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Boundedness of the multi-sublinear maximal operator with rough kernels on Morrey spaces

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Abstract. *In this paper the boundedness of multi-sublinear maximal operator $M_{\Omega,m}$ with rough kernels on product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ are study. The authors established the product Morrey boundedness of the multi-sublinear maximal operator with kernel $\Omega\in L^s(\mathbb{S}^{n-1})$, $1<s\le\infty$.*

1 Introduction

The classical Morrey spaces, introduced by Morrey [9] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations. They appeared to be quite useful in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces. See [3, **?**,**?**] for details. The boundedness of Hardy-Littlewood maximal operator on the classical Morrey spaces was studied by Chiarenza and Frasca [2].

The multilinear theory has been well developed in the past twenty years. Let $m\ge 1$ will denote an integer, $\theta_j\,(j=1,...,m)$ will be fixed, distinct, and nonzero real numbers and we denote $\mathbf{f}=(f_1,...,f_m)$. Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega\in L^s(\mathbb{S}^{n-1})$ with $1<s\le\infty$, where $\mathbb{S}^{n-1}$ denote the unit sphere of $\mathbb{R}^n$. Then the multi-sublinear maximal operator $M_{\Omega,m}$ with rough kernel Ω

$$M_{\Omega,m}\mathbf{f}(x)=\sup_{r>0}\frac{1}{r^n}\int_{B(x,r)}|\Omega(y)|\prod_{j=1}^{m}|f_j(x-\theta_j y)|dy.$$

If $m=1$, then $M_\Omega\equiv M_{\Omega,1}$ is the maximal operator with rough kernel Ω. When $m=1$ and $\Omega\equiv 1$, then $M\equiv M_{1,1}$ is the classical Hardy-Littlewood maximal operator.

In this work, we prove the boundedness of the multi-sublinear maximal operator with rough kernels $M_{\Omega,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$, if $1<p_1,\ldots,p_m<\infty$ and $1/p=1/p_1+\ldots+1/p_m$, and from the space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $WL^{p,\lambda}(\mathbb{R}^n)$, if $1\le p_1,\ldots,p_m<\infty$ and $1/p=1/p_1+\ldots+1/p_m$ and at least one exponent p_i, $1\le i\le m$ equals one.

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2 Morrey spaces

For $x\in\mathbb{R}^n$ and $r>0$, let $B(x,r)$ denote the open ball centered at x of radius r. Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n\ge 2)$ equipped with the normalized Lebesgue measure $d\sigma$.

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Morrey spaces $L^{p,\lambda}$, named after C. Morrey, were introduced by him in 1938 in [9] and defined as follows: For $\lambda \in \mathbb{R}, 0 < p \leq \infty, f \in L^{p,\lambda}$ if $f \in L^p_{loc}$ and

$$\|f\|_{L^{p,\lambda}} \equiv \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda} \|f\|_{Lp(B(x,r))} < \infty,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at the point $x \in \mathbb{R}^n$ of radius $r > 0$.

In other words $f \in L^{p,\lambda}$ if $f \in L^p_{loc}(\mathbb{R}^n)$ and there exists $c > 0$ (depending on f) such that for all $x \in \mathbb{R}^n$ and for all $r > 0$

$$\|f\|_{L^p(B(x,r))} \leq cr^{\lambda}.$$

The minimal value of c in this inequality is $\|f\|_{L^{p,\lambda}}$.

