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On Asymptotic Behavior Of The Mean Value Of The First Passage Time Of The Level By A Random Walk Described By Autoregression Process Of Order One ($AR\,(1)$)

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Abstract. *In the paper, the first order autoregression process $AR\,(1)$ with a discrete time is considered. The asymptotics for the mean value of the first passage time of the level by a random walk described by the process $AR\,(1)$ is found.*

1. Introduction. Let on some probability space $(\Omega, \mathcal{F}, P)$ be given a sequence of independent identically distributed random variables $\xi_n, n \geq 1$.

It is well known that the autoregression process of order one $AR\,(1)$ with discrete time is determined by the recurrent equality

$$X_n = \beta X_{n-1} + \xi_n, \tag{1}$$

$n \geq 1$, here it is assumed that the initial value X_0 is independent of $\xi_n, n \geq 1$.

As a rule the time series models that play a great role in applied problems of theory of random processes are described by autoregression processes ([2], [8]).

In the paper we consider a family of stopping moments of the form

$$\tau_a = \inf \{n : S_n \geq a\}, a > 0 \tag{2}$$

where $S_n = \sum\limits_{k=0}^{n} X_k^2$, and random variables X_k are determined by equality (1).

Similar families of stopping moments arise in applied problems of probability theory and mathematical statistics ([1-4], [7]).

The goal of the paper is to study asymptotic behavior of mathematical expectation $E\tau_a$ as $a \to \infty$.

The similar problem was studied in [6] for the family of passage times of the level by a random walk formed by the sums $\sum\limits_{k=0}^{n} X_k X_{k-1}, n \geq 1$.

2. **Formulation and proof of the basic result**. It holds

Theorem. *Let* $E\xi_n = 0,\ D\xi_n = 1, EX_0^2 < \infty$ *and* $|\beta| < 1$.

Assume that for some $\varepsilon, 0 < \varepsilon < \frac{1}{1-\beta^2} = \lambda$ *the following condition*

$$\sum_{n=1}^{\infty} P\left(S_n \leq n\left(\lambda - \varepsilon\right)\right) < \infty \tag{3}$$

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is fulfilled. Then

$$\frac{E\tau_a}{N_a} \to 1, \text{ as } a \to \infty,$$

where

$$N_a = a\left(1 - \beta^2\right).$$

Remark 1. Note that in the case $\beta = 0\,(X_n = \xi_n)$, from this theorem it follows that for the family of stopping times

$$t_a = \inf\left\{n : \sum_{k=0}^{n} \xi_k^2 > a\right\}, \ \ \xi_0 = X_0$$

it holds the relation

$$\frac{Et_a}{a} \to 1, \text{ as } a \to \infty,$$

that also may be obtained from theorem 4.4 of [10].

Note that in [7] some asymptotic properties of the sequences of sums $S_n = \sum\limits_{k=0}^{n} X_k^2$, $n \geq 1$ were studied.

In particular, in the paper [7] (see also [8]) it is proved that under the condition $EX_0^2 < \infty$ and $|\beta| < 1$ it holds the almost sure convergence

$$\frac{S_n}{n} \overset{a.s.}{\to} \frac{1}{1-\beta^2}, n \to \infty. \tag{4}$$

3. In the proof of the theorem the following result formulated in the form of a lemma plays a key role.

Lemma. *Let the family of random variables $Y_a, a \geq 0$ be uniformly integrable and converge in probability to some random variable Y, i.e.*

$$Y_a \overset{P}{\to} Y \text{ as } a \to \infty.$$

Then

$$EY_a \to EY \text{ as } a \to \infty.$$

The statement of this lemma follows from theorem 1.1 of [10], (see also theorem 4.5.4 from [9]).

Proof of the theorem. From the definition of the family of stopping moments τ_a we have

$$\frac{S_{\tau_a - 1}}{\tau_a} < \frac{a}{\tau_a} \leq \frac{S_{\tau_a}}{\tau_a}. \tag{5}$$

By theorem 2.1 of [9], from (4) and (5) it follows

$$\frac{\tau_a}{a} \overset{a.s.}{\to} 1 - \beta^2 \text{ as } a \to \infty, \tag{6}$$

as $\tau_a \overset{a.s.}{\to} \infty$ as $a \to \infty$ (see [5]).

