_2(R_+;H)}.$$

Hence we get:

$$\left\|A^2u\right\|_{L_2(R_+;H)} = \left\|\rho^{-1/2}\rho^{1/2}A^2u\right\|_{L_2(R_+;H)} \leq \max_t \rho^{-1/2}(t)\left\|\rho^{1/2}A^2u\right\|_{L_2(R_+;H)}$$

$$\leq \frac{1}{\alpha}\left\|\rho^{-1/2}P_0u\right\|_{L_2(R_+;H)} \leq \frac{1}{\alpha^2}\left\|P_0u\right\|_{L_2(R_+;H)}.$$

Let $\pi/4 \leq \varepsilon < \pi/2$. Then $\cos 2\varepsilon \leq 0$. In this case, from inequality (7) we get :

$$\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R_+;H)} \geq \left\|\rho^{-1/2}u''\right\|^2_{L_2(R_+;H)} + \left\|\rho^{1/2}A^2u\right\|^2_{L_2(R_+;H)}$$

$$+2\cos 2\varepsilon\left\|Au'\right\|^2_{L_2(R_+;H)} \geq \left\|\rho^{-1/2}u''\right\|^2_{L_2(R_+;H)} + \left\|\rho^{1/2}A^2u\right\|^2_{L_2(R_+;H)}$$

$$+2\cos 2\varepsilon\frac{1}{4\cos^2\varepsilon}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R_+;H)} = \left\|\rho^{-1/2}u''\right\|^2_{L_2(R_+;H)}$$

$$+\left\|\rho^{1/2}A^2u\right\|^2_{L_2(R_+;H)} + \frac{\cos 2\varepsilon}{2\cos^2\varepsilon}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R_+;H)}.$$

Thus

$$\left(1-\frac{\cos 2\varepsilon}{2\cos^2\varepsilon}\right)\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R_+;H)} \geq \left\|\rho^{-1/2}u''\right\|^2_{L_2(R_+;H)} + \left\|\rho^{1/2}A^2u\right\|^2_{L_2(R_+;H)}.$$

Consequently,

$$\frac{1}{2\cos^2\varepsilon}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(R_+;H)} \geq \left\|\rho^{1/2}A^2u\right\|^2_{L_2(R_+;H)}.$$

Hence we get

$$\left\|A^2u\right\|_{L_2(R_+;H)} \leq \frac{1}{\alpha}\left\|\rho^{1/2}A^2u\right\|_{L_2(R_+;H)} \leq \frac{1}{\alpha}\frac{1}{\sqrt{2}\cos\varepsilon}\left\|\rho^{-1/2}P_0u\right\|_{L_2(R_+;H)}$$

$$\leq \frac{1}{\alpha}\frac{1}{\sqrt{2}\cos\varepsilon}\frac{1}{\alpha}\left\|P_0u\right\|_{L_2(R_+;H)} = \frac{1}{\sqrt{2}\cos\varepsilon}\frac{1}{\alpha^2}\left\|P_0u\right\|_{L_2(R_+;H)}.$$

The theorem is proved.

Remark 1. It is easy to show that one can choose T so that the both dissipativity conditions in the condition of theorem 2 be fulfilled.

Indeed, let $T = A^{-1}$.Then $C = U^{-1}A,\ CT = U^{-1}AA^{-1} = U^{-1}$, $\mathrm{Re}U^{-1} \geq \cos\varepsilon E$. Obviously, $UAT = UAA^{-1} = U,\ \mathrm{Re}U > 0$.

Remark 2. Note that both of disssipativity conditions of the operators CT and UAT are independent.

Example 1^0. Let $T = A^{-1}U$, then $CT = U^{-1}\ AA^{-1}U = E > 0$ and $UAT = UAA^{-1}U = U^2$. But $U^2y = \sum\limits_{k=1}^{\infty} e^{2i\varphi_k}(y, e_k)e_k$, then

$$\mathrm{Re}UATy = \mathrm{Re}U^2y = \sum_{k=1}^{\infty} \cos 2\varphi_k(y, e_k)e_k \ ,$$

and $|\varphi_k| \leq \varepsilon < \pi/2$. Hence it follows that $\mathrm{Re}UAT \neq 0$. If $y = e_k$, then $\mathrm{Re}UAT = \cos 2\varphi_k < 0$ for $\varphi_k \in (\pi/4, \pi/2)$.