If $\lambda = 0$, then

$$L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n).$$

If $\lambda = \frac{n}{p}$, then

$$L^{p,\frac{n}{p}}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n).$$

If $\lambda > \frac{n}{p}$ or $\lambda < 0$, then

$$L^{p,\lambda} = \Theta,$$

where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.
So the admissible range of the parameters is

$$0 < p \leq \infty \quad \textit{and} \quad 0 \leq \lambda \leq \frac{n}{p}.$$

(If $p = \infty$ then the inequality for λ holds only if $\lambda = 0$ and $L^{\infty,0} = L^\infty$.)
Under these assumptions, which will always be assumed in the sequel, the space $L^{p,\lambda}$ is a Banach space for $1 \leq p \leq \infty$ and a quasi-Banach space for $0 < p < 1$.
Also the space $L^{p,\lambda}$ does not coincide with a Lebesgue space, if and only if

$$0 < p < \infty \quad \textit{and} \quad 0 < \lambda < \frac{n}{p}.$$

Furthermore,

$$L^\infty \cap L^p \subset L^{p,\lambda}.$$

If $f \in L^p$, then $f \in L^{p,\lambda}$ if and only if

$$\sup_{x\in\mathbb{R}^n, 0<r\leq 1} r^{-\lambda} \|f\|_{L^p(B(x,r))} < \infty,$$

hence under this assumption only local properties of f are of importance.

Also by $WL^{p,\lambda}$ we denote the weak Morrey space, the space the space of all functions $f \in WL^p_{loc}$ with finite quasi-norm

$$\|f\|_{WL^{p,\lambda}} \equiv \|f\|_{WL^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda} \|f\|_{WL^p(B(x,r))}.$$

Consider the Nikolskii space [1] $H^\lambda_p \equiv H^\lambda_p(\mathbb{R}^n)$ of functions possessing common smoothness of order λ measured in the L^p metrics. For $\lambda > 0, 1 \leq p \leq \infty$ they are defined in the following way: $f \in H^\lambda_p$ if $f \in L^p$ and

$$\|f\|_{H^\lambda_p} = \|f\|_{L^p} + \sup_{h\in\mathbb{R}^n, h\neq 0} |h|^{-\lambda} \|\Delta^\sigma_h f\|_{L^p} < \infty,$$

where $\Delta^\sigma_h f$ is the difference of f of order $\sigma \in \mathbb{N}$ with step h and $\sigma > \lambda$. (For different $\sigma > \lambda$ the definitions are equivalent.) One can prove that if $0 < \lambda < \frac{n}{p}$, then

$$H^\lambda_p \subset L^{p,\lambda}.$$

(For n = 1 see [8], for $n > 1$ [10], [11].)
Clearly the converse inclusion does not hold, because if $f \in L^{p,\lambda}$, then clearly $fg \in L^{p,\lambda}$ for any bounded measurable function g, which is not true for the case of the spaces H^λ_p.
So, $L^{p,\lambda}$ is not a space of functions possessing any kind of common smoothness of order λ, but the expressions $\|f\|_{L^p(B(x,r))}$ behave like the ones for functions f possessing certain smoothness of order λ.

[1] Detailed exposition of properties of these spaces can be found in [11], [1].

3 Boundedness of maximal operator $M_{\Omega,m}$

In this part, we investigate the boundedness of maximal operator $M_{\Omega,m}$ (see Section 1 on Morrey spaces defined by the following definitions.

Recall the definition of $M_{\Omega,m}$, as a special case when $m=1$, $\Omega\equiv 1$ and $\theta_1=1$, $M_{\Omega,m}$ is the Hardy-Littlewood maximal operator M. In [2] Chiarenza and Frasca obtained the boundedness of M on Morrey spaces.

Lemma 3.1 *[2] Let $1\le p<\infty$ and $0\le\lambda<n$.Then for $p>1$, M is bounded from $L^{p,\lambda}$ to $L^{p,\lambda}$ and for $p=1$, to M is bounded from $L^{1,\lambda}$ to $WL^{1,\lambda}$.*

The following theorem was proved by Ding, Lu in [6].