Then for obtaining the statement of the lemma from the indicated lemma it is necessary to show that the family $Y_a = \frac{\tau_a}{a}$, $a > 0$ is uniformly integrable, i.e. the following relation is fulfilled:

$$\sup_a \int\limits_{\frac{\tau_a}{a} > c} \frac{\tau_a}{a} dP \to o \ \text{ as } \ c \to \infty. \tag{7}$$

Let for $\varepsilon \in (0, \lambda)$ condition (3) be fulfilled. Assume

$$K_a = \left[\frac{a}{\lambda - \varepsilon}\right] + 1.$$

It is clear that for $n > K_a$ we have

$$n > \frac{a}{\lambda - \varepsilon} \quad \text{or} \quad a < n(\lambda - \varepsilon).$$

Therefore for $n > K_a$ it holds

$$P(\tau_a > n) \le P(S_n < a) \le P(S_n < n(\lambda - \varepsilon)). \tag{8}$$

It is easy to see that for sufficiently large c and a it holds

$$\int\limits_{\tau_a > ca} \frac{\tau_a}{a} dP \le \int\limits_{\tau_a > 2K_a} \tau_a dP. \tag{9}$$

It is clear that on the set $\{\tau_a > 2K_a\} = \{\omega : \tau_a > 2K_a\}$ it is fulfilled the inequality

$$\tau_a < 2(\tau_a - K_a).$$

Therefore we can write

$$\int\limits_{\tau_a > 2K_a} \tau_a dP \le \int\limits_{\tau_a > 2K_a} (\tau_a - K_a) dP \le 2 \int\limits_{\tau_a > K_a} (\tau_a - K_a) dP =$$

$$= 2\sum_{k=0}^{\infty} P(\tau_a > K_a + k) = 2 \sum_{n=K_a}^{\infty} P(\tau_a > n).$$

Hence, taking into account (8), we get

$$\int\limits_{\tau_a > 2K_a} \tau_a dP \le 2 \sum_{n=K_a}^{\infty} P(S_n < n(\lambda - \varepsilon)).$$

Hence, by condition (3) we have

$$\int\limits_{\tau_a > 2K_a} \tau_a dP \to o \quad \text{as} \quad a \to \infty.$$

This means that for any $\varepsilon > 0$ there exists a sufficiently large number a_0 such that

$$\int\limits_{\tau_a > 2K_a} \tau_a dP \le \varepsilon \quad \text{for} \quad a \ge a_0.$$

Then from (9) we get

$$\int\limits_{\frac{\tau_a}{a} > c} \frac{\tau_a}{a} dP \le \frac{1}{a_0} \int\limits_{\tau_a > ca_0} \tau_a dP \le \frac{1}{a_0} \int\limits_{\tau_a > 2K_{a_0}} \tau_a dP \le \varepsilon$$

for all $a \ge a_0$.

Hence it follows (7), i.e. we get that the family $\frac{\tau_a}{a}, a \ge a_0$ is uniformly integrable.

Thus the statement of the theorem follows from (6), (7) and from the lemma.

Remark 2. As is known [10], if the sum S_n is the sum of independent identically distributed random variables with finite variance and with positive mean value, then $\sum\limits_{n=1}^{\infty} P(S_n \le 0) < \infty$.

By means of this fact it is easy to see that if $D\xi_n^2 < \infty$, then condition (3) of the theorem is fulfilled for the case $\beta = 0$ (see remark 1).

Remark 3. The family of the stopping moments (2) arises in the problems of test of statistical hypothesis with respect to the parameter β. In these problems, τ_a plays the role of necessary observations (test) for accepting the hypothesis and it is necessary to know approximate values of mathematical expectation (mean value) $E\tau_a$.

$E\tau_a$ may be calculated according to the proved theorem by the approximate formula

$$E\tau_a \approx a\left(1-\beta^2\right)$$

for $a \geq a_0$, where $a_0 > 0$ is some beforehand chonse number.

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