Example 2^0. Let $T = A^{-1}U^{-1}$, then $UAT = UAA^{-1}U^{-1} = E$, and

$$CT = CA^{-1}U^{-1} = U^{-1}AA^{-1}U^{-1} = U^{-2} = (U^*)^2.$$

Since $(U^*)^2y = \sum\limits_{k=1}^{\infty} e^{-2i\varphi_k}(y, e_k)e_k$, then $\mathrm{Re}(U^*)^2y = \sum\limits_{k=1}^{\infty} \cos 2\varphi_k(y, e_k)e_k$. And as $|\varphi_k| \leq \varepsilon < \pi/2$, then there may happen that $\cos 2\varphi_k < 0$. It φ_{k_0} is such that $\cos 2\varphi_{k_0} < 0$, then $\mathrm{Re}((U^*)^2e_{k_0}, e_{k_0}) = \cos 2\varphi_{k_0} < 0$.

Now, apply the obtained estimations from theorem 2 to the solvability of boundary value problem (1), (2).

Definition. *If for any $f(t) \in L_2(R_+; H)$ there exists a vector- function $u(t) \in W_2^2(R_+; H)$ satisfying equation (1) almost everywhere in R_+ and boundary condition (2) in the sense of convergence*

$$\lim_{t\to+0} \left\|u(t) - Tu'(t)\right\|_{3/2} = 0$$

and it holds the estimation $\|u\|_{W_2^2(R_+;H)} \leq const\, \|f\|_{L_2(R_+;H)}$, then boundary value problem (1),(2) is called regularly solvable, and $u(t)$ a regular solution of boundary value problem (1),(2).

Boundary value problem (1),(2) was investigated for $\rho(t) = 1, A = A^* \geq cE, c > 0$ and $T = 0$ for example in the papers [2,3], and for $T \neq 0$ in the paper [4]. When $\rho(t)$ satisfies condition 2) boundary value problem (1),(2) was studied in the papers [5,6] for $T = 0$ and A is a normal operator, in the paper [7] for $T \neq 0$ and $A = A^* \geq cE, c > 0$. In the paper [8] the similar problem for $A = A^* \geq cE, c > 0$ was considered for higher order equations with scalar coefficients in the boundary conditions. Note that for $0 \leq \varepsilon < \pi/4$ boundary value problem (1),(2) was studied in the author's paper [9].

The following theorem is valid.

Theorem 3. *Let conditions 1)-4) be fulfilled, operator L be invertible in $H_{1/2}$, $\mathrm{Re}UAT \geq 0$ and $\mathrm{Re}CT \geq 0$ in $H_{1/2}$ and also it hold the inequality*

$$q(\varepsilon) = c_1(\varepsilon) \|B_1\| + c_0(\varepsilon) \|B_2\| < 1,$$

where the numbers $c_0(\varepsilon)$ and $c_1(\varepsilon)$ were determined in theorem 2. Then problem (1),(2) is regularly solvable.

Proof. Note that the operator P_0 isomorphically maps $W_{2,T}^2(R_+; H)$ onto $L_2(R_+; H)$. Therefore, having written problem (1),(2) in the form of the equation $P_0u + P_1u = f$, where $u(t) \in W_{2,T}^2(R_+; H)$, $f(t) \in L_2(R_+; H)$ after substitution $P_0u = \omega$ we get the equation $\left(E + P_1P_0^{-1}\right)\omega = f$ in the space

$L_2(R_+;H)$. Then using the estimates of the norms of intermediate derivatives operators from theorem 2 we get

$$\left\|P_1P_0^{-1}\omega\right\|_{L_2(R_+;H)} = \|P_1u\|_{L_2(R_+;H)} \le \sum_{j=0}^{1}\left\|B_{2-j}\right\|\left\|A^{2-j}u^{(j)}\right\|_{L_2(R_+;H)}$$

$$\le \sum_{j=0}^{1}\left\|B_{2-j}\right\| c_j(\varepsilon)\,\|P_0u\|_{L_2(R_+;H)} = q(\varepsilon)\,\|\omega\|_{L_2(R_+;H)}\,.$$

Since $q(\varepsilon) < 1$ the operator $E + P_1P_0^{-1}$ is invertible in $L_2(R_+;H)$ and
$u = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1}f.$ Then

$$\|u\|_{W_2^2(R_+;H)} \le const\,\|f\|_{L_2(R_+;H)}\ .$$

The theorem is proved.

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