Theorem 3.1 *Let p be the harmonic mean of $p_1,p_2,\dots,p_m>1$. Then we have the following conclusions.*
(i) If $p>1$, $\Omega\in L^s\left(\mathbb{S}^{n-1}\right)$, $s\ge 1$, then $M_{\Omega,m}(\mathbf{f})$ maps $L^{p_1}\times L^{p_2}\times\ldots\times L^{p_m}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.
(ii) If $p=1$, $\Omega\in L\log^+L\left(\mathbb{S}^{n-1}\right)$, then $M_{\Omega,m}(\mathbf{f})$ maps $L^{p_1}\times L^{p_2}\times\cdots\times L^{p_m}(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

When $m\ge 2$ and $\Omega\in L^s(\mathbb{S}^{n-1})$, we find out $M_{\Omega,m}$ also have the same properties by providing the following multi-version of the Lemma 3.1.

Theorem 3.2 *Let $\Omega\in L^s(\mathbb{S}^{n-1})$ with $1<s\le\infty$, $0\le\lambda<n$, p be the harmonic mean of $p_1,\dots,p_m>1$, $p\ge s'$ and satisfy*

$$\frac{\lambda}{p}=\sum_{j=1}^{m}\frac{\lambda_j}{p_j}\ \ for\ \ 0\le\lambda_j<n. \tag{3.1}$$

(i) If $p>s'$, there exists a positive constant C such that

$$\|M_{\Omega,m}\boldsymbol{f}\|_{L^{p,\lambda}}\le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p=s'$, there exists a positive constant C such that

$$\|M_{\Omega,m}\boldsymbol{f}\|_{WL^{p,\lambda}}\le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.$$

Here, we give the proof of Theorem 3.2.

Proof. Since $\Omega\in L^s(\mathbb{S}^{n-1})$ with $s>1$, Hölder's inequality yields that

$$\frac{1}{r^n}\int_{B(x,r)}|\Omega(y)|\prod_{j=1}^{m}|f_j(x-\theta_j y)|dy$$

$$\le\frac{1}{r^n}\left(\int_{B(x,r)}\prod_{j=1}^{m}|f_j(x-\theta_j y)^{s'}|dy\right)^{1/s'}\left(\int_{B(x,r)}|\Omega(y)|^s dy\right)^{1/s}$$

$$=\frac{1}{r^n}\left(\int_{B(x,r)}\prod_{j=1}^{m}|f_j(x-\theta_j y)^{s'}|dy\right)^{1/s'}\left(\int_0^r\int_{\mathbb{S}^{n-1}}|\Omega(\xi)|^s t^{n-1}d\xi dt\right)^{1/s}$$

$$=C\left(\frac{1}{r^n}\int_{B(x,r)}\prod_{j=1}^{m}|f_j(x-\theta_j y)^{s'}|dy\right)^{1/s'}$$

$$\le C\prod_{j=1}^{m}\left(\frac{1}{r^n}\int_{B(x,r)}|f_j(x-\theta_j y)|^{s'p_j/p}dy\right)^{p/s'p_j}\le C\prod_{j=1}^{m}\left[M(f_j^{s'p_j/p})(x)\right]^{p/s'p_j},$$

which implies a pointwise estimate

$$M_{\Omega,m}\mathbf{f}(x)\le C\prod_{j=1}^{m}\left[M(f_j^{s'p_j/p})(x)\right]^{p/s'p_j} \tag{3.2}$$

(i) If $p > s'$, by (3.2) and the Hölder inequality, we get

$$\frac{1}{t^\lambda}\int_{B(x,t)}|M_{\Omega,m}\mathbf{f}(y)|^p dy \le C\frac{1}{t^\lambda}\int_{B(x,t)}\prod_{j=1}^m\Big[M(f_j^{s'p_j/p})(y)\Big]^{p^2/s'p_j}dy$$

$$\le C\prod_{j=1}^m\Big(\frac{1}{t^\lambda}\int_{B(x,t)}\Big[M(f_j^{s'p_j/p})(y)\Big]^{p/s'}dy\Big)^{p/p_j}$$

for all $x\in\mathbb{R}^n$ and $t>0$. Taking the p-th root of both sides and applying Lemma 3.1 with $p/s'>1$ and the fact $f_j^{s'p_j/p}\in L^{p/s',\lambda_j}$, we get

$$\|M_{\Omega,m}\mathbf{f}\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{B(x,t)}|M_{\Omega,m}f(y)|^p dy\Big)^{1/p}$$

$$\le C\prod_{j=1}^m\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{B(x,t)}\Big[M(f_j^{s'p_j/p})(y)\Big]^{p/s'}dy\Big)^{1/p_j} = C\prod_{j=1}^m\Big\|M(f_j^{s'p_j/p})\Big\|_{L^{p/s',\lambda_j}}^{p/s'p_j}$$

$$\le C\prod_{j=1}^m\Big\|f_j^{s'p_j/p}\Big\|_{L^{p/s',\lambda_j}}^{p/s'p_j} = C\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}},$$

which is the desired inequality.

(ii) If $p=s'$, for any $\beta>0$, let $\varepsilon_0=\beta$, $\varepsilon_m=1$ and $\varepsilon_1,\varepsilon_2,...,\varepsilon_{m-1}>0$ be arbitrary which will be chosen later. From the pointwise estimate (3.2), we get

$$\{y\in B(x,t):|M_{\Omega,m}\mathbf{f}(y)|>\beta\}\subset\bigcup_{j=1}^m\Big\{y\in B(x,t):\Big[M(f_j^{s'p_j/p})(y)\Big]^{p/s'p_j}>\frac{\varepsilon_{j-1}}{t^{(\lambda-\lambda_j)/p_j}\varepsilon_j}\Big\}.$$

Let us now take $\varepsilon_1,\varepsilon_2,...,\varepsilon_{m-1}>0$ such that

$$\frac{\varepsilon_j}{\varepsilon_{j=1}} = \frac{\Big[\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}\Big]^{s'/p_j}}{\beta^{s'/p_j}\|f_j\|_{L^{p_j,\lambda_j}}},\qquad j=1,2,...,m.$$

Then, applying Lemma 3.1 with $p/s'=1$ and the fact $f_j^{p_j}\in L^{1,\lambda_j}$, we get

$$\Big|\Big\{y\in B(x,t):\big|M_{\Omega,m}\mathbf{f}(y)\big|>\beta\Big\}\Big|$$

$$\le C\sum_{j=1}^m\Big|\Big\{y\in B(x,t):M(f_j^{p_j})(y)>\Big(\frac{\varepsilon_{j-1}}{t^{(\lambda-\lambda_j)/p_j}\varepsilon_j}\Big)^{p_j}\Big\}\Big|$$

$$\le C\sum_{j=1}^m t^{\lambda_j}\Big(\frac{t^{(\lambda-\lambda_j)/p_j}\varepsilon_j}{\varepsilon_{j-1}}\Big)^{p_j}\|f_j^{p_j}\|_{L^{1,\lambda_j}} = C\sum_{j=1}^m t^\lambda\Big(\frac{\varepsilon_j}{\varepsilon_{j-1}}\Big)^{p_j}\|f_j\|_{L^{p_j,\lambda_j}}^{p_j}$$

$$= C\sum_{j=1}^m t^\lambda\Big[\Big(\frac{\varepsilon_j}{\varepsilon_{j-1}}\Big)\|f_j\|_{L^{p_j,\lambda_j}}\Big]^{p_j} = C\sum_{j=1}^m t^\lambda\left(\frac{1}{\beta}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}\right)^{s'} = Ct^\lambda\left(\frac{1}{\beta}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}\right)^{p}.$$

Hence, we obtain the following inequality

$$\|M_{\Omega,m}\mathbf{f}\|_{WL^{p,\lambda}} = \sup_{\beta>0}\beta\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\Big|\Big\{y\in B(x,t):|M_{\Omega,m}\mathbf{f}(y)|>\beta\Big\}\Big|\Big)^{\frac{1}{p}}\le C\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}.$$

This is the conclusion (ii) of Theorem 3.2.

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